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Massive scalar field in de Sitter space: a two-loop calculation and a comparison with the stochastic approach

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Introduction and Motivations

- The de Sitter universe is a spacetime with positive constant 4-curvature that is homogeneous and isotropic in both space and time.
- It is completely characterized by only one constant H and has as many symmetries as the flat (Minkowski) spacetime.
- De Sitter universe plays a central role in understanding the properties of cosmological inflation.
- Inflation is a stage of accelerated expansion of the early Universe. The expansion is quasi-exponential, and at lowest order it can be approximated by de Sitter space.
- The inflationary stage allows the growth of quantum fluctuations, which are necessary to explain observed large-scale structure of the Universe. So it is important to study quantum field theory in de Sitter background.

Introduction and Motivations

- Scalar fields in de Sitter space are of particular importance for understanding the period of inflation and the growth of quantum fluctuations.
- In the massive case if $m^2 \ll H^2$, the leading contribution to $\langle \phi^2 \rangle_{\rm ren}$

$$\langle \phi^2(\vec{x}, t) \rangle_{\rm ren} = \frac{3H^4}{8\pi^2 m^2} + \mathcal{O}\left(\left(m^2/H^2 \right)^0 \right),$$

  derives entirely from the long-wavelength modes.
- When there is a self-interaction, each successive term in the weak coupling perturbative expansion contains higher and higher powers of H^2/m^2. The perturbation theory breaks down when the value of H^2/m^2 overwhelms the smallness of the coupling constant.

Introduction and Motivations

- A non-perturbative method for calculating the expectation values of the coarse-grained theory, containing only the long-wavelength fluctuations of a scalar field, was proposed by Starobinsky.
- The expectation values can be determined by using a probability distribution function that is a solution to a simple Fokker-Planck equation.
- In this work we consider a massive scalar field with a quartic self-interaction.
- We calculate a long-wavelength part of the two-point function up to two-loop order using the "in-in" formalism.

Introduction and Motivations

- We compare our results with the Hartree-Fock approximation and with the stochastic approach.
- We argue that the perturbative expression for the two-point function can be reorganized into a sum of exponential functions that depend on the two given points in a de Sitter invariant way.

De Sitter Space in Flat Coordinates

We consider the de Sitter spacetime represented as an expanding spatially flat homogeneous and isotropic universe with the following metric

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2) \ ,$$

where the scale factor $a(t)$ is

$$a(t) = e^{Ht} \ , -\infty < t < \infty,$$

and H is the Hubble constant that characterizes the rate of expansion.

If we introduce a conformal time coordinate, given by $\eta(t) \equiv \int dt\, a^{-1}(t)$

$$ds^2 = a^2(\eta)(d\eta^2 - dx^2 - dy^2 - dz^2),$$
$$a(\eta) = -\frac{1}{H\eta}, \; -\infty < \eta < 0.$$

Physical distances: $\ell_{phys} = a(\eta)\ell = -\ell/(H\eta)$.

Physical energy or momentum: $k_{phys} = k/a(\eta) = -kH\eta$.

Massive Scalar Field in de Sitter space

We will study a massive scalar field with a quartic self-interaction

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\phi^4\right) .$$

When $\lambda = 0$, the equation of motion for the rescaled field $\chi \equiv a(\eta)\phi$ is

$$\chi'' - \nabla^2\chi - \frac{1}{\eta^2}\left(2 - \frac{m^2}{H^2}\right)\chi = 0 .$$

We expand the field $\phi(\vec{x}, t)$ in terms of creation and annihilation operators

$$\phi(\vec{x}, t) = \int \frac{d^3\vec{k}}{(2\pi)^3}\left\{\phi_k(\eta)e^{i\vec{k}\cdot\vec{x}}a_{\vec{k}} + \phi_k^*(\eta)e^{-i\vec{k}\cdot\vec{x}}a^\dagger_{\vec{k}}\right\} .$$

