Entanglement dynamics in 1+1 dimensional systems: the Ising spin chain

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Thanks to: O. Castro-Alvaredo, M. Lencses and I. Szczesny

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Quantum quench protocol

1. Prepare a system at $t < 0$ in the GS $|\Psi_0\rangle$ of a Hamiltonian $H(\{\lambda\})$.
   Example (Ising spin chain):

$$H(\{\lambda\}) = -\sum_n \left[\sigma_n^x \sigma_{n+1}^x + h_z \sigma_n^z + h_x \sigma_n^x\right], \quad \{\sigma_n^\alpha, \sigma_m^\beta\} = 2\delta_{n,m}\delta_{\alpha,\beta}$$

2. At time $t = 0$, suddently modify the Hamiltonian $H(\{\lambda\}) \to H(\{\lambda'\})$.
   Example:

$$h_x \to h_x + \delta_{h_x}; \quad \text{longitudinal field quench}$$

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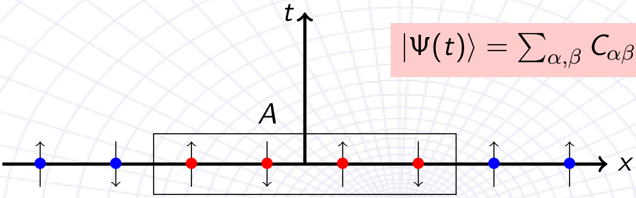
3. Study the **unitary** time evolution for positive times

$$\boxed{|\Psi(t)\rangle = e^{-iH(\{\lambda'\})t}|\Psi_0\rangle}$$

- Try to infer **large time** behaviour of local observables (Ex. σ_n^x, $\sigma_n^z, \ldots$) and entanglement entropies (relaxation, thermalization, etc...)

Entanglement evolution in 1+1 d (Definitions)

- Consider the spacial bipartition of the Hilbert space: A $\otimes$ B



$$|\Psi(t)\rangle = \sum_{\alpha,\beta} C_{\alpha\beta}(t)|\alpha\rangle_A \otimes |\beta\rangle_B$$

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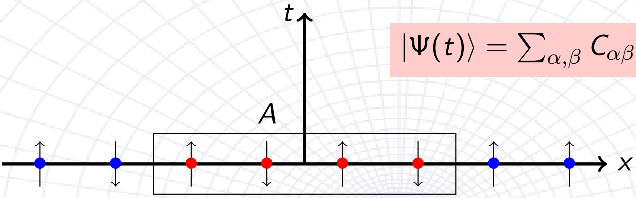


$$|\Psi(t)\rangle = \sum_{\alpha,\beta} C_{\alpha\beta}(t)|\alpha\rangle_A \otimes |\beta\rangle_B$$

- From the rectangular matrix $C(t)$ we obtain the reduced density matrix on the right

$$\rho_A(t) := \mathrm{Tr}_B|\Psi(t)\rangle\langle\Psi(t)|$$

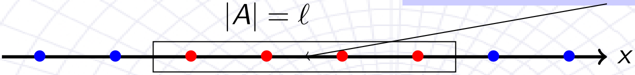
- From the eigenvalues of ρ_A, one builds usual entanglement measures

$$\Delta S_A(t) := S_A(t) - S_A(0);\;\; S_A(t) = -\mathrm{Tr}_A[\rho_A(t)\log\rho_A(t)]$$

Entanglement and thermalization: basics

- How to construct a **statistical ensemble** for local correlations inside A after the system relaxes?

$$\rho_{\text{stat}} = \lim_{t\to\infty} \mathrm{Tr}_B |\Psi(t)\rangle\langle\Psi(t)|$$



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$$|A| = \ell$$

- Stationary entropy is the **entanglement** entropy!
- Suppose to **quench** h_z in the Ising spin chain, keeping $h_x = 0$

$$|\Psi(t)\rangle = e^{it\sum_n \overbrace{\sigma_n^x\sigma_n^x + (h_z+\delta_{h_z})\sigma_n^z}^{-H}} \underbrace{|\Psi_0\rangle}_{\text{GS for } h_z}$$

