GENERALIZING THE COUPLING BETWEEN SPACETIME AND MATTER

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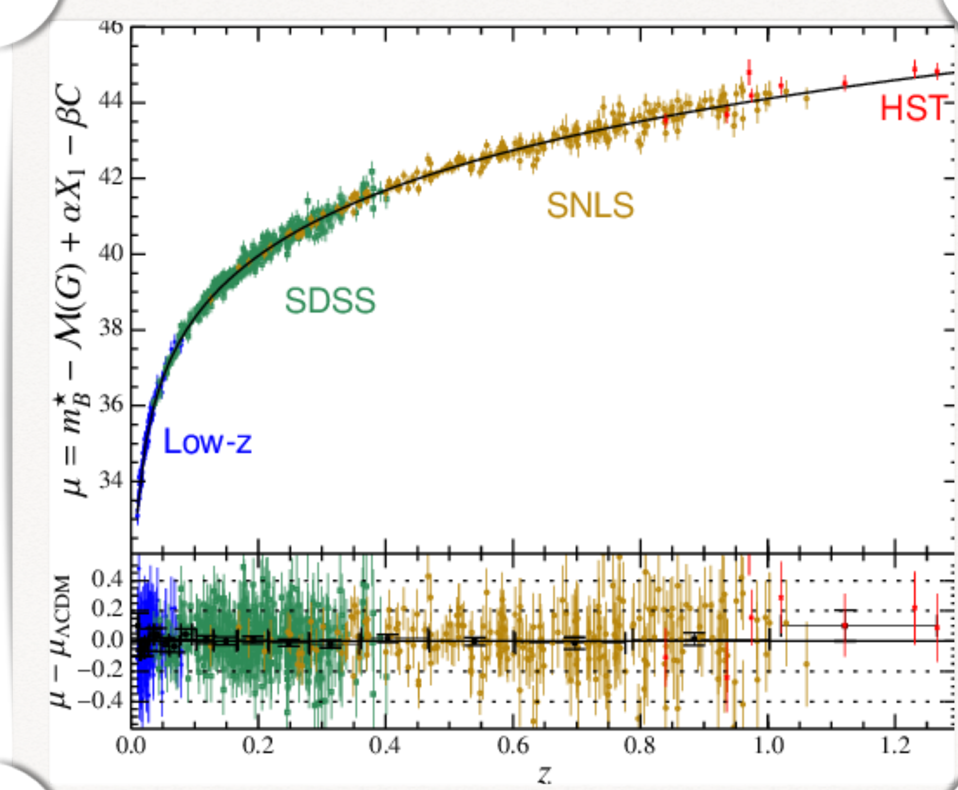
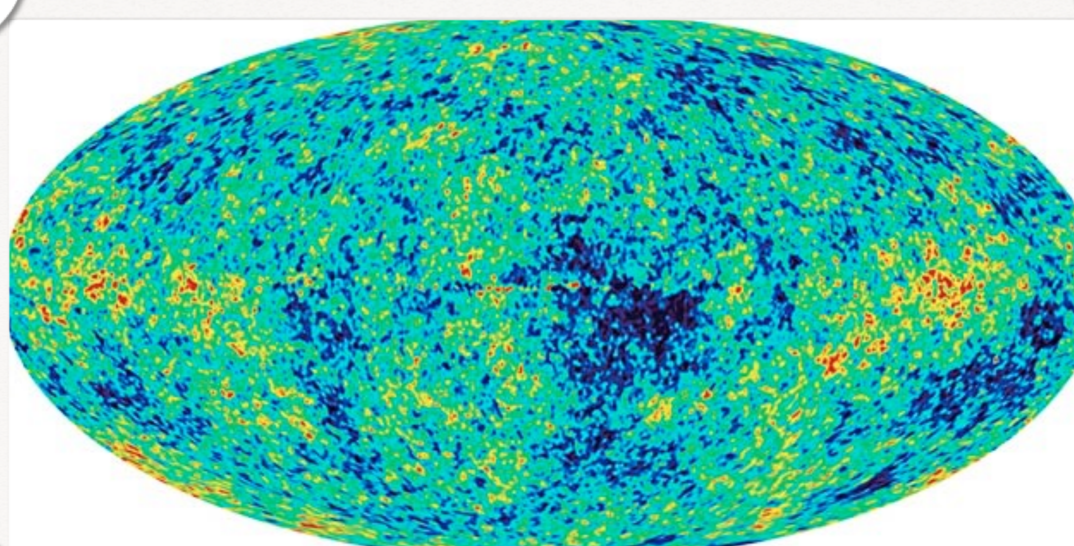
THE DARK UNIVERSE: DARK ENERGY

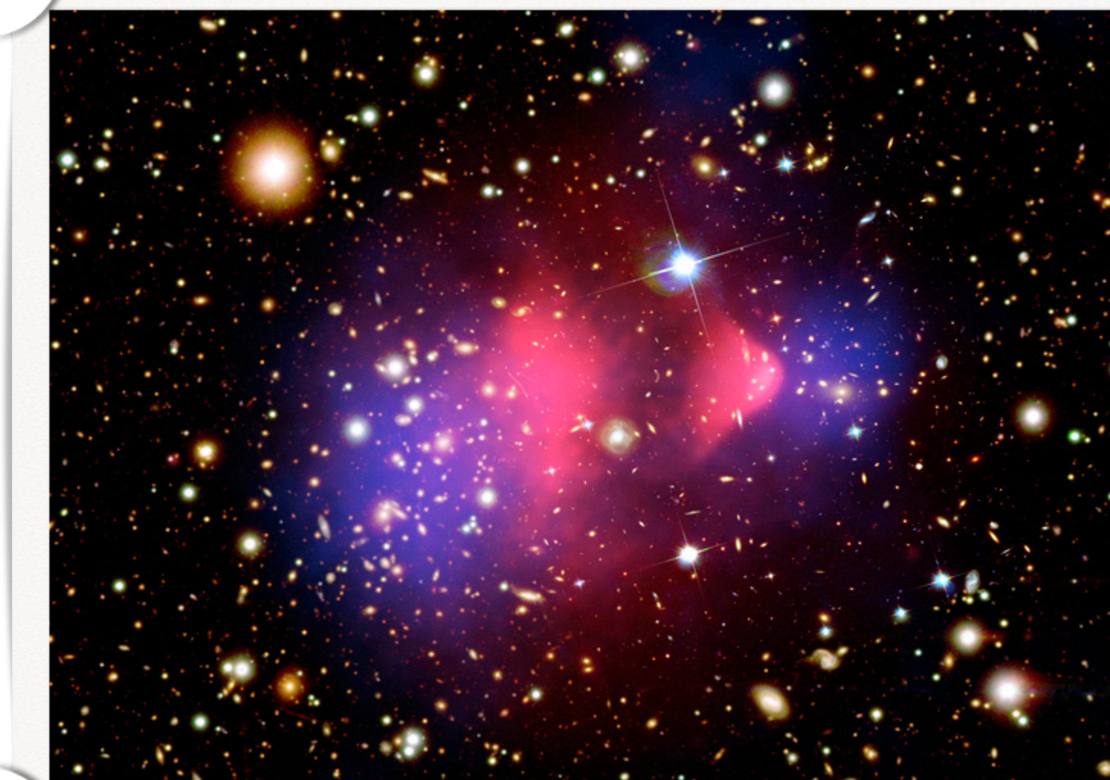
- The expansion rate of the universe is not what it is supposed to be (as from GR)
- One way to explain this is that the Universe is filled with a new form of "energy" i.e. a fluid with pressure called Dark Energy (DE)
- Dark Energy has negative pressure!
- DE is the main constituent of the Universe and dominates its evolution, but is YET undetected