The mode functions $\chi_k \equiv a(\eta)\phi_k$ obey the differential equation

$$\chi_k'' + k^2 \left[1 - \frac{1}{k^2\eta^2}\left(2 - \frac{m^2}{H^2}\right)\right]\chi_k = 0 \, ,$$

where $k = |\vec{k}|$. The general solution of this equation can be expressed as a linear combination of Hankel functions:

$$\chi_k(\eta) = \sqrt{-k\eta}\left[A_k\, \mathcal{H}_\nu^{(1)}(-k\eta) + B_k\, \mathcal{H}_\nu^{(2)}(-k\eta)\right] \, ,$$

$$\nu = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \, .$$

The choice of the coefficients A_k and B_k defines a vacuum state $|0\rangle$ annihilated by $a_{\vec{k}}$.

$$a_{\vec{k}}|0\rangle = 0 \;\; \text{for any } \vec{k} \, .$$

If one wants to have a vacuum that in the remote past $\eta \to -\infty$ (or, equivalently, for modes with very short physical wavelength, $-kH\eta \gg H$) behaves like the vacuum in Minkowski spacetime,

$$\chi_k(\eta) \to \frac{e^{-ik\eta}}{\sqrt{2k}} ,$$

one should choose

$$\chi_k(\eta) = -\frac{\sqrt{\pi}}{2}\sqrt{-\eta}\,\mathcal{H}_\nu^{(1)}\left(-k\eta\right) .$$

Such a choice is called the Bunch-Davies vacuum. As long as $m \neq 0$, this state is de Sitter invariant. If $m^2 \ll H^2$, then

$$\nu \approx \frac{3}{2} - u, \;\; \text{with} \;\; u \equiv \frac{m^2}{3H^2} \ll 1 .$$

Perturbative calculation of the two-point correlation function

We present a perturbative calculation of the long-wavelength part of the two-point function. There are two reasons why it is meaningful to consider exclusively the long-wavelength modes.

The first reason is physical. The fluctuations relevant for the formation of the observed large-scale structure of the universe are those whose wavelength, by the end of inflation, has been stretched to a size much larger than the Hubble horizon.

The second reason is mathematical. Calculations are much simpler if instead of the exact modes, one uses their long-wavelength limit. At the same time, in many cases the results can reflect the behavior of the untruncated theory. In the small mass limit, the long-wavelength two-point function matches with the untruncated one for large separations and for coinciding spacetime points.

The free two-point function in a vacuum state $|0\rangle$ is

$$\begin{aligned}\langle\phi(\vec{x},t_1)\phi(\vec{y},t_2)\rangle_{\lambda^0} &\equiv \langle 0|\phi(\vec{x},t_1)\phi(\vec{y},t_2)|0\rangle \\ &= \int \frac{d^3\vec{k}}{(2\pi)^3}\phi_k(\eta_1)\phi_k^*(\eta_2)e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \\ &= \frac{1}{2\pi^2}\int_0^\infty dk\ k^2\frac{\sin kr}{kr}\phi_k(\eta_1)\phi_k^*(\eta_2)\ ,\end{aligned}$$

Its long-wavelength part consists of modes with physical momenta much less than H:

$$\frac{k}{a(\eta_1)H} = -k\eta_1 < \epsilon\ ,\quad \frac{k}{a(\eta_2)H} = -k\eta_2 < \epsilon\ ,$$

where $\epsilon \ll 1$. In this limit

$$\phi_k(\eta) \approx \frac{iH}{\sqrt{2}}(-\eta)^{3/2}(-k\eta)^{-\nu} = \frac{iH}{\sqrt{2k^3}}(-k\eta)^{u}.$$