- The model maps to **free Majorana fermions** of mass $m \propto (1-h_z)$

$$H = \sum_k \varepsilon(k) b^\dagger(k) b(k), \;\; \{b^\dagger(k), b(k')\} = \delta_{kk'}$$

Entanglement and thermalization: basics

- For a transverse field quench the **stationary density matrix** is

$$\rho_{\mathrm{stat}} = \frac{1}{Z} e^{-\sum_k \beta(k) b^\dagger(k) b(k)}$$

- The coefficients β's are fixed by the **initial state**

$$\mathrm{Tr}[\rho_{\mathrm{stat}} b^\dagger(k) b(k)] = \langle \Psi_0 | b^\dagger(k) b(k) | \Psi_0 \rangle \equiv \alpha(k)$$

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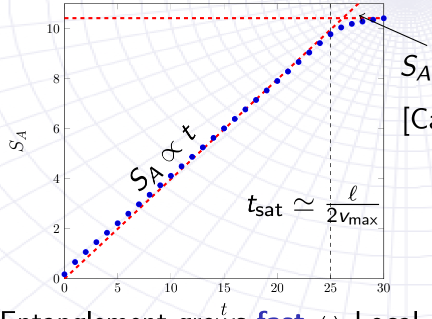
- Entanglement entropy is $O(1)$ at $t = 0$ and $O(\ell)$ at $t = \infty$.



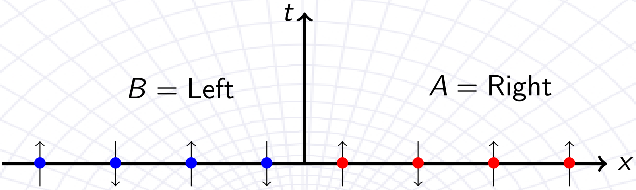
$$S_A(\infty) = -\mathrm{Tr}[\rho_{\text{stat}} \log(\rho_{\text{stat}})]$$

[Calabrese, Fagotti and Essler (2012)]

- Entanglement grows **fast** $\Leftrightarrow$ Local observables **relax** to equilibrium

Left-Right entanglement

- **Simpler** version of the problem: A=Right (R) and B=Left (L)



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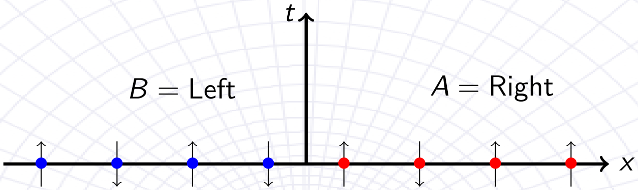

- At $t = 0$, the system is the ground state of $H(\{\lambda\})$, $|\Psi_0\rangle$
- How does entanglement propagate from **Left to Right**?

$$\Delta S_R(t) = S_R(t) - S_R(0);\ S_R(t) = -\mathrm{Tr}_R[\rho_R(t)\log\rho_R(t)]$$

- According to the heuristics, it should grow linearly in time **forever**

$$\Delta S_R(t) \overset{t\gg 1}{=} \Gamma t$$

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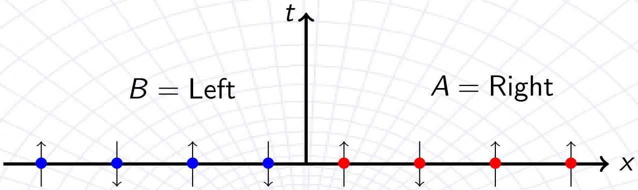

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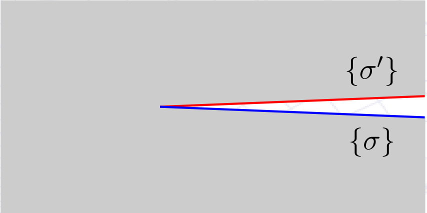
- This setup can be studied **analytically** [O. Castro-Alvaredo, M. Lencses, I. Szczesny and JV; JHEP 2019, PRL 2020]