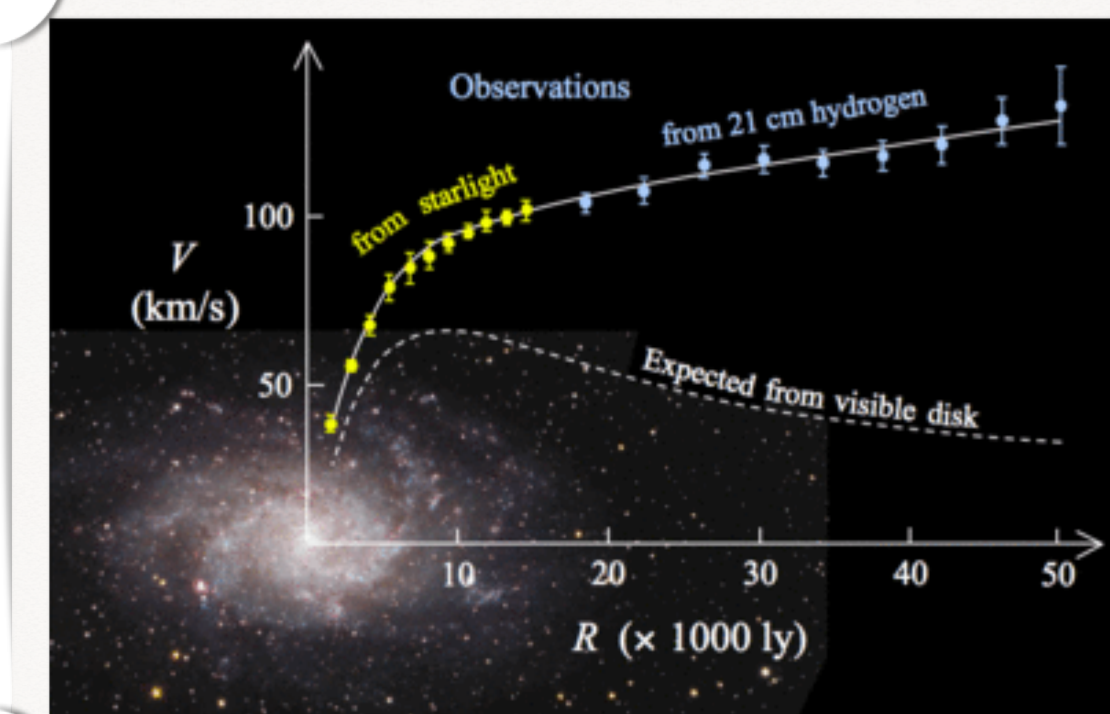




THE DARK UNIVERSE: DARK MATTER

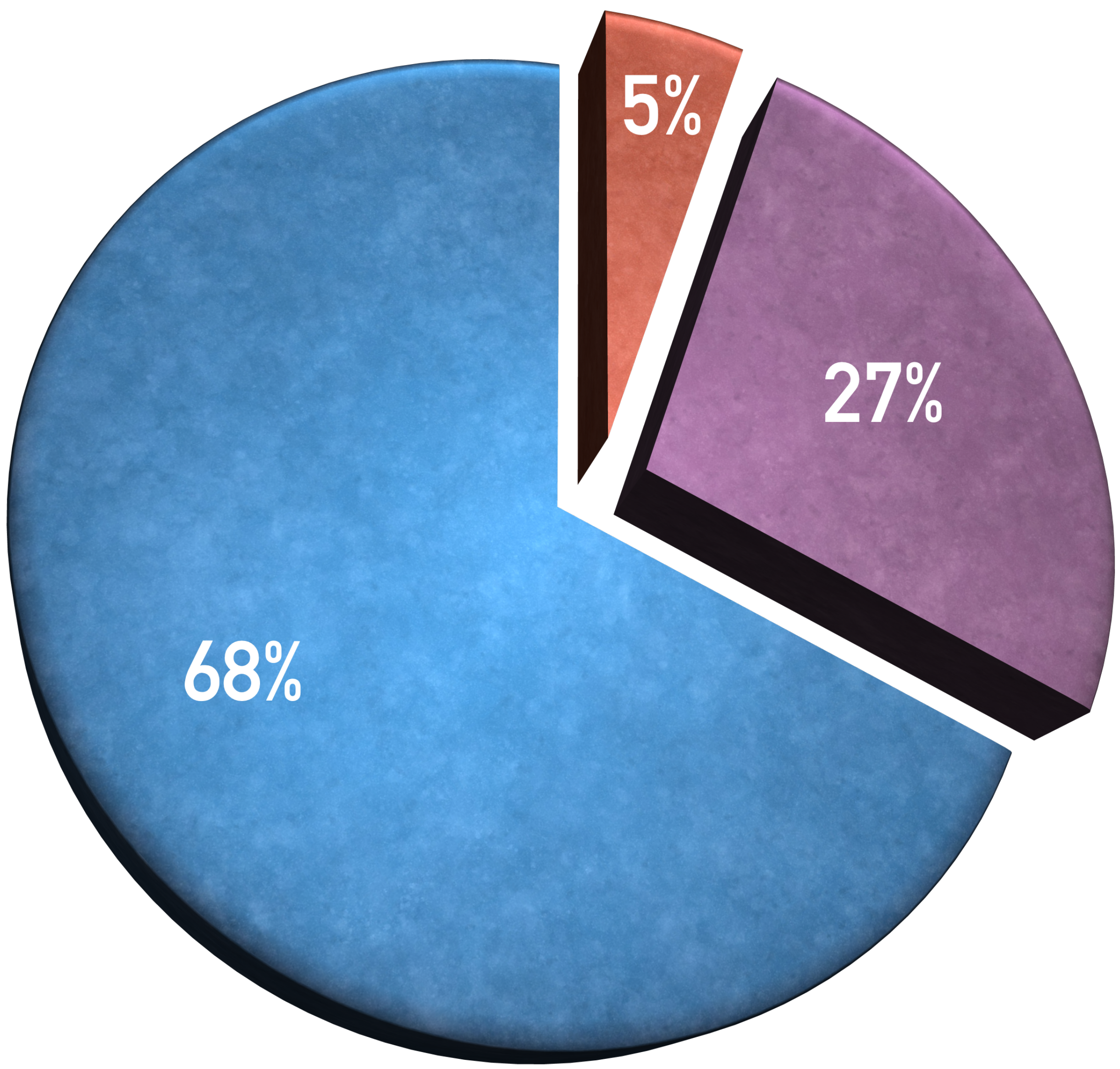
- Astrophysical objects do no seem to move according to the Newtonian laws of motion (and by extension GR)
- One way to explain this is that these objects are made mostly by a form of matter (fluid with zero pressure) called Dark Matter (DM)
- DM does not to interact with light
- Also structures in the Universe do not form at the correct rate without an additional zero pressure component
- Primordial Nucleosynthesis seems to require DM
- DM is the second most abundant constituent of the universe but it has defied detection so far.





DARK UNIVERSE

The picture that emerges is as fascinating as unsettling…



MODELLING THE DARK SECTOR

New Fields

- *Quintessence,*
- *Phantom fields,*
- *S-matter*

Change in the Geometry

- *Averaging*
- *Tolman-Bondi*
- *Special Solutions*

Dark Sector

Modify the Law of Gravitation

- *Higher order,*
- *Nonminimally coupled,*
- *Auxiliary fields*

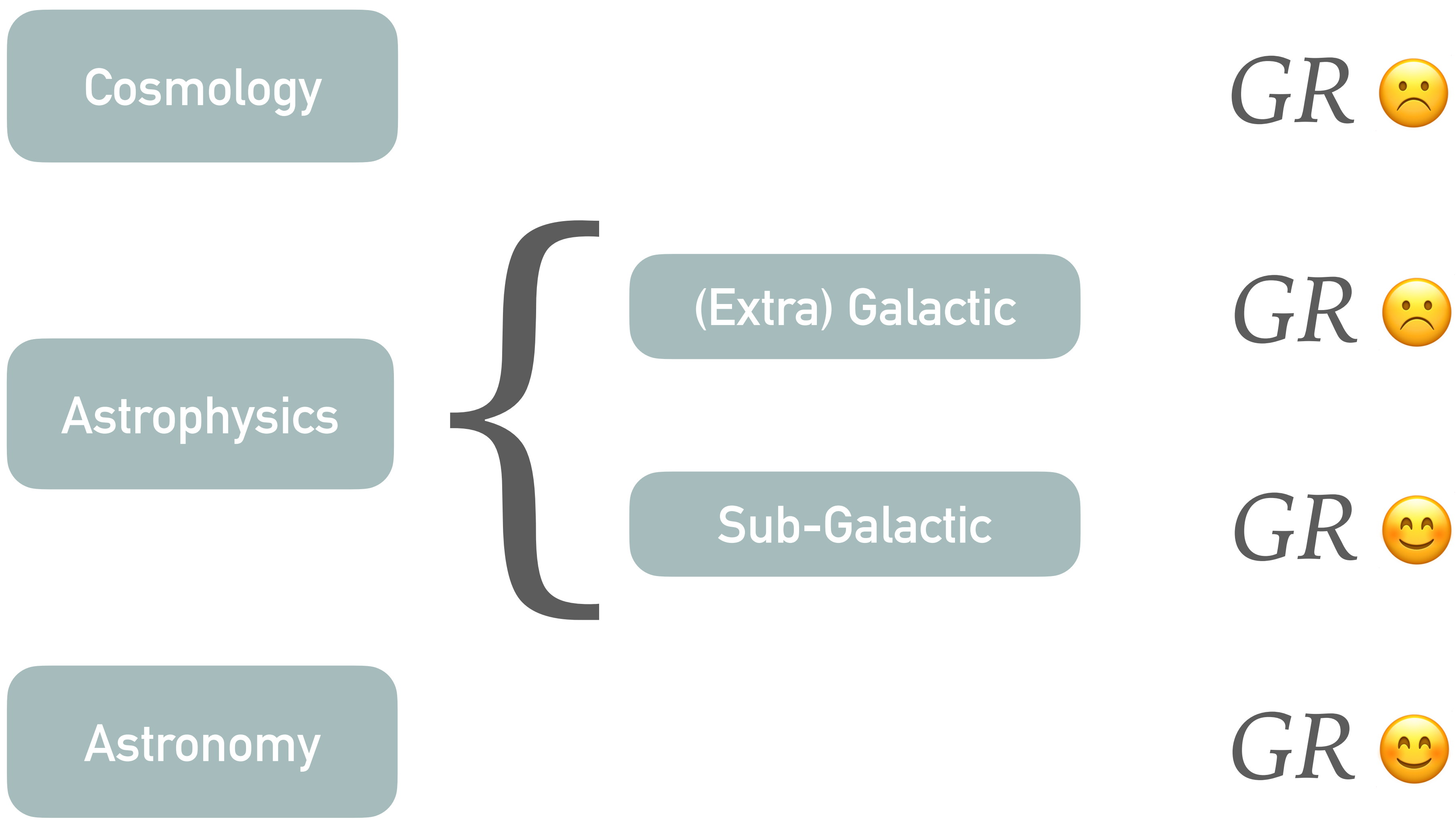
Anything else??

