FLRW QUANTUM COSMOLOGY

A TOMOGRAPHIC ANALYSIS

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Papers on quantum cosmology and tomography 1

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- Manko, Marmo and Stornaiolo GRG 37, 2003 (2005)
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- Capozziello, Manko, Marmo and Stornaiolo Phys. Scr. 80, 046906 (2009)

Papers on quantum cosmology and tomography 2

- Stornaiolo C., Phys.Scripta 90 (2015) no.7, 074032
- Stornaiolo C., Tomographic analysis of the De Sitter models in Quantum Cosmology *Int.J.Geom.Meth.Mod.Phys.* 16 (2018) 01, 1950012
- Stornaiolo C. , Emergent classical universes from initial quantum states in a tomographical description, *Int.J.Mod.Phys.D* 28 (2019) 16, 2040009
- Stornaiolo C., Emergent classical universes from initial quantum states in a tomographical description to appear in *Int.J.Geom.Meth.Mod.Phys. 17 (2020) 11, 2050167 arXiv:2007.03726[gr-qc]*
- *Stornaiolo C. FLRW Quantum cosmology from a tomographic perspective, in preparation.*

The definition of quantum and classical tomogram

$$\mathscr{W}_Q\left(X,\mu,\nu\right) = \int W(q,p)\delta(X-\mu q-\nu p)dxdp$$

$$\mathscr{W}_C\left(X,\mu,\nu\right) = \int f(q,p)\delta(X-\mu q-\nu p)dq\,dp$$

Both definitions are Radon transform respectively of a Wigner function and a Boltzmann function. They define quantum and classical states with the same class of functions

The definition of a tomogram from a wave function

$$\mathscr{W}\left(X,\mu,\nu\right)=\frac{1}{2\pi\hbar\,|\,\nu\,|}\left|\int\psi\left(y\right)\exp\left[i\left(\frac{\mu}{2\hbar\nu}y^{2}-\frac{X}{\hbar\nu}y\right)\right]dy\right|^{2}.$$

which equivalent to the previous definition in terms of Radon transform

Interpretation of the tomogram as probability functions

$$|\psi(q)|^2 \longrightarrow |\tilde{\psi}(p)|^2$$

$$|\psi(X,\mu,\nu)|^2$$

$$\psi(X,\mu,\nu) = \int \psi(q) e^{-iG(q,X)} dq$$

$$G(q,X) = -\frac{\mu}{2\nu}\left(q^2 + X^2\right) + \frac{qX}{\nu} \qquad X = \mu q + \nu p$$

Fundamental condition for the tomogram

$$\int \mathscr{W}(X, \mu, \nu) dX = 1$$

Construction of the classical de Sitter tomogram

$$f(q,p) = \delta\left(-4p^2 + \lambda q - 1\right)$$

$$\mathscr{W}\left(X,\mu,\nu\right) = \int \delta\left(-4p^2 + \lambda q - 1\right)\delta(X - \mu q - \nu p)dqdp$$

$$= \frac{1}{2|\mu|}\frac{1}{\left|\sqrt{\frac{\lambda^2\nu^2}{16\mu^2} + \frac{\lambda X}{\mu} - 1}\right|}$$

Classical and quantum de Sitter tomograms

$$\mathscr{W}_{class.} = \frac{1}{2|\mu|}\frac{1}{\left|\sqrt{\frac{\lambda^2\nu^2}{16\mu^2}+\frac{\lambda X}{\mu}-1}\right|}$$

$$S = \frac{1}{3\hbar\lambda}\left(1-\frac{\lambda X}{\mu}-\frac{\lambda^2}{16}\frac{\nu^2}{\mu^2}\right)^{3/2} - \frac{\pi}{4}$$

$$W_{HH}(q,p) = \frac{2^{1/3}A^2}{\pi(\hbar\lambda)^{1/3}}Ai\left[\frac{4p^2-\lambda q+1}{(\hbar\lambda)^{2/3}}\right] \approx \frac{1}{2|\mu|}\frac{1}{\left|1-\frac{\lambda X}{\mu}-\frac{\lambda^2}{16}\frac{\nu^2}{\mu^2}\right|^{1/2}}\left|\cos\left(\frac{2}{3}\left|S\right|^{3/2}-\frac{\pi}{4}\right)\right|^2$$