Entanglement evolution in 1+1 d: Twist Fields

- At **real** time $t = 0$, the element

$$\rho_R^{\{\sigma,\sigma'\}}(t=0)$$

is a partition function on the complex plane with a **slit**



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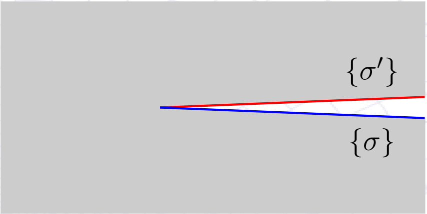

- $e^{(1-n)\times \text{Reny Entropy}}$ is a **partition function** on a n-sheeted Riemann surface. This can be calculated using the **twist-field**

$$\mathrm{Tr}_R \rho_R^n(t=0) \propto {}^{\otimes n}\langle \Psi_0 | \mathcal{T}(0,0) | \Psi_0 \rangle^{\otimes n}$$

- **Time evolving** in real time with $H(\{\lambda'\})$ for $t>0$ on each replica

$$\mathrm{Tr}_R \rho_R^n(t) \propto {}^{\otimes n}\langle \Psi_0 | e^{it\sum_{r=1}^n H^{(r)}(\{\lambda'\})} \mathcal{T}(0,0) e^{-it\sum_{r=1}^n H^{(r)}(\{\lambda'\})} | \Psi_0 \rangle^{\otimes n}$$

Example: Ising spin chain close to criticality

- Consider again the Ising spin chain

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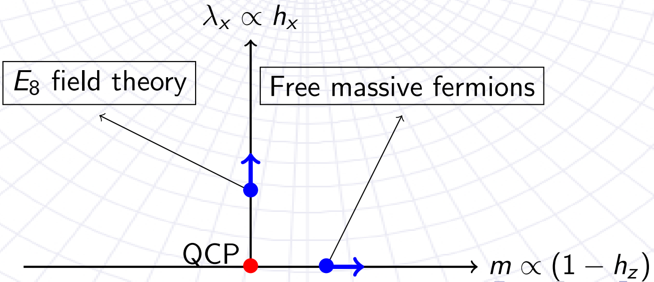
$$H(\{\lambda\}) = -\sum_n \left[\sigma_n^x \sigma_{n+1}^x + h_z \sigma_n^z + h_x \sigma_n^x\right]$$

- Close to the critical point $\lambda_x = \lambda_z = 0$, fluctuations are described by the effective action

$$\mathcal{A}_{eff} = \int dtdx\ \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - \lambda_x \int dtdx\ \sigma\ ; \quad m \propto (1 - h_z);\ \ h_x \propto \lambda_x$$

- **Problem:** How entanglement grows after quenching the couplings?



E_8 symmetry and Ising chain in a longitudinal field

- **Mass spectrum** conjectured long ago [A. B. Zamolodchikov 1989]
- It consists of 8 stable particles: $m_1 < m_2 < m_3 \cdots < m_8$.
- Measured **experimentally** from FT of the two-point function of σ^x

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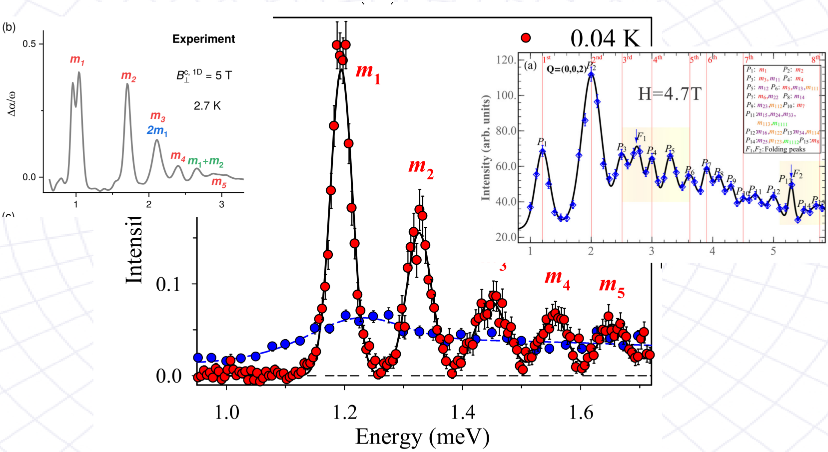
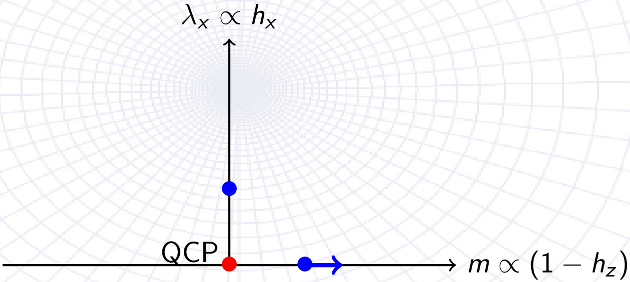