The long-wavelength part of the two-point function is

$$\langle \phi(\vec{x}, t_1)\phi(\vec{y}, t_2)\rangle_{\lambda^0, L} = \frac{H^2(\eta_1\eta_2)^u}{4\pi^2} \int_0^{-\epsilon/\eta_{\rm m}} \frac{dk}{k} \frac{k^{2u}\sin(kr)}{kr} \ ,$$

where $\eta_{\rm m}$ is the earliest time that accompanies the momentum $\vec{k}$: $\eta_{\rm m} \equiv \min(\eta_1, \eta_2)$. In the case of coinciding spacetime points

$$\langle \phi^2(\vec{x}, t)\rangle_{\lambda^0, L} = \frac{H^2(-\eta)^{2u}}{4\pi^2} \int_0^{-\epsilon/\eta} \frac{dk}{k^{1-2u}} = \frac{H^2}{8\pi^2}\frac{\epsilon^{2u}}{u}.$$

If

$$\exp\left(-u^{-1}\right) \ll \epsilon \ll 1,$$

then ϵ^{2u} may be replaced by 1 and

$$\langle \phi^2(\vec{x}, t)\rangle_{\lambda^0, L} = \frac{H^2}{8\pi^2 u} = \frac{3H^4}{8\pi^2 m^2} \ .$$

Given two points in de Sitter space, there is a de Sitter invariant function associated with them:

$$Z(X, Y) = -H^2 \eta_{\mu\nu} X^\mu Y^\nu,$$

where X and Y represent coordinates in five-dimensional Minkowski embedding space with the metric $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1, -1)$.
In spatially flat coordinates

$$Z(\vec{x}_1, \eta_1; \vec{y}, \eta_2) = \frac{\eta_1^2 + \eta_2^2 - |\vec{x} - \vec{y}|^2}{2\eta_1\eta_2} .$$

If points are timelike separated, then $Z > 1$; if points are lightlike separated, then $Z = 1$, and if points are spacelike separated, then $Z < 1$.

If $(\vec{x}, t_1)$ and $(\vec{y}, t_2)$ are related in such a way that

$$Z > 1 - \frac{1}{2\epsilon^2} \ ,$$

then

$$-r/\eta_{\rm m} < 1/\epsilon \ .$$

We have

$$\langle \phi(\vec{x}, t_1)\phi(\vec{y}, t_2)\rangle_{\lambda^0, L} \quad = \quad \frac{H^2}{8\pi^2 u} e^{-uH|t_1 - t_2|} \ .$$

There are two important subcases of for which the two-point function is given by the above expression. One is when points $(\vec{x}, t_1)$ and $(\vec{y}, t_2)$ are timelike or lightlike separated; the other is when these points have coinciding time coordinates and the physical spatial distance between them satisfies
$a(t)r < (\epsilon H)^{-1}$.
In the latter case, we obtain the same result as that for the coinciding spacetime points.
This means that as far as the long-wavelength correlation function is concerned, there is no difference between coinciding spacetime points and points on a constant time hypersurface that are separated by a proper distance less than $(\epsilon H)^{-1}$.

Let us consider the case when

$$(-r/\eta_{\rm m}) > 1/\epsilon \ ,$$

or

$$Z < 1 - \frac{1}{2\epsilon^2} \ll -1 \ .$$

It corresponds to the regime of large spacelike separation between points. Then

$$\begin{aligned}\langle\phi(\vec{x},t_1)\phi(\vec{y},t_2)\rangle_{\lambda^0,L} &= \frac{H^2}{8\pi^2 u}\left(\frac{\eta_1\eta_2}{r^2}\right)^u \\ &= \frac{H^2}{8\pi^2 u} e^{-uH(t_1+t_2)}(rH)^{-2u} \ .\end{aligned}$$

In this regime the equal-time two-point function,

$$\langle\phi(\vec{x},t)\phi(\vec{y},t)\rangle_{\lambda^0,L} = \frac{3H^4}{8\pi^2 m^2}(RH)^{-\frac{2m^2}{3H^2}} \ ,$$

depends only on the physical spatial distance $R \equiv re^{Ht}$.