$$\mathscr{W}_L(X,\mu,\nu) \approx \frac{1}{2|\mu|}\frac{1}{\left|1-\frac{\lambda X}{\mu}-\frac{\lambda^2}{16}\frac{\nu^2}{\mu^2}\right|^{1/2}} \times \left|e^{iS}\right|^2$$

$$W_{Linde}(q,p) = \frac{2^{1/3}A^2}{\pi(\hbar\lambda)^{1/3}}Bi\left[\frac{4p^2-\lambda q+1}{(\hbar\lambda)^{2/3}}\right] \approx \frac{1}{2|\mu|}\frac{1}{\left|1-\frac{\lambda X}{\mu}-\frac{\lambda^2}{16}\frac{\nu^2}{\mu^2}\right|^{1/2}}\left|\sin\left(\frac{2}{3}\left|S\right|^{3/2}-\frac{\pi}{4}\right)\right|^2$$

The universe in a tomogram

- Previous results
- We found the relation between the quantum and classical tomograms for a de Sitter universe
- The classical limit ($\hbar \to 0$) coincided with the limit $\lambda \to 0$. Ruling out the Hartle and Hawking and Linde models which have no classical limits
- Only the classical limit of the Vilenkin model coincided with the classical one.

- But $\lambda \neq 0$ suggests that also the HH and Linde models can have a quasi-classical limit and can be viable cosmological models to be studied
- Therefore the decay of the cosmological constant can be responsible of the quantum to classical transition More results were considered with a generic potential in the Lagrangian
- It was found that in many models the decay of the cosmological constant produced the transition to a classical state
- But this is not general

Extending to models with perfect fluids

- In this lecture I explore the properties of the tomograms when in quantum cosmological models with a cosmological fluid
- To this aim the fluid is represented in terms of velocity potentials
- One of these potentials is thermasy, we will see its properties and its role in the formulation of the Wheeler-De Witt equation
- We will take some well-knows solutions and derive its tomograms
- Finally we will consider the properties of the tomograms near $a = 0$ and see how the initial singularity is avoided in the tomographic formulation at any time.

First principle of thermodynamics

$$\delta q \equiv T ds = d\Pi + p d\frac{1}{\rho_0}$$

$$\downarrow$$

$$dp = \rho_0 d\mu - \rho_0 T ds\,.$$

$$\mu = \frac{d\rho}{d\rho_0} = \frac{\rho + p}{\rho_0} = 1 + \Pi + \frac{p}{\rho_0}\,.$$

$$p = p(\mu, s)$$

$$n = \frac{n}{V} \qquad \rho_0 = \frac{Nm}{V} = mn \qquad \rho = \rho_0(1+\Pi) \qquad \Pi = \frac{U}{Nm}$$

s is the specific entropy T is the temperature

Equations of state

$$p = w\rho$$

- $$\rho_0 = \frac{\mu^{1/w}}{(1+w)^{1/w}} \exp\left(-\frac{s}{w}\right)$$

$$\rho = \frac{\mu^{1+1/w}}{(1+w)^{1+1/w}} \exp\left(-\frac{s}{w}\right)$$

$$p = \lambda \frac{\mu^{1+1/w}}{(1+w)^{1+1/w}} \exp\left(-\frac{s}{w}\right)$$

Fluids in general relativity

$$S = \int_M d^4x\sqrt{-g}R + \int_{\partial M} d^3x\sqrt{h}h_{ab}K^{ab} + \int 16\pi G\sqrt{-g}pd^4x$$

$$U_a = \mu^{-1}\left(\partial_a\varphi + \vartheta\partial_a s + \alpha^A\partial_a\beta_A\right).$$

$$-\mu^2 = g^{ab}\left(\partial_a\varphi + \vartheta\partial_a s\right)\left(\partial_b\varphi + \vartheta\partial_b s\,.\right)$$

Introduced in General Relativity by B.F Schutz

Fluids in General Relativity

B.F Schutz PRD 2 (1970), 2762,

B.F. Schutz PRD 4 (1971), 3559

J.D. Brown CQG 10(1993) 1571 (Alternative lagrangians and analysis of the potentials)