S.C. PLB 766 (2017) 55-58

J. Feng et al. PRD 101 (2020) 6, 064002,

J. Feng et al. PRD 103 (2021) 8, 084055

MODIFICATIONS OF GR: "HOW TO" GUIDE

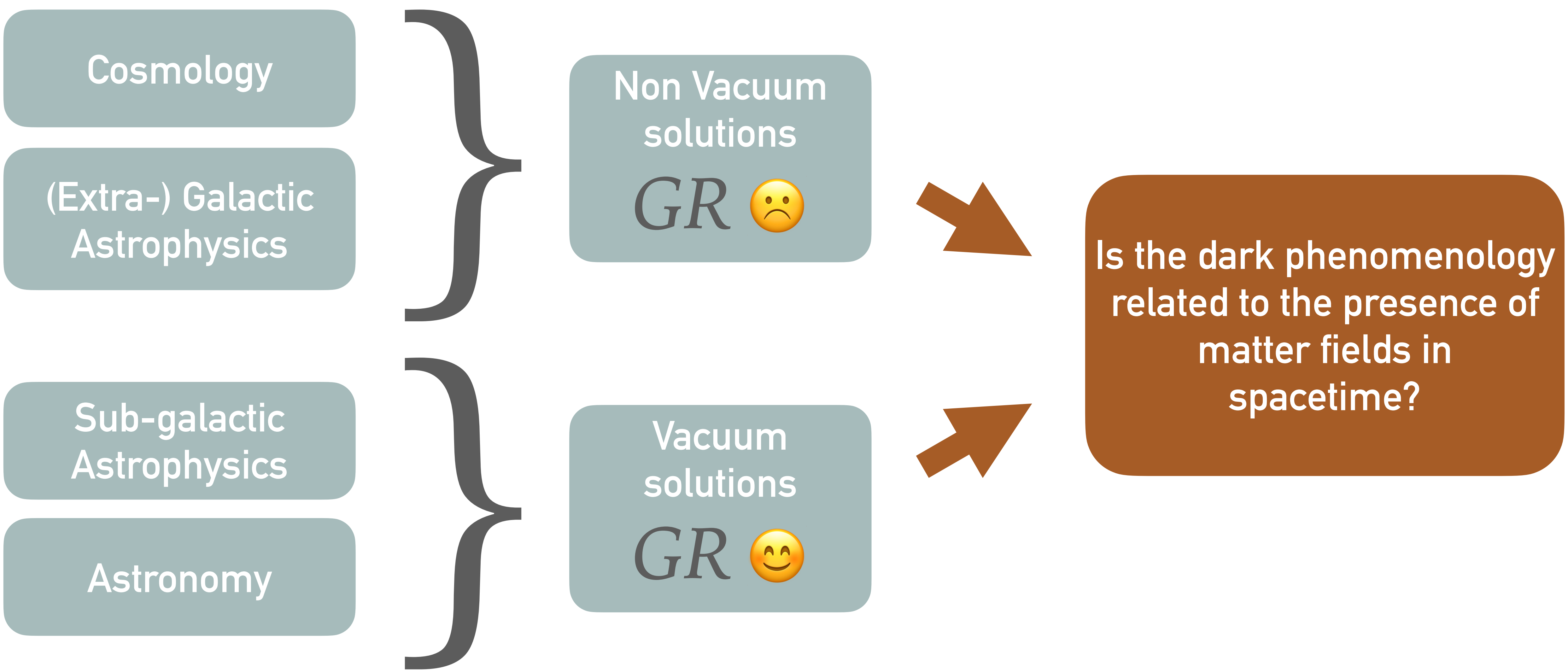


So, we need to have a modification of the gravitational interaction which is relevant only above EG astrophysics.

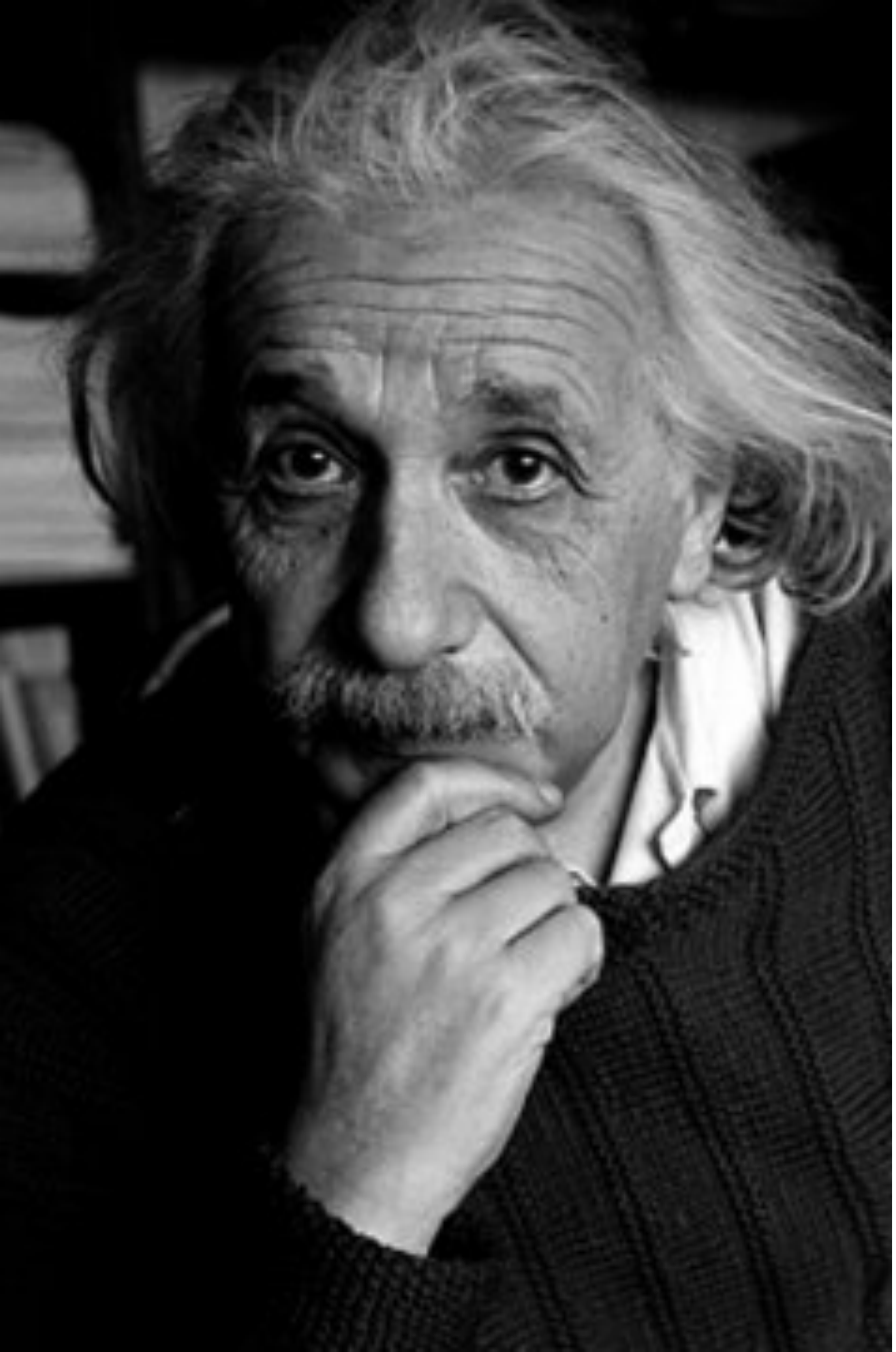
This is complicated, as changes in a non-linear theory will, in general, affect all scales.