The exact (untruncated) two-point correlator function is known, it is expressed through a hypergeometrical functions and its leading in the parameter u terms give the results coinciding with the long-wavelength correlators, presented above for the cases of coinciding spacetime points $Z = 1$ and the points separated by large timelike or spacelike intervals $|Z| \gg 1$. Just as in flat spacetime, the expectation value of

the commutator of two fields vanishes for spacelike separated points and is nonzero for timelike separated points. However, for the long-wavelength fields

$$\langle[\phi(\vec{x}, t_1), \phi(\vec{y}, t_2)]\rangle_{\lambda^0, L} = 0$$

both for timelike and spacelike related points. The vanishing of this commutator indicates that the long-wavelength part of the field in a sense behaves like a classical quantity.

Schwinger–Keldysh technique

- Schwinger-Keldysh or "in-in" or "closed time path" formalism serves for the calculations of expectation values of operators when only the initial state of the system is given.
- In contrast to the "in-out" formalism there are four types of the propagators and two types of vertices, characterizing the quantum fields on the way forward in time and "back in time".
- After some calculations one remains with the integrals including Wightman functions and theta-functions.
- Diagrams that correspond to these integrals look similar to Feynman diagrams.

The one-loop correction to the two-point function is given by the following diagram:

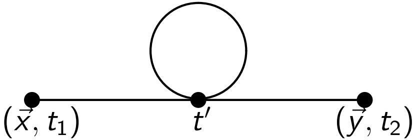

Fig.1

In the case of the timelike, lightlike or small spacelike separation between the points we have

$$\langle \phi(\vec{x}, t_1)\phi(\vec{y}, t_2)\rangle_{\lambda,L} = -\frac{\lambda H^2}{64\pi^4 u^3}\left(1 + uH|t_1 - t_2|\right)e^{-uH|t_1-t_2|} .$$

In the case of coinciding spacetime points one finds

$$\langle \phi^2(\vec{x}, t) \rangle_{\lambda,L} = -\frac{27\lambda H^8}{64\pi^4 m^6} \; .$$

The long-wavelength correlation function at large spacelike separations is

$$\langle \phi(\vec{x}, t_1)\phi(\vec{y}, t_2) \rangle_{\lambda,L}$$
$$= -\frac{\lambda H^2}{64\pi^4 u^3}\left\{1 + u \ln\left(r^2 H^2 e^{H(t_1+t_2)}\right)\right\} e^{-uH(t_1+t_2)}(rH)^{-2u} \; .$$

The effective parameter of the perturbative expansion is not λ but λ/u^2, so the perturbation theory is valid as long as $\lambda \ll m^4/H^4$.

To calculate the two-loop contribution to the two-point correlator, we should consider three diagrams. First of them is the diagram with two independent loops.

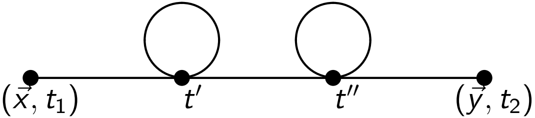

Fig.2

When $Z > 1 - (2\epsilon^2)^{-1}$, we obtain

$$\langle\phi(\vec{x}, t_1)\phi(\vec{y}, t_2)\rangle^{(1)}_{\lambda^2, L} = \frac{\lambda^2 H^2}{512\pi^6 u^5}\left(1 + uH|t_1 - t_2| + \frac{1}{2}u^2 H^2 |t_1 - t_2|^2\right) e^{-uH|t_1 - t_2|} \,,$$

which for coinciding spacetime points becomes

$$\langle\phi^2(\vec{x}, t)\rangle^{(1)}_{\lambda^2, L} = \frac{243\lambda^2 H^{12}}{512\pi^6 m^{10}} \,.$$

In $Z < 1 - (2\epsilon^2)^{-1}$ regime, we have

$$\begin{aligned}
&\langle \phi(\vec{x}, t_1)\phi(\vec{y}, t_2)\rangle^{(1)}_{\lambda^2, L} \\
&= \frac{\lambda^2 H^2}{512\pi^6 u^5} \left\{1 + u \ln\left(r^2 H^2 e^{H(t_1+t_2)}\right)\right. \\
&\left. + \frac{1}{2} u^2 \ln^2\left(r^2 H^2 e^{H(t_1+t_2)}\right)\right\} e^{-uH(t_1+t_2)} (rH)^{-2u} \ .
\end{aligned}$$

The second diagram can be called “snowman”.