- Coldea et al. (2011); Zhang et al. (2020); Zou et al. (2020).

Results for the Ising spin chain close to criticality



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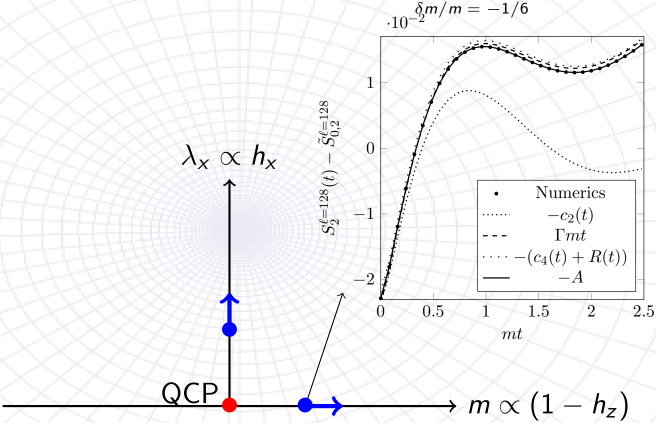

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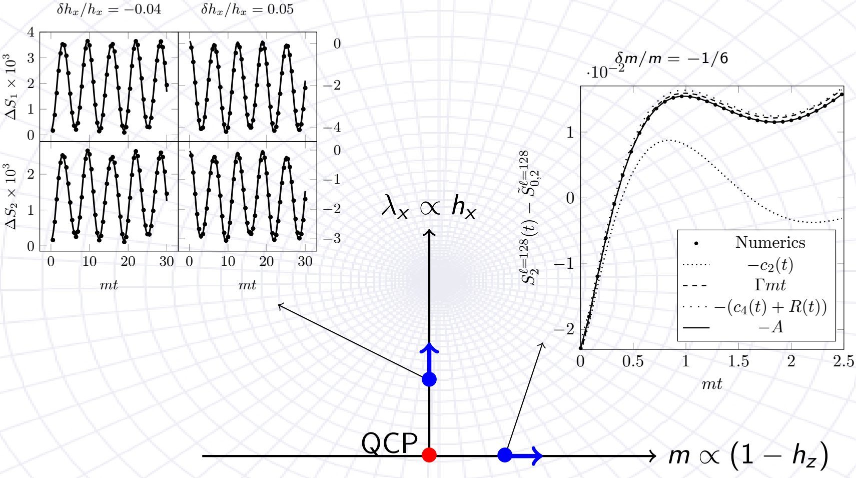