MATTERS OF GRAVITY (COUPLING)

What if we reason in terms of matter content of the spacetime?



If the answer is yes, then the key must be in the structure of the Einstein equations with matter…



SPACETIME-MATTER COUPLING

- One of the most important foundational assumptions of General Relativity is how geometry and matter interact.
- Einstein chose a constant proportionality relation

$$G_{\mu\nu} = \kappa T_{\mu\nu}$$

- This choice was motivated by:
  - ⋆ Poisson like equations
  - ⋆ Variational structure of the vacuum equations
  - ⋆ Standard conservation equations for matter

GENERALISED COUPLING

We can preserve the above features also if we consider a more general version of the EFE:

$$G_{ab} = \chi_{ab}{}^{cd} T_{cd}$$

The tensor χ represent a generalised coupling between matter and spacetime. It has the following properties:

$$\chi_{(ab)(cd)} = \chi_{abcd}$$

$$\chi_{abcd} \quad \text{invertible}$$

$$\chi_{abcd} = \chi_{abcd}(\mu_0) \leftrightarrow \chi_{abcd}(0) = 0$$

The tensor χ is also dynamical. Its evolution is given by the Bianchi identities and reads

$$\left(\nabla^b \chi_{ab}{}^{cd}\right) T_{cd} + \chi_{ab}{}^{cd} \nabla^b T_{cd} = 0 \qquad (\nabla_b T^b_a = 0)$$

GENERALISED COUPLING

Because for the properties above, in cases in which the r.h.s of the field equations is negligible the new field equations reduce to

$$G_{ab} = 0$$

In other words the new theory:

- ✦ reproduces all the vacuum phenomenology of standard GR
- ✦ satisfies all the tests for relativistic gravitation in sub-galactic astrophysics (GW, BH, pulsar, etc)
- ✦ satisfies all the local tests for relativistic gravitation in Astronomy (Newtonian and PPN)

Deviation from General Relativity can be then observed only when matter is present (non vacuum spacetimes).

A FORM FOR THE COUPLING TENSOR

The most important problem to solve is to understand the nature of the coupling tensor.

To simplify things we assume that the fourth order tensor χ can be written in terms of second order tensors:

$$\chi_{\mu\nu}{}^{\alpha\beta} = \Psi(A_{\cdot}{}^{\cdot})\, A_{\mu}{}^{\alpha} A_{\nu}{}^{\beta}$$

We will also assume that the fields $A_{\mu}{}^{\alpha}$ are non-dynamical in the sense that they are generated by an action without a kinetic term.

Indeed we can assume that the action for the coupling field $A_{\mu}{}^{\nu}$ is

$$S_A[A_{\cdot}{}^{\cdot}, g^{\cdot\cdot}] = -\frac{\lambda}{\kappa}\int d^4x\sqrt{-g}\,F(A_{\cdot}{}^{\cdot})$$

where F is a generic analytic function.

ACTION FOR GENERALIZED COUPLING THEORIES.

The field equations with generalized couplings can then be obtained if we consider an action of the type

$$S[\phi, g^{\cdot\cdot}, A_{\cdot}^{\cdot}] = \int d^4x \left\{ \left(R - 2\left[\Lambda - \lambda(1 - F)\right]\right)\sqrt{-g} + 2\,\kappa\,\mathscr{L}_m[\phi, \mathfrak{g}^{\cdot\cdot}]\sqrt{-\mathfrak{g}} \right\}$$

where we have introduced the tensor

$$\mathfrak{g}_{\mu\nu} = \chi_{\mu\nu}{}^{\alpha\beta}\, g_{\alpha\beta} = \Psi(A_{\cdot}^{\cdot})\, A_{\mu}{}^{\alpha} A_{\nu}{}^{\beta}\, g_{\alpha\beta}$$

This suggests that the introduction of a non trivial coupling between matter and geometry can be realized within a bi-metric framework.

Matter couples with spacetime via an effective metric $\mathfrak{g}_{\mu\nu}$ rather than $g_{\mu\nu}$ and this feature can be used to include additional degrees of freedom to the classical field description used in GR (e.g. quantum properties).

THE GENERAL FIELD EQUATIONS

We can then write the file equations as

$$G_{\mu\nu} + \left[\Lambda - \lambda\,(1 - F)\right]\, g_{\mu\nu} = \kappa\,\Psi\,|\,A\,|\,\bar{A}^{\alpha}{}_{\mu}\,\bar{A}^{\beta}{}_{\nu}\,\mathfrak{T}_{\alpha\beta}$$

$$A_{\beta}{}^{\alpha} - \delta_{\beta}{}^{\alpha} = q\left[\frac{1}{4}\mathfrak{T}\,A_{\beta}{}^{\alpha} - \mathfrak{T}_{\beta\nu}\,\mathfrak{g}^{\alpha\nu}\right]$$

where

$$\mathfrak{T}_{\alpha\beta} = -\,2\frac{\partial L_m}{\partial \mathfrak{g}^{\alpha\beta}} + \mathfrak{g}_{\alpha\beta}L_m\,.$$

What is the form of the function F?

A particularly significative choice, which gives $A_{\mu}{}^{\nu} = \delta_{\mu}{}^{\nu}$ in vacuum, is a vacuum energy term related to $\mathfrak{g}_{\mu\nu}$

$$S_A[A^{\cdot}_{\cdot}, g^{\cdot\cdot}] = -\,\frac{\lambda}{\kappa}\int d^4x\sqrt{-\mathfrak{g}} = -\,\frac{\lambda}{\kappa}\int d^4x\sqrt{-g}\,|\,A\,|$$

THE MINIMAL EXPONENTIAL MEASURE MODEL

The model we obtain is MEMe. It is characterized by the equations

$$G_{\mu\nu} + \left[\Lambda - \lambda\,\left(1 - |A|\right)\right]\, g_{\mu\nu} = \kappa\,|A|\,\bar{A}^{\alpha}{}_{\mu}\,\bar{A}^{\beta}{}_{\nu}\,\mathfrak{T}_{\alpha\beta}$$

$$A_{\beta}{}^{\alpha} - \delta_{\beta}{}^{\alpha} = q\left[\frac{1}{4}\mathfrak{T}\,A_{\beta}{}^{\alpha} - \mathfrak{T}_{\beta\nu}\,\mathfrak{g}^{\alpha\nu}\right] \qquad \mathfrak{g}_{\mu\nu} = A_{\mu}{}^{\alpha}\,A_{\nu}{}^{\beta}\,g_{\alpha\beta}$$

If the matter sector is defined by a perfect fluid we can write

$$G_{\mu\nu} = \kappa\left[\,|A(\boxed{q})|\,(p+\rho)\,U_{\mu}\,U_{\nu} + \left(\frac{|A(\boxed{q})|\,(p\,\boxed{q} - 1) + 1}{\boxed{q}} - \frac{\Lambda}{\kappa}\right) g_{\mu\nu}\right]$$

$$A_{\mu}{}^{\alpha} = Y\,\delta_{\mu}{}^{\alpha} - 4(1-Y)\,U_{\mu}\,U^{\alpha} \qquad Y = \frac{4(1-p\,\boxed{q})}{4-\boxed{q}\,(3\,p-\rho)}$$

$$|A(q)| = \frac{(1-p\,\boxed{q})^{3}(\boxed{q}\,p+1)}{\left[1-\frac{\boxed{q}}{4}(3p-\rho)\right]^{4}} \qquad g^{\mu\nu}U_{\mu}U_{\nu} = -1 \qquad \boxed{q = \frac{\kappa}{\lambda}}$$

THE MINIMAL EXPONENTIAL MEASURE MODEL

If $|A| = 0$, since $\mathfrak{g}_{\mu\nu}$ cannot be inverted, MEMe breaks down as a bi-metric theory. This occurs at a scale $\lambda_* = \kappa/q_*$ such that

$$\rho_* \approx \frac{1}{|q|_*} \qquad q < 0 \qquad\qquad p_* \approx \frac{1}{q_*} \qquad q > 0$$

At the scale λ_* the tensor $\mathfrak{g}_{\mu\nu}$ cannot be considered as a metric, and no field theory on $\mathfrak{g}_{\mu\nu}$ can be constructed. Thus λ_* resembles a Planck length even though it can have any value, as it is not related to $g_{\mu\nu}$.