Fluids

- V.G. Lapchinskii and V.A Rubakov Teor. Mat. Fiz. 33 (1977), 364
- N. Lemos JMP 37(1996), 1449
- F.G Alvarenga & al. GRG 34 (2002), 351
- A.B. Batista &al. PRD 65 (2002), 063511

Equations

$$G_{ab} = 8\pi \left((\rho + p)U_a U_b - \frac{1}{2} g_{ab} p \right)$$

$$\nabla_a(\rho_0 U^a) = 0$$

$$U^a \partial_a s = 0$$

$$U^a \partial_a \vartheta = T$$

$$U^a \partial_a \varphi = -\mu$$

FLRW universe

$$ds^2 = \sigma\left(-N^2dt^2 + a^2(t)\left(\frac{dr^2}{1-Kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right)\right)$$

$$\sigma = 1/6$$

$$L = NKa - \frac{a\dot{a}^2}{N} - \frac{16\pi GNa^3}{36}p$$

$$U^a = (-1,0,0,0) \qquad U_a = (N,0,0,0)$$

$$\mu = \frac{\dot{\varphi} + \vartheta\dot{s}}{N} \qquad p = \lambda\frac{\mu^{1+1/w}}{(1+w)^{1+1/w}}\exp\left(-\frac{s}{w}\right)$$

The action and the momenta

$$S = \int \left(NKa - \frac{a\dot{a}^2}{N} + 16\pi G \frac{N^{-\frac{1}{w}} a^3}{36} \frac{w}{(1+w)^{1+1/w}} \left(\dot{\varphi} + \vartheta \dot{s}\right)^{1+1/w} \exp\left(-\frac{s}{w}\right)\right) dt$$

$$p_a = \frac{\partial L_T}{\partial \dot{a}} = -2\frac{a\dot{a}}{N}$$

$$p_\varphi = \frac{N^{-\frac{1}{w}} a^3}{36} \frac{1}{(1+w)^{1/w}} \left(\dot{\varphi} + \vartheta \dot{s}\right)^{1/w} \exp\left(-\frac{s}{w}\right)$$

$$p_\vartheta = 0$$

$$p_s = \vartheta p_\varphi$$

Hamiltonian formalism

$$\mathscr{H} = NH = N\left(-\frac{p_a^2}{4a} - Ka + \frac{1}{(16\pi G)}\frac{p_\varphi^{\lambda+1}}{a^{3\lambda}}\exp(s) \right)$$

$$\tau = 16\pi G\, p_s\, p_\varphi^{-(w+1)}\exp(-s)$$

$$p_\tau = \frac{1}{(16\pi G)} p_\varphi^{(w+1)}\exp(s)$$

$$\bar{\varphi} = \varphi - (w+1)\frac{p_s}{p_\varphi}$$

$$p_{\bar{\varphi}} = p_\varphi$$

Hamiltonian

$$\mathcal{H} = NH = N\left(-\frac{p_a^2}{4a} - Ka + \frac{p_\tau}{a^{3w}}\right),$$

ADM Hamiltonian= N x Superhamiltonian

$$\frac{\hbar^2}{4a}\frac{\partial^2\Psi}{\partial a^2} - Ka\Psi = \frac{i\hbar}{a^{3w}}\frac{\partial\Psi}{\partial\tau}$$

What is τ?

$$\tau = 16\pi G\,\vartheta\, p_\varphi^{-\lambda} \exp(-s)$$

$$= 16\pi G\,\vartheta \left[\frac{a^3}{36}\rho_0 \exp\left(-\frac{s}{w}\right)\right]^{-w} \exp(-s)$$

$$= (36)^\lambda 16\pi G\,\vartheta \left[\frac{a^3}{36}\rho_0\right]^{-w}$$

$\propto constant \times \eta$

τ is the <u>thermasy</u> up to a constant

In a universe with radiation it is the <u>conformal time</u>

The concept of thermasy

- Thermasy ϑ was introduced by Van Dantzig
- Infinitesimally it is $d\vartheta = Tdt$ $\qquad \dot{\vartheta} = T$
- The thermasy has convective derivative equal to T and together with the specific entropy describes a convective motion in terms of velocity potentials
- Relativistic thermasy $d\vartheta = Tds$ (the line element or the proper time instead of time)
- D. Van Dantzig "On the phenomenological thermodynamics of moving matter" Physica VI (1939) 673-704

Thermasy as matter time

- Kijowski and coworkers called it "matter time"
- They interpreted it as the time delay of a particle under brownian motion with respect to the time of propagation of a fluid along the flux lines
- Measure of temperature from the variation of the decay time of a fluid of radioactive atoms

Kijowski J, Sm6lski A and G6mick A 1990 Phys. Rev. D 41 1875

- A particle accelerated in a vacuum should have a different thermasy with respect to a particle in a uniform motion due to the Unruh effect.

