New equations in relativistic cosmology

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Outline

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- 2. Our effective energy-momentum tensor
- 3. Modifying perfect fluid space-times
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1. Conceptual steps

- Einstein equations:

$$E_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \qquad \kappa = \frac{8\pi G}{c^4},$$

1. Conceptual steps

- But Einstein did not know that matter fields are quantum fields, nor did he know that quantum fields are operator-valued distributions (A.S. Wightman, Fortsch. Phys. 44, 143 (1996)).
- Instead of modifying the Einstein-Hilbert Lagrangian or adding new matter fields, we have investigated a new way of coupling gravity with matter.
- For this purpose, we recall that quantum Yang-Mills in curved space-time leads to contraction of Ricci with the energy-momentum tensor (B.S. DeWitt, «The global approach to quantum field theory», Oxford University Press, 2003).

1. Conceptual steps

- Our assumption: such a contraction might contribute to the desired field equation, in such way that

$$E_{\mu\nu} = \kappa T_{\mu\nu} + B R_{(\mu}{}^{\alpha} T_{|\alpha|\nu)}.$$

1. Dimensional analysis

- On denoting by b a dimensionless parameter, we can write that

$$B = b\kappa(\ell_P)^2,$$

2. Our effective energy-momentum tensor

- What is covariantly conserved is now the tensor

$$\tau_{\mu\nu} \equiv T_{\mu\nu} + AR_{(\mu}{}^{\alpha}T_{|\alpha|\nu)},$$

2. Our effective ...

- In particular, for perfect fluids,

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu},$$

2. Our effective …

- And the Ricci tensor can be written in the form

$$R^{\perp\perp}_{\mu\nu} + 2R^{\perp||}_{(\mu}u_{\nu)} + R^{||||}u_\mu u_\nu,$$

2. Our effective …

- Where, on defining the projector

$$\Pi(u)_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu$$

2. Our effective …

- We find

$$R^{\perp\perp}_{\mu\nu} \equiv [\Pi(u)R]_{\mu\nu} = \Pi(u)_\mu{}^\alpha \Pi(u)_\nu{}^\beta R_{\alpha\beta},$$

$$R^{\perp||}_{\mu} \equiv -\Pi(u)_\mu{}^\alpha R_{\alpha\beta} u^\beta = -\Pi(u)_\mu{}^\alpha R_{\alpha u},$$

$$R^{||||} \equiv u^\alpha u^\beta R_{\alpha\beta} = R_{uu}.$$

2. Our effective …

- And hence a pure energy part

$$[\tau_{\text{en}}]_{\mu\nu} = \rho(1 - AR^{||||})u_\mu u_\nu \equiv \rho_{\text{eff}} u_\mu u_\nu,$$

2. Our effective …

- A mechanical stress part

$$[\tau_{\mathrm{mec}}]_{\mu\nu} = p[\Pi(u)_{\mu\nu} + AR^{\perp\perp}_{\mu\nu}],$$

2. Our effective …

- And a thermal part

$$[\tau_{\mathrm{th}}]_{\mu\nu} = -A(p-\rho)R^{\perp\|}_{(\mu}u_{\nu)},$$

2. Our effective …

- Fundamental remark: since there are no a priori sign restrictions on R parallel in Ricci, we can expect either positive or negative values for the effective energy density. This property is of basic importance in order to obtain models of dark matter or dark energy.

3. Modifying perfect-fluid space-times

- The Dunn and Tupper space-time was discovered by looking for solutions of the Einstein-Maxwell equations for source-free electromagnetic fields. The resulting metric is

$$ds^2 = -2du\,dr + u^{-2n} r^{-2m} dy^2 + u^{-2m} r^{-2n} dz^2,$$

3. Perfect-fluid …

- Having defined

$$m = \frac{(\sqrt{3} - 1)}{4}, \qquad n = -\frac{(\sqrt{3} + 1)}{4}.$$

3. Perfect-fluid …

- In this solution, the principal null congruences of the electromagnetic field are geodesic, and the corresponding null tetrad undergoes parallel propagation along these congruences. Upon performing the coordinate transformation

$$t = \sqrt{2ur}, \qquad x = \frac{(m-n)}{2}\log\left(\frac{r}{u}\right),$$

$$y \to 2^{\frac{(m+n)}{2}} y, \qquad z \to 2^{\frac{(m+n)}{2}} z,$$

3. Perfect-fluid …

- The metric takes eventually the form

$$ds^2 = -dt^2 + \frac{t^2}{(m-n)^2}dx^2 + t^{-2(m+n)}[e^{-2x}dy^2 + e^{2x}dz^2]$$

3. Perfect-fluid …

- Such a metric is an exact solution of the Einstein equations with vanishing cosmological constant and sourced by a perfect fluid provided that the dimensionless constants m and n satisfy the condition

$$m(2m+1)+n(2n+1)=0.$$

3. Perfect-fluid …

- In the particular (but non-trivial) case m=0, n=-1/2, we find the exact solution where only the «tt» metric component is modified, i.e.

$$g_{tt} = -1 + \epsilon f_{tt},$$

3. Perfect-fluid …

- Jointly with energy density and pressure (calligraphic L being a length scale)

$$\kappa \mathcal{L}^2 \rho(t) = \frac{1}{t^2} + \epsilon \kappa \mathcal{L}^2 \rho_1(t), \qquad p(t) = \epsilon p_1(t),$$

3. Perfect-fluid ...

- And the exact solution is

$$f_{tt} = \frac{1}{\epsilon} + \frac{t^{3/4}}{C_1 - \frac{(11\epsilon+12t^2)t^{3/4}}{33}},$$

$$\kappa\mathcal{L}^2\rho_1(t) = -\frac{(\epsilon - t^2)}{t^2\epsilon^2},$$

$$\kappa\mathcal{L}^2 p_1(t) = -\frac{1}{\epsilon^2},$$

3. Perfect-fluid …

- Where epsilon is not necessarily small and C_1 is an integration constant.