Close to λ_* the theory can be approximated as

$$G_{\mu\nu} \approx \kappa(\lambda_* - \Lambda)g_{\mu\nu}$$

i.e. we have a (anti-)deSitter behavior, depending on the relative value of λ_* and Λ.

λ_* corrects the "bare" vale of the cosmological constant in this regime.

HOW DO WE CONSTRAINT THE PARAMETER q ?

In MEMe gravitational waves propagate according to $g_{\mu\nu}$ while light propagates via $\mathfrak{g}_{\mu\nu}$. Therefore there is a natural discrepancy between light signals and gravitational waves, but no causality violations (Babichev2008).

The relation between the two speeds is

$$\left(\frac{c_g}{c}\right)^2 = 1 - \frac{q\,(p+\rho)}{1+q\rho} \approx 1 - q\,(p+\rho) + O(q^2)$$

Using the timing uncertainty in GW170817, which also traversed Earth before being detected, we find $q < 0$ and the upper limit

$$|q| < 2 \times 10^{-14}\, m^3/J$$

For comparison, the inverse of the higher density probed in accelerators is $\sim 10^{-36}\,\mathrm{m^3/J}$ (Pasechnik2017; Mitchell2016).

COSMOLOGY

We can grasp the behavior of MEMe on cosmological spacetimes using the Dynamical Systems Approach. In FLRW

$$ds^2 = -dt^2 + S^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]$$

The cosmological equations read, for a barotropic fluid with $p = w\rho$,

$$3q\left(H^2 + \frac{k}{S^2}\right) = \frac{\kappa(1-pq)^3(q\rho+1)^2}{\left[1+\frac{q}{4}(\rho-3p)\right]^4} + q\Lambda - \kappa$$

$$6q\left(\dot{H} + H^2\right) = -\frac{\kappa(pq-1)^3(q\rho+1)[2-q(\rho+3p)]}{\left[1+\frac{q}{4}(\rho-3p)\right]^4} + 2(q\Lambda - \kappa)$$

COSMOLOGY

Defining the dimensionless variables

$$\chi = \frac{qH^2}{\kappa}, \qquad K = \frac{k}{H^2S^2}, \qquad \Omega = \frac{\kappa\rho}{3H^2}, \qquad L = \frac{\Lambda}{3H^2}, \qquad \mathcal{N} = \ln\frac{S}{S_0}$$

The cosmological equations can be written as

$$\chi_{,\mathcal{N}} = 2(L-1)\chi + \chi R(\Omega,\chi)\frac{[3\chi\Omega(1+3w)-2]}{3\chi\Omega+1} - \frac{2}{3},$$

$$L_{\mathcal{N}} = \frac{1}{3}L\left\{\frac{2}{\chi} + 6(1-L) - R(\Omega,\chi)\right\}$$

$$\Omega_{\mathcal{N}} = \frac{1}{3}\Omega\left\{\frac{2}{\chi} + 6(1-L) - \chi R(\Omega,\chi)\frac{[3\chi\Omega(1+3w)-2]}{3\chi\Omega+1} - Q(\Omega,\chi)\right\}$$

$$L - K - R(\Omega,\chi) - \frac{1}{3\chi} = 1$$

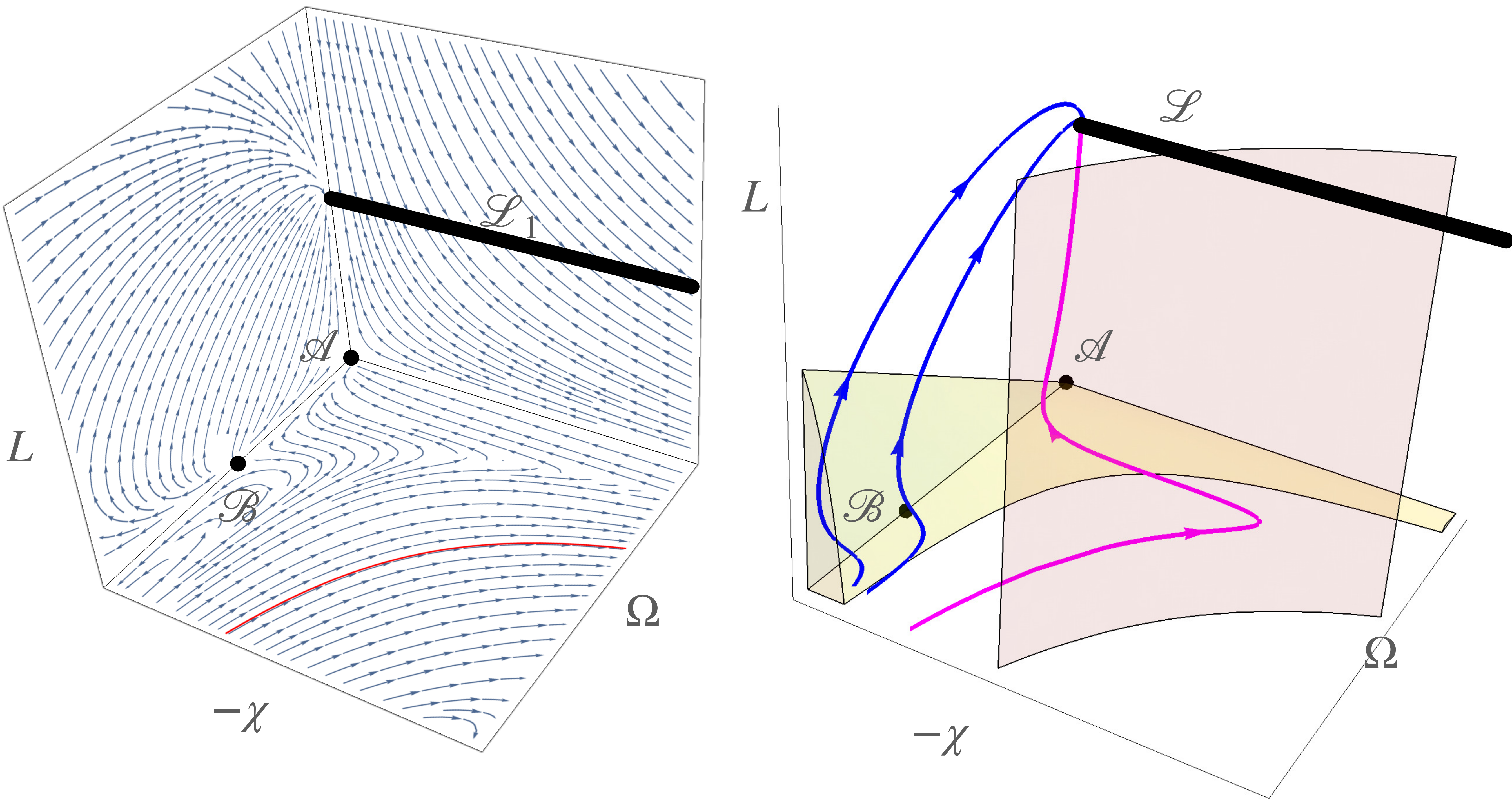
COSMOLOGY

Point	$\{\chi, \Omega, L, K\}$	Attractor	Repeller	Scale factor	Energy density
$\mathcal{A}$	$\{0, 0, 0, -1\}$	Never	Never	$S = S_0(t - t_0)$	$\rho = 0$
$\mathcal{B}$	$\{0, 1, 0, 0\}$	Never	Never	$S = S_0(t - t_0)^{\frac{2}{3(w+1)}}$	$\rho = \frac{\rho_0}{(t-t_0)^2}$
$\mathcal{L}_1$	$\{\chi_0, 0, 1, 0\}$	$0 \leq w \leq 1$	Never	$S = S_0 e^{\gamma_1 (t-t_0)}$	$\rho = 0$
$\mathcal{L}_2$	$\{\frac{1}{3w\Omega_0}, \Omega_0, 1 + w\Omega_0, 0\}$	$0 < w \leq 1$	Never	$S = S_0 e^{\gamma_2 (t-t_0)}$	$\rho = \rho_0$

If we assume high densities at an early time, one expects initial conditions to be close to the subspace $3\Omega\chi = -1$ which corresponds to $\rho q = -1$. This condition naturally implies an inflationary phase.