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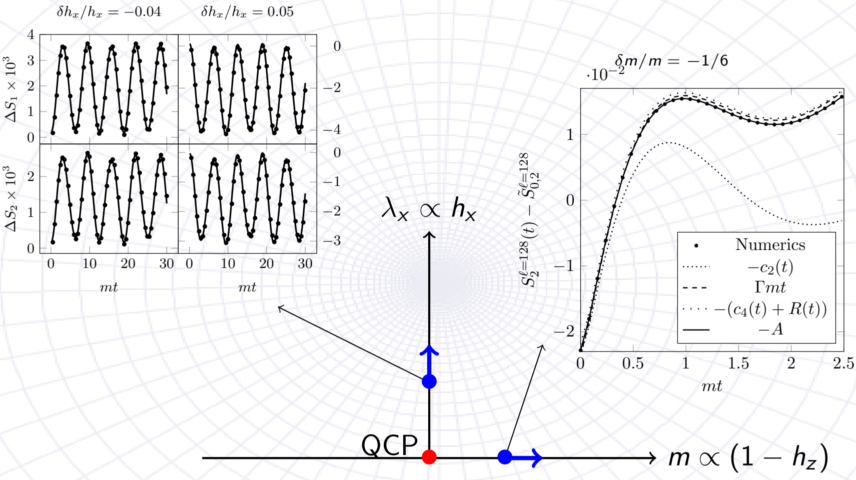

- **Perturbative** representation of the initial state ($\delta\lambda_x/\lambda_x \ll 1$ or $\delta m/m \ll 1$) & **spectral** decomposition of the 1pt function
- **Numerical** tests through MPS and exact lattice results

Entanglement spectroscopy along the E_8 line

- From the FT transform of the entanglement entropy we can extract the **masses** of E_8 field theory

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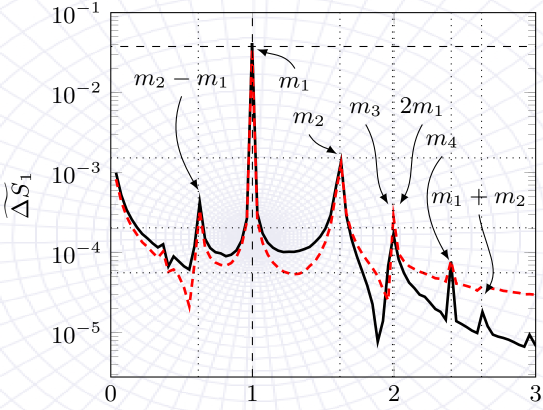

- These can be compared with the LR entanglement entropy obtained **numerically (MPS)** from the state:

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Will entanglement eventually grow?

- Let me recap/conclude

$$H(\{\lambda\}) = -\sum_n \left[\sigma_n^x \sigma_{n+1}^x + h_z \sigma_n^z + h_x \sigma_n^x\right]$$

- QFT results obtained for $\delta_{h_x}/h_x \ll 1$ suggest that entanglement growth is strongly **suppressed** after a quench of the longitudinal field
- Notice instead that for a quench of the transverse field δ_{h_z}/h_z entanglement grows **linearly**
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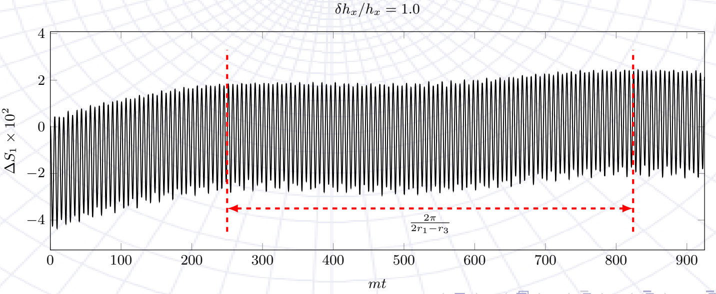

Conclusions and perspectives

- **Take Home Message:** Entanglement dynamics is a powerful diagnostic to understand subsystem thermalization at large times
- Entanglement grows fast in time ($\propto t$) $\Rightarrow$ extensive Shannon entropies ($\propto L$) for $t \gg L$. Some systems fail to relax.
- **TODO[*]:** Application of the formalism to other 1d chains: Ising in imaginary longitudinal field, XYZ scaling limit of Sine-Gordon
- **TODO[**]:** A physical lattice picture that explains absence of entanglement growth for small longitudinal field in Ising (see Milsted, Liu, Preskill and Vidal 2012.07243)
- **TODO[***]:** Calculate second order contributions in δ_λ/λ to the Rényi entropies by using QFT
- **TODO[***]:** Role of measurements on entanglement evolution, can we formulate them in a QFT setting?