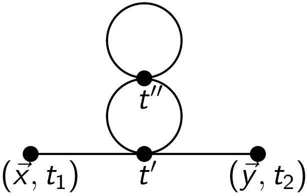

Fig. 3

In the case $Z > 1 - (2\epsilon^2)^{-1}$ it gives

$$\langle \phi(\vec{x}, t_1)\phi(\vec{y}, t_2)\rangle^{(2)}_{\lambda^2, L} = \frac{\lambda^2 H^2}{512\pi^6 u^5}\Big(1 + uH|t_1 - t_2|\Big) e^{-uH|t_1 - t_2|} \ ,$$

which for coinciding spacetime points reduces to

$$\langle \phi^2(\vec{x}, t)\rangle^{(2)}_{\lambda^2, L} = \frac{243\lambda^2 H^{12}}{512\pi^6 m^{10}} \ .$$

When $Z < 1 - (2\epsilon^2)^{-1}$, we obtain

$$\langle \phi(\vec{x}, t_1)\phi(\vec{y}, t_2)\rangle^{(2)}_{\lambda^2, L}$$
$$= \frac{\lambda^2 H^2}{512\pi^6 u^5} \left\{1 + u \ln\left(r^2 H^2 e^{H(t_1+t_2)}\right)\right\} e^{-uH(t_1+t_2)}(rH)^{-2u} \, .$$

The last two-loop diagram is "sunset".

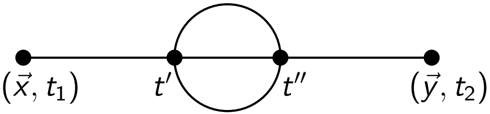

Fig. 4

For $Z > 1 - (2\epsilon^2)^{-1}$, it gives

$$\langle \phi(\vec{x}, t_1)\phi(\vec{y}, t_2)\rangle^{(3)}_{\lambda^2, L} = \frac{\lambda^2 H^2}{1024\pi^6 u^5}\Big(1 + 2uH|t_1 - t_2|\Big)e^{-uH|t_1-t_2|}$$
$$+\frac{\lambda^2 H^2}{3072\pi^6 u^5}e^{-3uH|t_1-t_2|} \ .$$

For coinciding spacetime points, it reduces to

$$\langle \phi^2(\vec{x}, t) \rangle^{(3)}_{\lambda^2, L} = \frac{81\lambda^2 H^{12}}{256\pi^6 m^{10}} \, .$$

In the opposite regime, $Z < 1 - (2\epsilon^2)^{-1}$, we obtain

$$\begin{aligned}
&\langle \phi(\vec{x}, t_1)\phi(\vec{y}, t_2) \rangle^{(3)}_{\lambda^2, L} \\
&= \frac{\lambda^2 H^2}{1024\pi^6 u^5} \left\{ 1 + 2u \ln \left(r^2 H^2 e^{H(t_1+t_2)} \right) \right\} e^{-uH(t_1+t_2)} (rH)^{-2u} \\
&+ \frac{\lambda^2 H^2}{3072\pi^6 u^5} e^{-3uH(t_1+t_2)} (rH)^{-6u} \, .
\end{aligned}$$