Relevant orbits stemming from this subspace "bounce" against the general Friedmann fixed point ($\mathscr{B}$) and then, maintaining a low value for χ, the orbits evolve toward one of the attractors of the lines $\mathscr{L}$.

COSMOLOGY



Thus MEMe is able to offer a unified framework for dark energy and inflation.

CONCLUSION AND PERSPECTIVES

- An alternative way to look at the problem of dark phenomenology is in terms of matter content of a spacetime, rather than scales.
- This idea leads to the definition of a new class of extensions of GR, called Generalized Coupling Theories. We focused on a specific model belonging to this class, called **M**inimal **E**xponential **Me**asure model.
- The MEMe model is a modification of GR that introduces a single new parameter and is able to pass all the observational tests for gravitation in vacuum (astronomical motions and GW)
- The MEMe model can be proven to be able to mimic dark energy phenomenology as well as inflation
- Signatures of the MEMe model are unlikely to appear in galactic rotation curves and diluted matter distributions, but may still produce measurable differences in the interior of neutron stars (at least at PPN level).

CONCLUSION AND PERSPECTIVES

- Much to understand yet. For example:
  - The the structure of relativistic compact object and nature of junction conditions will more likely bring even more constraints on q
  - The corrections to the Newtonian potential appear very close to the non vacuum extension of proposal of corpuscular gravity by Dvali and Gomez.
  - This connection suggests, together with the properties of the MEMe model at high densities, that in this framework the gravitational collapse do not result in a singularity.
  - How the idea of generalized coupling fits with other theories of (quantum) gravity, e.g. conformal gravity?

ADDITIONAL MATERIAL

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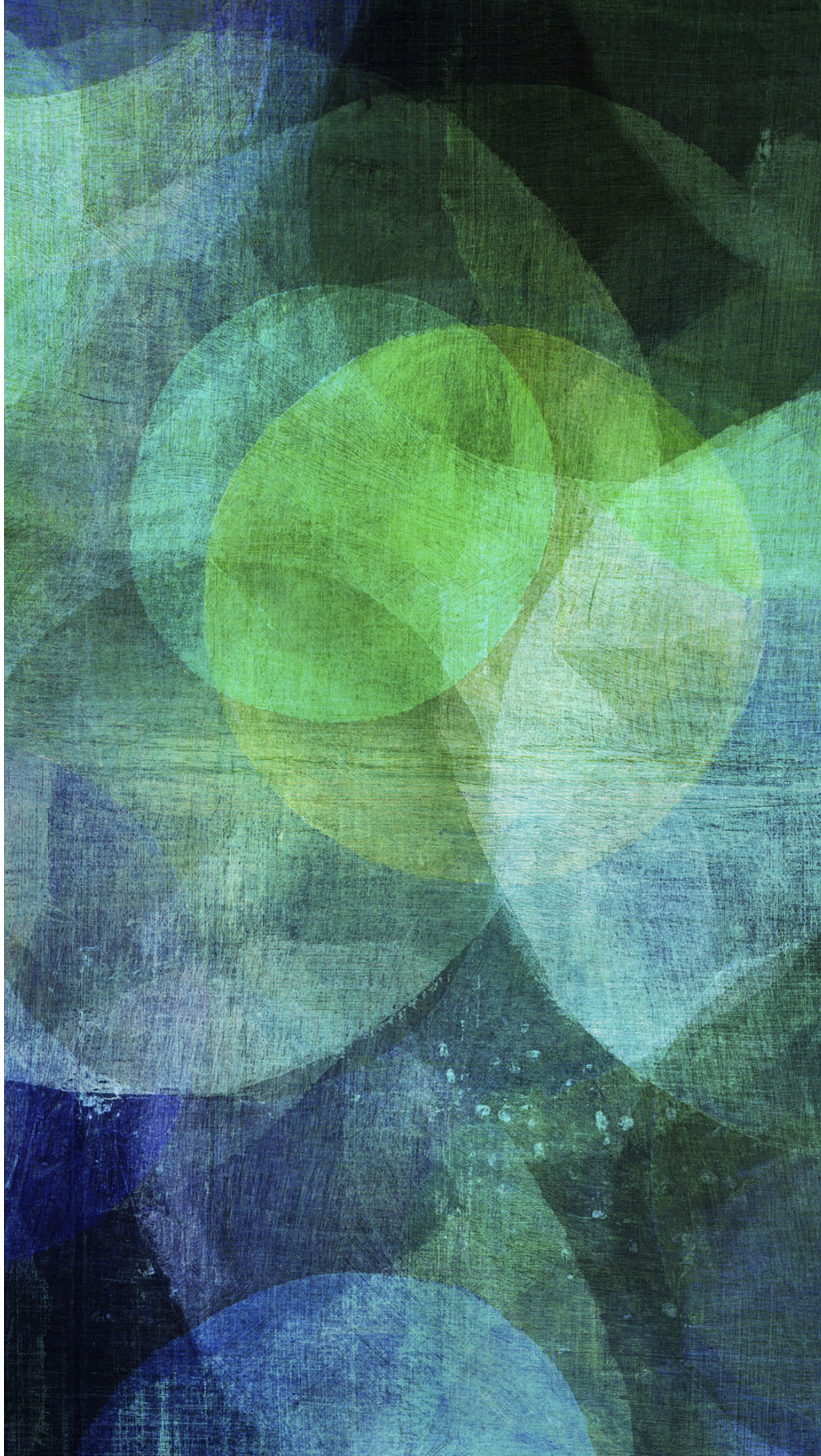
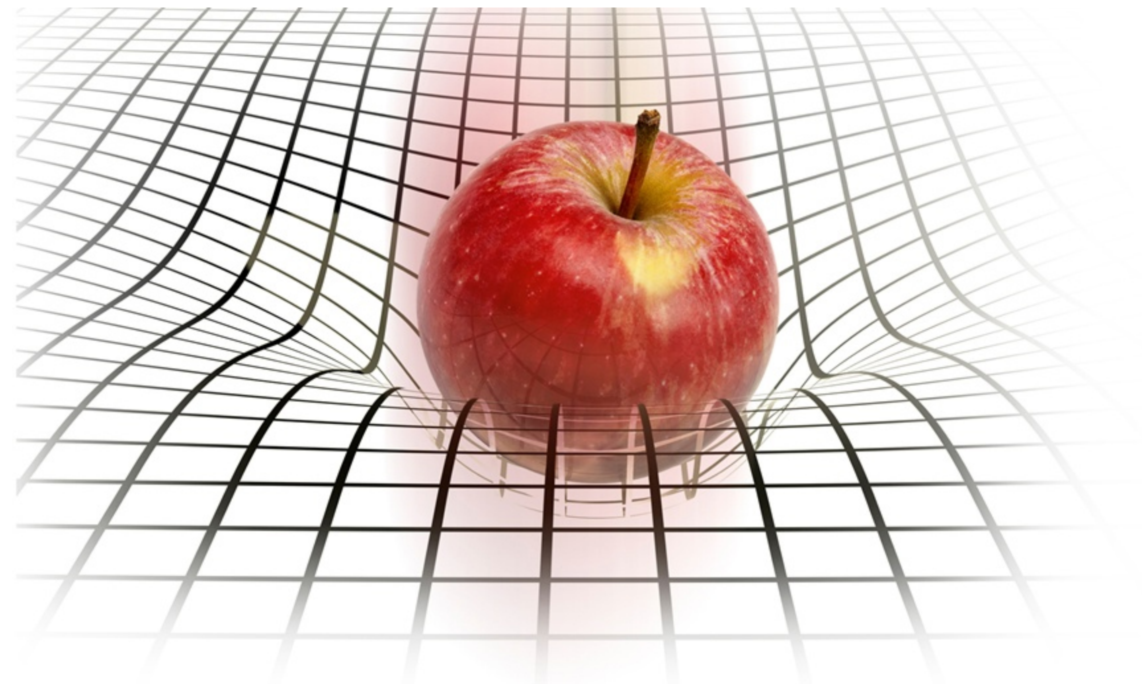


NEWTONIAN LIMIT



To understand how MEMe behaves in weakly gravitating astrophysical systems, like galaxies, we need to investigate the Newtonian limit of the theory inside a matter distribution.