Thermasy in a radiation universe

$$P = \frac{1}{3}\rho\,.$$

$$\rho = \frac{\pi^2}{30} g(T) T^4$$

$$\bar{s} = ns = \frac{2\pi^2}{45} g(T) T^3$$

$$\nabla_a T^{ab} = 0 \longrightarrow a^{-3} T \frac{\partial}{\partial t}\left(\frac{\rho + p}{T} a^3\right) = a^{-3} T \frac{\partial \bar{s} a^3}{\partial t} = 0.$$

e.g. S. Dodelson, Modern Cosmology, Amsterdam, Netherlands: Academic Pr. (2003) 440

Thermasy and conformal time

$$T \propto \frac{1}{a}.$$

$$\vartheta = \int kT dt \propto \int \frac{dt}{a(t)} = \eta$$

In conclusion in a radiation universe thermasy is proportional to the conformal time and so is τ. The Wheeler De Witt is a legitimate Schroedinger equation

The Wheeler De Witt equation

The Wheeler-DeWitt equation

$$\frac{\hbar^2}{4a}\frac{\partial^2\Psi}{\partial a^2} - Ka\Psi = \frac{i\hbar}{a^{3w}}\frac{\partial\Psi}{\partial\tau}$$

can be expressed in different equivalent ways

Wheeler DeWitt equation expressed in terms of the thermasy

$$\frac{\hbar^2}{4a}\frac{\partial^2\Psi}{\partial a^2} - Ka\Psi = \frac{i\hbar}{a^{3w}}\alpha\frac{\partial\Psi}{\partial\vartheta}$$

$$\frac{\hbar^2}{4a}\frac{\partial^2\Psi}{\partial a^2} - Ka\Psi = \frac{i\hbar}{a^{3w}}\frac{\alpha}{kT}\frac{\partial\Psi}{\partial t}$$

or

$$\frac{\hbar^2}{4}\frac{\partial^2\Psi}{\partial a^2} - Ka^2\Psi = \frac{i\hbar}{a^{3w-1}}\frac{\alpha}{kT}\frac{\partial\Psi}{\partial t}$$

Modified Wheeler DeWitt equation

Notice that for $T \to \infty$ the equation looses the time dependence. I.e. the Wheeler De Witt takes its usual form which derives from this singular condition

$$\frac{\hbar^2}{4}\frac{\partial^2\Psi}{\partial a^2} - Ka^2\Psi + af(\Lambda, \phi, \ldots)\Psi = \frac{i\hbar}{a^{3w-1}}\frac{\alpha}{kT}\frac{\partial\Psi}{\partial t}$$

$$\Big\downarrow \; T \to \infty$$

$$\frac{\hbar^2}{4}\frac{\partial^2\Psi}{\partial a^2} - Ka^2\Psi + af(\Lambda, \phi, \ldots)\Psi = 0$$

The Wheeler De Witt equation in a radiation dominated universe

We have seen that in a radiation universe $(w = \frac{1}{3}$

$$\tau \propto \vartheta \propto \eta$$

$$\frac{\hbar^2}{4a}\frac{\partial^2\Psi}{\partial a^2} - Ka\Psi = \frac{i\hbar}{a}\frac{\partial\Psi}{\partial\eta}$$

$$\frac{\hbar^2}{4}\frac{\partial^2\Psi}{\partial a^2} - Ka^2\Psi = i\hbar\frac{\partial\Psi}{\partial\eta}$$ or

using the definition of conformal time. $d\eta = \frac{1}{a}da$

$$\frac{\hbar^2}{4}\frac{\partial^2\Psi}{\partial a^2} - Ka^2\Psi = i\hbar\, a\frac{\partial\Psi}{\partial t}$$