Comparison with the Hartree-Fock approximation and with the stochastic approach

Starting with the Klein-Gordon equation and using the Hartree-Fock (Gaussian) approximation

$$\langle \phi^4 \rangle = 3 \langle \phi^2 \rangle^2 ,$$

we arrive to the following equation for the two-point correlator:

$$\frac{\partial}{\partial t} \langle \phi^2 \rangle_L = \frac{H^3}{4\pi^2} - \frac{2m^2}{3H} \langle \phi^2 \rangle_L - \frac{2\lambda}{H} \langle \phi^2 \rangle_L^2 \ .$$

As $t \to \infty$, all of the solutions to this equation approach an equilibrium value that satisfies

$$\frac{H^3}{4\pi^2} - \frac{2m^2}{3H} \langle \phi^2 \rangle_L - \frac{2\lambda}{H} \langle \phi^2 \rangle_L^2 = 0 \ .$$

For $\lambda = 0$, we have

$$\langle \phi^2 \rangle_L = \frac{3H^4}{8\pi^2 m^2} \, .$$

When $\lambda \neq 0$, we have

$$\langle \phi^2 \rangle_L = \frac{m^2}{6\lambda} \left(\sqrt{1 + \frac{9\lambda H^4}{2\pi^2 m^4}} - 1 \right) ;$$

we chose the root that coincides with the preceding expression in the limit $\lambda \to 0$. Assuming that $\lambda H^4 / m^4 \ll 1$, and expanding the preceding expression yields

$$\langle \phi^2 \rangle_L = \frac{3H^4}{8\pi^2 m^2} - \frac{27\lambda H^8}{64\pi^4 m^6} + \frac{243\lambda^2 H^{12}}{256\pi^6 m^{10}} + \mathcal{O}(\lambda^3) \, .$$

Comparing this expansion with the results obtained by the field-theoretical methods, we see that they match at zeroth- and first-order in λ, but there is a mismatch at second order: the λ^2-term omits the contribution of the sunset diagram and is equal to the sum of other two diagrams. Hence, it can be concluded that the Hartree-Fock approximation resums all cactus type diagrams of the perturbation theory.

The stochastic approach argues that the behavior of the long-wavelength part of the quantum field $\phi(\vec{x}, t)$ in de Sitter space can be modelled by an auxiliary classical stochastic variable φ with a probability distribution $\rho(\varphi, t)$ that satisfies the Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} = \frac{H^3}{8\pi^2} \frac{\partial^2 \rho}{\partial \varphi^2} + \frac{1}{3H} \frac{\partial}{\partial \varphi} \left(\frac{\partial V}{\partial \varphi} \rho(t, \varphi) \right).$$

In our case the potential has the form

$$V(\varphi) = \frac{1}{2} m^2 \varphi^2 + \frac{\lambda}{4} \varphi^4 \ .$$

At late times any solution of the Fokker-Planck equation approaches the static equilibrium solution

$$\rho_{\rm eq}(\varphi) = N^{-1} \exp\left(-\frac{8\pi^2}{3H^4} V(\varphi)\right),$$

where N is the normalization fixed by the condition

$$\int_{-\infty}^{\infty} \rho_{\rm eq}(\varphi)\, d\varphi = 1 .$$

In our case we can calculate this normalization explicitly

$$\begin{aligned} N &= \int_{-\infty}^{\infty} \exp\left[-\frac{8\pi^2}{3H^4}\left(\frac{\lambda\varphi^4}{4} + \frac{m^2\varphi^2}{2}\right)\right] d\varphi \\ &= \frac{m}{\sqrt{2\lambda}} \exp(z) \mathcal{K}_{\frac{1}{4}}(z) , \end{aligned}$$

where $\mathcal{K}_{\frac{1}{4}}(z)$ is a modified Bessel function of the second kind, and $z \equiv \frac{\pi^2 m^4}{3\lambda H^4}$.