As we are in a matter distribution, the curvature of spacetime is measured by $\mathfrak{g}_{\mu\nu}$ and we will refer to this tensor for our expansion.

The gravitational potential associated to $\mathfrak{g}_{\mu\nu}$ can be written

$$\Phi_J = \Phi_E + \frac{q}{2\kappa}\Delta\Phi_E \, .$$

where Φ_E is the standard Newtonian potential.

Notice that in the previous expression the term $\Delta\Phi_E$ might lead to divergencies if ρ has a sharp discontinuity (e.g. boundary of a star)

POST NEWTONIAN LIMIT

The issue with the Newtonian potential becomes worse if we go to the Post-Newtonian approximation.

We can solve this problem by introducing one additional potential and some counterterms (Feng, Mukhoyama, C. 2021). In particular:

$$
\begin{aligned}
\mathfrak{g}_{00} =& \tilde{g}_{00} + c_0 \Delta U + c_1 \Delta \Phi_1 + c_2 \Delta \Phi_2 + c_3 \Delta \Phi_3 \\
& + c_4 \Delta \Phi_4 + c_\Psi \Delta \Psi + c_w \Delta \Phi_W \\
\mathfrak{g}_{0j} =& g_{0j}^{PPN} + d_V \Delta V_j + d_W \Delta W_j \\
\mathfrak{g}_{ij} =& g_{ij}^{PPN} + e_0 \Delta U \delta_{ij}
\end{aligned}
$$

Where

$$\tilde{g}_{00} = g_{00}^{PPN} + 2\nu\Psi \qquad\qquad \Delta\Psi \approx -4\pi G^2 \rho^{*2}$$

and ρ^* is the conserved rest mass in the sense of Will (Will 2018).

POST NEWTONIAN LIMIT

and

$$
\begin{aligned}
g_{00}^{PPN} =& -1 + 2U - 2\beta U^2 + (2\gamma + 1 + \alpha_3 + \zeta_1 - 2\xi)\Phi_1 \\
& + 2(1 - 2\beta + \zeta_2 + \xi)\Phi_2 + 2(1 + \zeta_3)\Phi_3 + 2(3\gamma + 3\zeta_4 - 2\xi)\Phi_4 \\
& - (\zeta_1 - 2\xi)\Phi_6 - 2\xi\Phi_W \\
g_{0j}^{PPN} =& -\tfrac{1}{2}(4\gamma + 3 + \alpha_1 - \alpha_2 + \zeta_1 - 2\xi)V_j - \tfrac{1}{2}(1 + \alpha_2 - \zeta_1 + 2\xi)W_j \\
g_{ij}^{PPN} =& (1 + 2\gamma U)\delta_{ij}
\end{aligned}
$$

with

$$
\begin{aligned}
\Delta U &= -4\pi G\,\rho^* & \text{where:} \quad \rho^* &:= \sqrt{-g}\,u^0\,\rho \\
\Delta V_i &= -4\pi G\,\rho^*\,v_i & \Delta W_i &= -4\pi G\,\rho^*\,v_i + 2\partial_i\partial_t U \\
\Delta\Phi_1 &= -4\pi G\,\rho^*\,v^2 & \Delta\Phi_2 &= -4\pi G\,\rho^*\,U \\
\Delta\Phi_3 &= -4\pi G\,\rho^*\,\Pi & \Delta\Phi_4 &= -4\pi G\,p
\end{aligned}
$$

POST NEWTONIAN LIMIT

$$\Phi_6 = G\int \rho^{*\prime}\frac{\left[\vec{v}\cdot(\vec{x}-\vec{x}')\right]^2}{|\vec{x}-\vec{x}'|^3}d^3x'$$

$$\Phi_W = G\int\int \rho^{*\prime}\rho^{*\prime\prime}\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\cdot\left[\frac{\vec{x}'-x''}{|\vec{x}'-x''|}\right]d^3x'd^3x''$$
$$-\int\int \rho^{*\prime}\rho^{*\prime\prime}\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\cdot\left[\frac{\vec{x}-x''}{|\vec{x}'-x''|}\right]d^3x'd^3x''$$

Substituting into the field equations we find the coefficients for the PPN expansion. In the case of the MEMe model we have

$$c_0 = \frac{3q}{8\pi G}, \quad c_1 = \frac{5q}{16\pi G}, \quad c_2 = -\frac{3(3\gamma+2)q}{8\pi G} \quad c_3 = \frac{3q}{8\pi G},$$
$$c_4 = \frac{3q}{8\pi G}, \quad c_\Psi = \frac{21q^2}{64\pi G^2}, \quad c_w = 0 \quad e_0 = \frac{q}{8\pi G},$$
$$d_V = -\frac{q}{2\pi G}, \quad d_W = 0 \quad \nu = \frac{3q}{2G}$$

We focus now on the physical consequences of this PPN expansion.

MONOPOLE TERM

Assuming that the matter distribution is compact, static and spherically symmetric, only the monopole terms of the PPN expansion are relevant. The effective gravitational potential reads:

$$\Phi(x) = \frac{G\bar{M}}{r} + \nu\,\frac{G^2\mu^2}{r} + O(r^{-2})$$

$$\bar{M} = \int \bar{\rho}_e(x')\,d^3x' \qquad \mu^2 = \int \bar{\rho}_e(x')^2\,d^3x'$$

$$\rho_e = \rho^*\left[1 + 2\beta_2\,U + \beta_3\,\Pi + 3\beta_4\,p/\rho^* + \nu\,G\,\rho\right].$$

$$\beta_2 := \frac{1}{2}\left(1 - 2\beta + \zeta_2\right) \quad \beta_3 := 1 + \zeta_3 \quad \beta_4 := \gamma + \zeta_4$$

When we are outside a matter distribution $\rho_e \approx \rho^*$ and the mass terms can be simply rescaled to include the corrections.

MONOPOLE TERM

However, inside the matter distribution the term μ^2 depends on the internal structure of the distribution itself. For example, in the case of the Gaussian profile

$$\bar{\rho}_e(x) = \frac{\bar{M}}{\left(\sqrt{2\pi}\,\sigma\right)^3} \exp\left[-\frac{r^2}{2\sigma^2}\right] \qquad \Rightarrow \qquad \mu^2 = \frac{\bar{M}^2}{8\,\pi^{3/2}\,\sigma^3}\,.$$

Which depends on the width σ of the distribution. As a consequence one can define the effective gravitational constant

Since ν is negative, the gravitating mass of an object outside matter sources is less than the sum of its parts.

$$G_{eff} = G_0 \left[1 + \frac{\nu\, G_0\, \bar{\rho}_C}{6\sqrt{\pi}}\right] \qquad \bar{\rho}_C := \frac{3\bar{M}}{4\pi\sigma^3}$$

and we can constrain q with error on measurements of G_{eff} :

$$|\,q\,| \lessapprox 10^{-24}\ m^3/J$$

CIRCULAR ORBITS

Let us now consider the case of circular orbits. We consider the gravitational field generated by a spherically symmetric and stationary source. The line element has the form