Conditions for the solutions

- Extended minisuperspace (Lapchinski-Rubakov) $-\infty < a < +\infty$
- Hermitian operators when $0 < a < +\infty$
- Square integrable wave functions
- This imply that also the tomograms are well defined

Boundary conditions

$$\left.\frac{\partial \Psi}{\partial a}\right|_{a=0} = c\,\Psi(0), \qquad c \in (-\infty, +\infty)$$

$$\left.\frac{\partial \Psi}{\partial a}\right|_{a=0} = 0, \qquad c = 0$$

$$\Psi(0) = 0, \qquad c = +\infty$$

Lapchinski-Rubakov initial conditions

$$w = \frac{1}{3}$$

$$\Psi(a,0) = C\left[\exp\left(-\frac{(a-a_0)^2}{\beta^2}\right) - \exp\left(-\frac{(a+a_0)^2}{\beta^2}\right)\right]$$

$$\mathscr{W}\left(X,\mu,\nu,0\right) = \frac{1}{2\pi|\nu|}A^2(\mu,\nu)e^{-X^2/2\tilde{\beta}}$$

$$\times\left[\exp\left(\frac{-2Xa_0\mu}{\tilde{\beta}^2}\right) + \exp\left(\frac{2Xa_0\mu}{\tilde{\beta}^2}\right) - 2\cos\left(\frac{Xa_0\nu}{\nu^2+\mu^2\beta^2}\right)\right]$$

$-\infty < a < +\infty$ (extended minisuperspace)

Time evolution and propagators

$$\mathscr{W}\left(X,\mu,\nu,\tau\right)=\int\Pi\left(X,\mu,\nu,\tau,X',\mu',\nu'\right)\mathscr{W}(X',\mu',\nu',0)\,dX'\,d\mu'\,d\nu'$$

$$\Pi^{osc.}\left(X,\mu,\nu,t,X',\mu',\nu'\right)=\delta(X-X')\delta\left(\mu'-\mu\cos\omega\tau+\nu\sin\omega\tau\right)$$

$$\times\,\delta\left(\nu'-\nu\cos\omega\tau-\frac{\mu}{\omega}\sin\omega\tau\right)$$

$$\Pi^{flat}\left(X,\mu,\nu,t,X',\mu',\nu'\right)=\delta(X-X')\delta\left(\mu'-\mu\right)\delta\left(\nu'-\nu-\mu\tau\right)$$

$$\Pi^{antiosc.}\left(X,\mu,\nu,t,X',\mu',\nu'\right)=\delta(X-X')\delta\left(\mu'-\mu\cosh\omega\tau+\nu\sinh\omega\tau\right)$$

$$\times\,\delta\left(\nu'-\nu\cosh\omega\tau-\frac{\mu}{\omega}\sinh\omega\tau\right)$$

Evolution of the tomogram

We notice that for $T \to 0$
both $\Pi^{osc.}$ and $\Pi^{nonosc.}$
tend to Π^{flat} then when the
universe cools down one has

$$\mathscr{W}\left(X, \mu, \nu, t\right) = \frac{1}{2\pi|\nu - \mu t|} A^2(\mu, \nu) e^{-X^2/2\tilde{\beta}}$$

$$\times \left[\exp\left(\frac{-2Xa_0\mu}{\tilde{\beta}^2}\right) + \exp\left(\frac{2Xa_0\mu}{\tilde{\beta}^2}\right) - 2\cos\left(\frac{Xa_0(\nu - \mu t)}{(\nu - \mu t)^2 + \mu^2\beta^2}\right)\right]$$

Conclusions

- Tomograms are very useful to analyse the transition from quantum to classical universe
- They were studied only for a de Sitter universe
- Therefore it is necessary to extend the tomograms to more general models with material sources
- But revisiting the models with perfect fluids it appears that the fiducial time is an effective time measurable with the clocks of the observers.
- In particular we found out that in a radiation dominated universe the Wheeler De Witt equation is a Schroedinger equation
- The tomograms that describe the initial state generalise the conditions for the absence of a singularity in any frame of the phase space..
- Future work extend these results to models with cosmological constant and scalar fields
- Find general solutions with approximation techniques