Using this equilibrium distribution, we obtain

$$\langle \varphi^2 \rangle = \frac{m^2}{2\lambda} \frac{\mathcal{K}_{\frac{3}{4}}(z)}{\mathcal{K}_{\frac{1}{4}}(z)} - \frac{m^2}{2\lambda}.$$

Expanding this in the limit $\lambda H^4/m^4 \ll 1$ (which corresponds to $z \gg 1$) gives

$$\langle \varphi^2 \rangle = \frac{3H^4}{8\pi^2 m^2} - \frac{27\lambda H^8}{64\pi^4 m^6} + \frac{81\lambda^2 H^{12}}{64\pi^6 m^{10}} + \mathcal{O}(\lambda^3) \,.$$

This result is in agreement with the result of the quantum field theory calculations, and unlike the Hartree-Fock approximation, it includes the contribution of the sunset diagram.

The long-wavelength two-point function of $\phi(\vec{x}, t)$ too can be calculated by using the classical stochastic variable φ: if the points $(\vec{x}, t_1)$ and $(\vec{y}, t_2)$ are timelike or lightlike related, this correlation function is given by

$$\langle \phi(\vec{x}, t_1)\phi(\vec{y}, t_2)\rangle_L = \langle \varphi(t_1)\varphi(t_2)\rangle \ .$$

If the correlation function $\langle \varphi(t_1)\varphi(t_2)\rangle$ depends only on the absolute value of the time difference $T \equiv |t_1 - t_2|$, it can be expressed as

$$\langle \varphi(t_1)\varphi(t_2)\rangle = \int_{-\infty}^{\infty} \varphi\, \Xi(\varphi, T) d\varphi \ ,$$

where the function $\Xi(\varphi, T)$ satisfies the Fokker-Planck equation,

$$\frac{\partial \Xi}{\partial T} = \frac{H^3}{8\pi^2}\frac{\partial^2 \Xi}{\partial \varphi^2} + \frac{1}{3H}\frac{\partial}{\partial \varphi}\left(\frac{\partial V}{\partial \varphi}\Xi(\varphi, T)\right) \ ,$$

with the initial condition

$$\Xi(\varphi, 0) = \varphi \rho_{\rm eq}(\varphi) \ .$$

Derivatives of $\langle\varphi(t_1)\varphi(t_2)\rangle$ at $T = 0$ can be computed by using the equations above:

$$\frac{\partial}{\partial T}\langle\varphi(t_1)\varphi(t_2)\rangle\Big|_{T=0} = -\frac{H^3}{8\pi^2} ,$$

$$\frac{\partial^2}{\partial T^2}\langle\varphi(t_1)\varphi(t_2)\rangle\Big|_{T=0} = \frac{H^2}{24\pi^2}\Big(3\lambda\langle\varphi^2\rangle + m^2\Big) ,$$

and so on. It is easy to confirm that the T-derivatives of the two-point correlation function presented earlier (for $Z > 1 - \frac{1}{2\epsilon^2}$ case) satisfy these equalities as well.

Exponentiation of the perturbative series

The expression for the two-point correlation function can be presented in the following way: (the case $Z > 1 - \frac{1}{2\epsilon^2}$):

$$\langle\phi(\vec{x}, t_1)\phi(\vec{y}, t_2)\rangle_L = \frac{H^2}{8\pi^2 u}\left(1 - \frac{\lambda}{8\pi^2 u^2} + \frac{5\lambda^2}{128\pi^4 u^4}\right) e^{-uHT}$$
$$-\frac{\lambda H^3 T}{64\pi^4 u^2}\left(1 - \frac{3\lambda}{8\pi^2 u^2}\right) e^{-uHT}$$
$$+\frac{\lambda^2 H^4 T^2}{1024\pi^6 u^3} e^{-uHT} + \frac{\lambda^2 H^2}{3072\pi^6 u^5} e^{-3uHT} + \mathcal{O}(\lambda^3)$$
$$= \frac{H^2}{8\pi^2 u}\left(1 - \frac{\lambda}{8\pi^2 u^2} + \frac{5\lambda^2}{128\pi^4 u^4} + \mathcal{O}(\lambda^3)\right)$$
$$\times\left[1 - \frac{\lambda HT}{8\pi^2 u} + \frac{\lambda^2 HT}{32\pi^4 u^3} + \frac{1}{2}\left(\frac{\lambda HT}{8\pi^2 u}\right)^2 + \mathcal{O}(\lambda^3)\right] e^{-uHT}$$
$$+\frac{\lambda^2 H^2}{3072\pi^6 u^5} e^{-3uHT} + \dots ,$$