4. Modifying FLRW space-times

- Let us take a background space-time belonging to the FLRW family with metric

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{(1-kr^2)} + f^2(r)(d\theta^2 + \sin^2\theta d\phi^2)\right],$$

4. FLRW …

- The original Einstein equations have the familiar form

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Lambda}{3} - \frac{k}{a^2} + \frac{\kappa}{3}\rho_0,$$

$$\frac{\ddot{a}}{a} = \frac{\Lambda}{3} - \frac{\kappa}{2}\left(\frac{1}{3}\rho_0 + p_0\right),$$

4. FLRW …

- Supplemented by the compatibility condition

$$\dot{\rho}_0 = -3\frac{\dot{a}}{a}(\rho_0 + p_0).$$

4. FLRW …

- Our modified field equations become

$$\frac{\ddot{a}}{a} = \frac{1}{3}\frac{2A\kappa p(\kappa\rho + \Lambda) - \kappa(\rho + 3p) + 2\Lambda}{2 - A\kappa(\rho + p) + 2A^2\kappa^2 p\rho},$$

$$\frac{\dot{a}^2}{a^2} = \frac{[2\kappa^2\rho p - \kappa\Lambda(3\rho + p)]A + 2\kappa\rho + 2\Lambda}{3[2 - A\kappa(p + \rho) + 2A^2\kappa^2\rho p]} - \frac{k}{a^2}.$$

4. FLRW …

- In the spatially flat case, the unperturbed solution is

$$\kappa t_0^2 \rho_0 = \frac{4t_0^2}{3\gamma^2 t^2},$$

$$p_0 = (\gamma - 1)\rho_0,$$

$$a(t) = t_0 \left(\frac{t}{t_0}\right)^{\frac{2}{3\gamma}},$$

4. FLRW …

- We now rescale the time variable: T=t/t_0, and look for perturbative solutions to first order in B=kappa L**2 epsilon (epsilon being dimensionless). Hence we find

$$\kappa t_0^2 \rho_1(t) = -\frac{8(7\gamma-6)}{9\gamma^3(3\gamma-2)T^4} - \frac{2p_1\kappa t_0^2}{(2+\gamma)} + T^{-\frac{(2+\gamma)}{\gamma}},$$

$$p_1(t) = p_1,$$

$$\frac{a_1(t)}{t_0} = C_1 T^{\frac{2}{3\gamma}} - \frac{\gamma\kappa t_0^2 p_1}{4(2+\gamma)} T^{\frac{2(1+3\gamma)}{3\gamma}} + \frac{\gamma^2}{4(\gamma-2)} T^{-\frac{4}{3\gamma}+1} - \frac{2(\gamma-1)(\gamma-2)}{9(3\gamma-2)\gamma^3} T^{\frac{2}{3\gamma}-2},$$

4. FLRW …

- Where C_1 is an integration constant. In particular, if gamma=1 and C_1 vanishes, we find

$$\kappa t_0^2 \rho_1(t) = -\frac{8}{9T^4} - \frac{2}{3} t_0^2 \kappa p_1 + \frac{1}{T^3},$$

$$p_1(t) = p_1,$$

$$\frac{a_1(t)}{t_0} = -\frac{\kappa t_0^2 p_1}{12} T^{\frac{8}{3}} - \frac{1}{4} T^{-\frac{1}{3}},$$

4. FLRW …

- If p_1 vanishes, we avoid the blow-up of the scale factor, and find

$$\kappa t_0^2 \rho_{(1)}(t) = -\frac{8}{9T^4} + \frac{1}{T^3},$$

$$p_1(t) = 0,$$

$$\frac{a_1(t)}{t_0} = -\frac{1}{4} T^{-\frac{1}{3}},$$

4. FLRW …

- This solution should be compared with the unperturbed values when gamma=1, i.e.

$$\kappa t_0^2 \rho_0 = \frac{4}{3T^2}, \qquad p_{(0)} = 0, \qquad a_0(t) = t_0 T^{\frac{2}{3}}.$$

4. FLRW …

- We obtain therefore, to linear order in epsilon,

$$\kappa t_0^2 \rho = \frac{4}{3T^2} + \epsilon \left(-\frac{8}{9T^4} + \frac{1}{T^3} \right),$$

$$\frac{a(t)}{t_0} = T^{\frac{2}{3}} - \frac{\epsilon}{4} T^{-\frac{1}{3}},$$

5. Concluding remarks

- Our modified field equations are tensorial but describe a hybrid scheme, where the variation of the Einstein-Hilbert action yields a tensor that is coupled to the energy-momentum tensor in a novel way. Such a partial differential equation, as far as we can see, is not variational (cf. S.C. Anco, Fields Inst. Commun. 79, 119 (2017)).
- The resulting equation reduces to general relativity in vacuum, and is able to lead to dark matter and dark energy (more detailed comparison with observations is in order).

5. Concluding remarks

- Our modified field equations contain, as a particular case, some f[R] theories, when the dimensionless parameter b approaches zero.
- The contraction of the Einstein tensor with the energy-momentum tensor can be studied as well.