$$ds^2 = f\,dt^2 + h\left(dr^2 + r^2\,d\Omega^2\right)$$

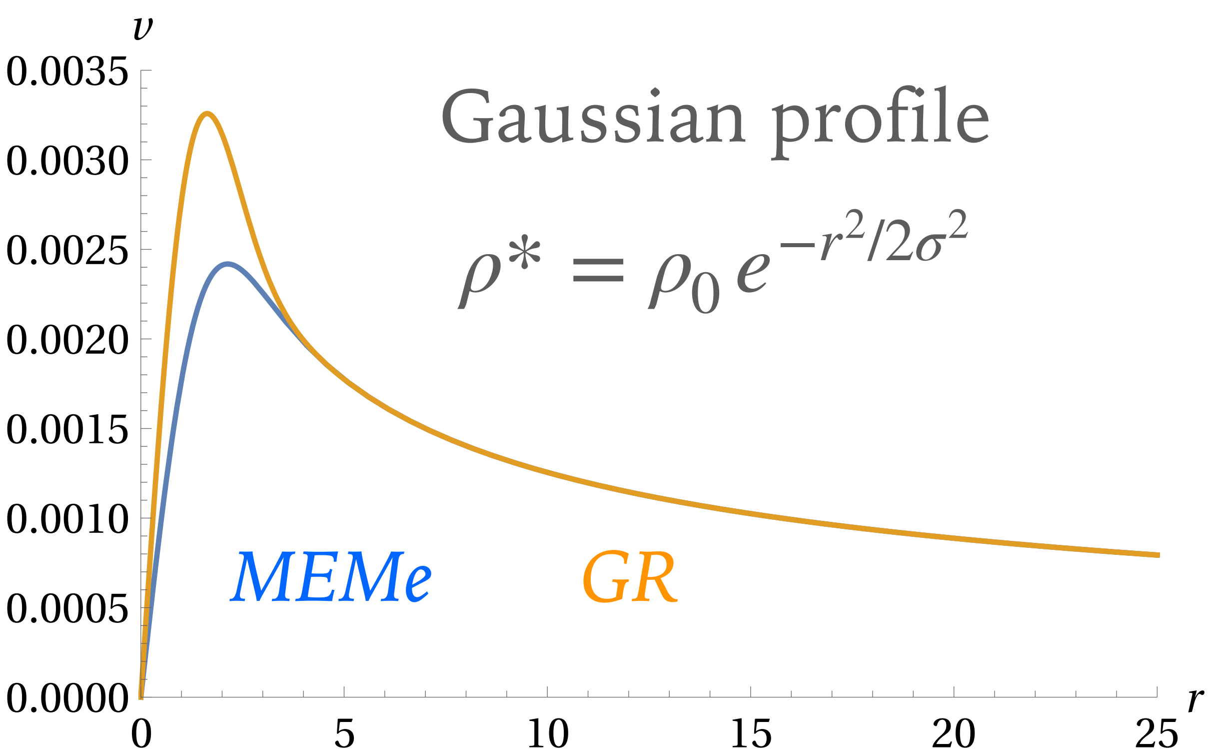
where, neglecting the internal energy density an pressure, we have

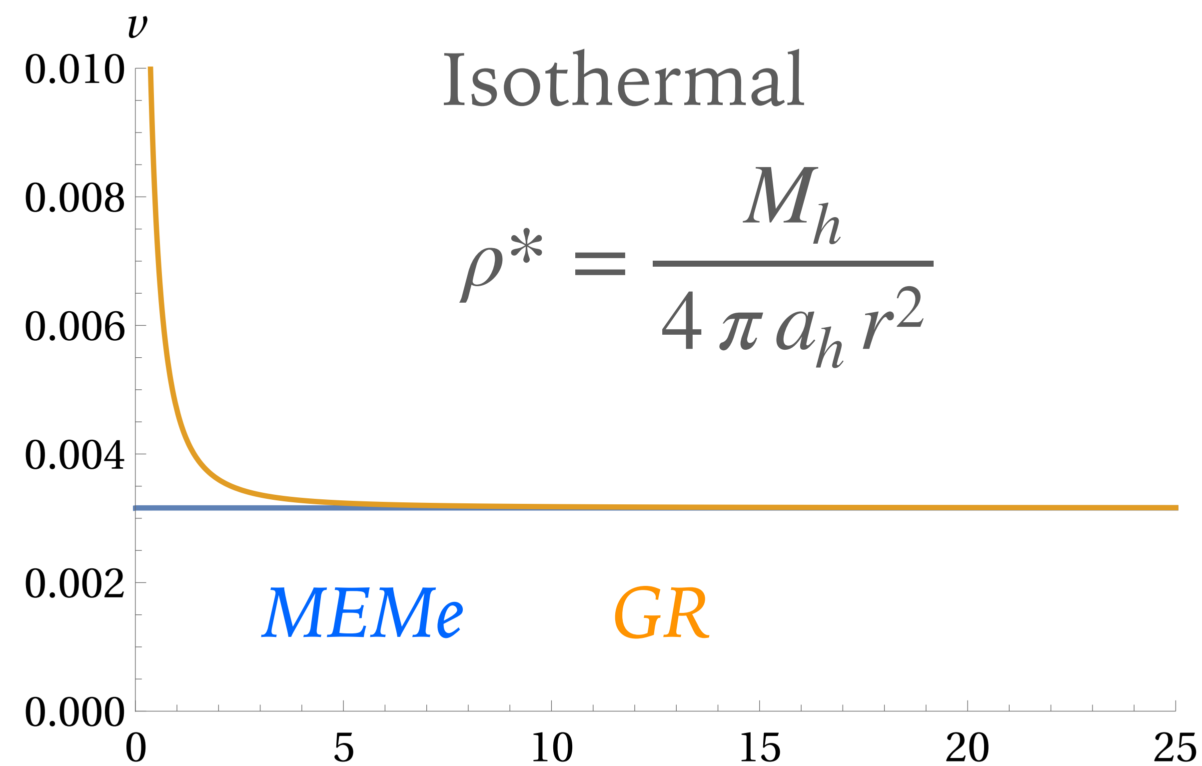
$$\begin{aligned} f =& -1 + 2U - 2U^2 + c_0\Delta U - 2\Phi_2 + c_2\Delta\Phi_2 \\ & + 2\nu\Psi + c_\Psi\Delta\Psi, \\ h =& 1 + 2U - e_0\Delta U. \end{aligned}$$

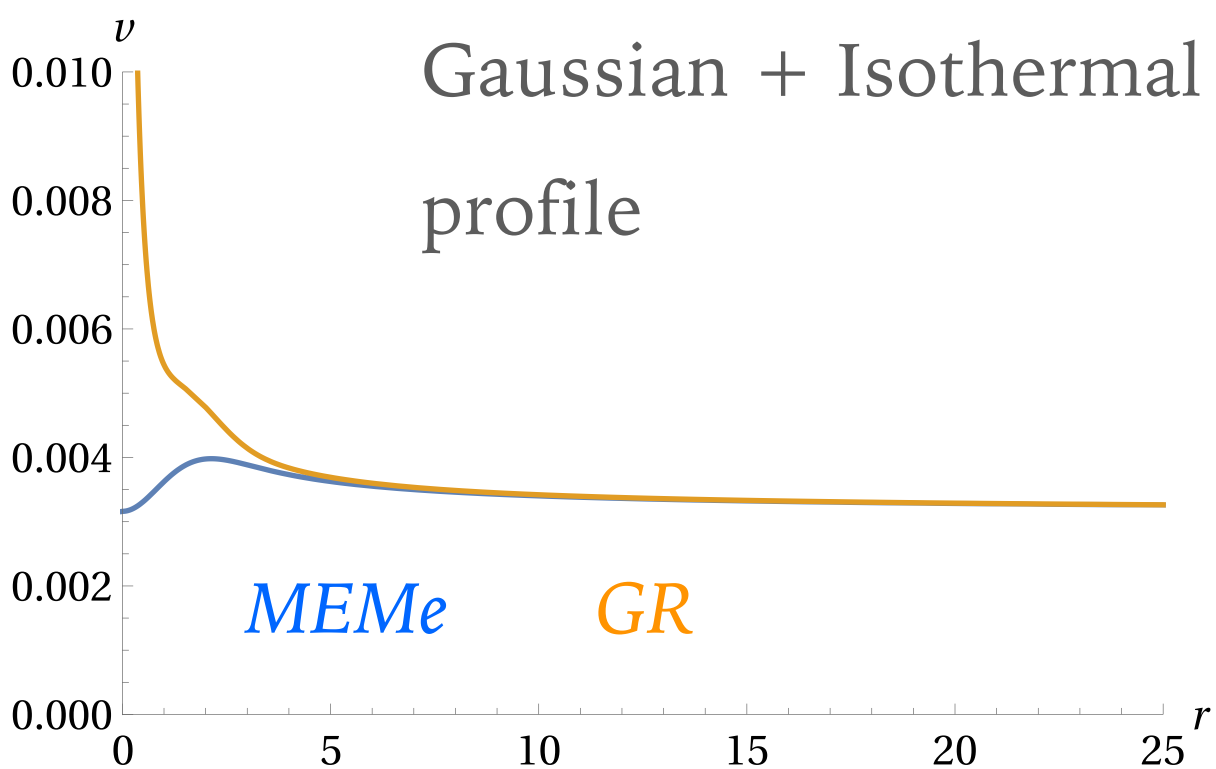
Given a density profile, we can calculate the functions above. The tangential velocities along a circular orbit will be

$$v(r) = \sqrt{\frac{-rf'(r)}{rh'(r) + 2h(r)}}$$

CIRCULAR ORBITS: EXAMPLES







As the velocities tend to GR for large distances, it turns out that MEMe cannot account for flattening of the rotation curves of galaxies.