To second order in λ, the expression in squared brackets matches with the first three terms in the Taylor series of the exponential function

$$\exp\left[-\frac{\lambda HT}{8\pi^2 u}\left(1-\frac{\lambda}{4\pi^2 u^2}\right)\right],$$

so it is plausible that an infinite series of diagrams may be resummed into this exponent. With this assumption, we arrive at

$$\langle\phi(\vec{x},t_1)\phi(\vec{y},t_2)\rangle_L = \frac{H^2}{8\pi^2 u}\left(1-\frac{\lambda}{8\pi^2 u^2}+\frac{5\lambda^2}{128\pi^4 u^4}+\mathcal{O}(\lambda^3)\right)$$
$$\times\exp\left[-uHT\left(1+\frac{\lambda}{8\pi^2 u^2}-\frac{\lambda^2}{32\pi^4 u^4}+\mathcal{O}(\lambda^3)\right)\right]$$
$$+\frac{\lambda^2 H^2}{3072\pi^6 u^5}\exp\left[-3uHT\right]+\ldots\,.$$

Analogously for $Z < 1 - \frac{1}{2\epsilon^2}$:

$$\langle\phi(\vec{x},t_1)\phi(\vec{y},t_2)\rangle_L = \frac{H^2}{8\pi^2 u}\left(1 - \frac{\lambda}{8\pi^2 u^2} + \frac{5\lambda^2}{128\pi^4 u^4} + \mathcal{O}(\lambda^3)\right)$$
$$\times \left(r^2 H^2 e^{H(t_1+t_2)}\right)^{-u\left(1+\frac{\lambda}{8\pi^2 u^2} - \frac{\lambda^2}{32\pi^4 u^4} + \mathcal{O}(\lambda^3)\right)}$$
$$+ \frac{\lambda^2 H^2}{3072\pi^6 u^5}\left(r^2 H^2 e^{H(t_1+t_2)}\right)^{-3u} + \dots ,$$

In this regime the equal-time correlation function depends only on the physical spatial distance $R \equiv re^{Ht}$.

We see that the perturbative corrections don't change the long-wavelength part of the commutator: just as in the free theory case, it is equal to zero both for timelike and spacelike related points.

As $T \to \infty$, the two-point function decays with the characteristic correlation time

$$T_c \sim \frac{1}{uH} = \frac{3H}{m^2} \gg \frac{1}{H} \ .$$

Similarly, as $R \to \infty$, the equal-time correlation function decays with the characteristic correlation length

$$R_c \sim \frac{1}{H} \exp\left(\frac{3H^2}{2m^2}\right) \ .$$

This behavior differs from a much faster exponential decay of the equal-time correlation function in flat spacetime:
$\langle \phi(\vec{x}, t)\phi(\vec{y}, t)\rangle_{\rm flat} \sim \sqrt{m/r^3}e^{-mr}$ as $r \to \infty$.

Conclusions

- We have calculated—up to two loops—the long-wavelength two-point function for a scalar theory with a small mass and a quartic interaction.
- It has been shown that it is de Sitter invariant for coinciding points as well as at large spacelike and large timelike separations.
- We have demonstrated that the commutator of the long-wavelength part of the field is equal to zero both at the free theory level and with the perturbative corrections.
- Our results are in agreement with Starobinsky's stochastic approach in which the coarse-grained quantum field is equivalent to a classical stochastic quantity.
- It would be interesting but more difficult to consider similar problems on more general backgrounds.