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Exact solutions in statistical quantum field theory

Based on

F.B., M. Buzzegoli, A. Palermo, JHEP 02, 101 (2021), arXiv:2007.08249
A. Palermo, M. Buzzegoli, F.B., arXiv 2106.08340
D. Rindori, L. Tinti, F. B., D. Rischke, arxiv:2102.09106

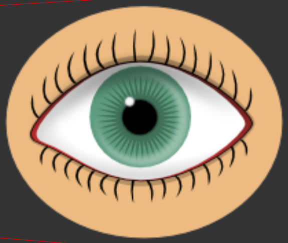
OUTLINE

- Introduction
- Exact solutions at global equilibrium with rotation and acceleration
- An exact solution for a longitudinally boost invariant expansion
- Outlook and conclusions

SIGRAV 2021, Urbino

Gravity-related motivations

$$G^{\mu\nu} = 8\pi G\, T^{\mu\nu}$$



In FRW universe, the form of the stress-energy tensor is dictated by symmetry

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu - p g^{\mu\nu}$$

Semi-classical equation:

$$T^{\mu\nu}(x) \equiv \mathrm{Tr}(\widehat{\rho}\widehat{T}^{\mu\nu}(x))_{\mathrm{ren}}$$

This definition implies quantum corrections to the classical relation between energy density and pressure depending on the density operator:

$$\rho = \rho_{class} + \hbar \Delta\rho \qquad p = p_{class} + \hbar \Delta p$$

The density operator

Local equilibrium density operator

Zubarev 1979 Weert 1982

$$\widehat{\rho} = \frac{1}{Z} \exp\left[- \int_\Sigma \mathrm{d}\Sigma_\mu \; \left(\widehat{T}^{\mu\nu} \beta_\nu - \zeta \widehat{j}^\mu \right) \right]$$

This operator is obtained by maximizing the entropy with the constraints of given energy density and momentum density. β and ζ are Lagrange multiplier *functions* with a thermodynamic meaning

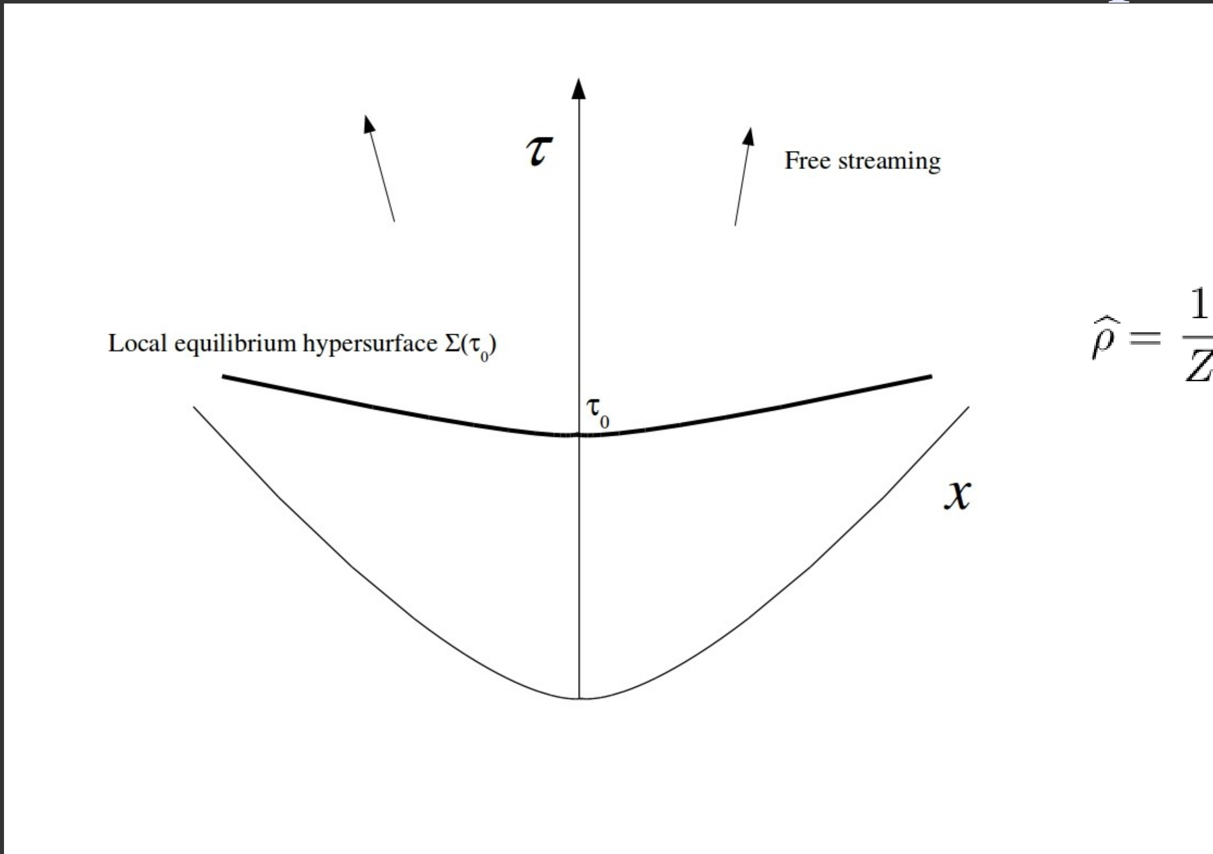
$\beta^\mu = (1/T) u^\mu$ is the four-temperature vector

$T = 1/\sqrt{\beta^2}$ is the proper or comoving temperature

$\zeta = \mu/T$ is the ratio between chemical potential and temperature

$$u^\mu = \beta^\mu / \sqrt{\beta^2}$$

True statistical operator



T_B = Belinfante stress-energy tensor

$$\widehat{\rho} = \frac{1}{Z} \exp\left[- \int_{\Sigma(\tau_0)} \mathrm{d}\Sigma_\mu \left(\widehat{T}_B^{\mu\nu} \beta_\nu - \zeta \widehat{j}^\mu \right) \right] .$$

With the Gauss theorem

$$\widehat{\rho} = \frac{1}{Z} \exp\left[- \int_{\Sigma(\tau)} \mathrm{d}\Sigma_\mu \left(\widehat{T}_B^{\mu\nu} \beta_\nu - \zeta \widehat{j}^\mu \right) + \int_\Theta \mathrm{d}\Theta \, \left(\widehat{T}_B^{\mu\nu} \nabla_\mu \beta_\nu - \widehat{j}^\mu \nabla_\mu \zeta \right) \right] ,$$

Local equilibrium, non-dissipative terms

Dissipative terms

Search for exact solutions

$$T^{\mu\nu}(x) \equiv \mathrm{Tr}(\widehat{\rho}\widehat{T}^{\mu\nu}(x))_{\mathrm{ren}}$$

GOAL: exact solution of the free (scalar) field in FRW universe (hard)

SOLUTIONS FOUND FOR THREE CASES:

- Free scalar field for a general global equilibrium in flat spacetime
- Free Dirac field for a general global equilibrium in flat spacetime
- Free scalar field for a longitudinally boost invariant expansion in flat spacetime (non-equilibrium)

PROBLEM 1
General covariant *global* equilibrium in flat spacetime

F.B., M. Buzzegoli, A. Palermo, JHEP 02, 101 (2021), arXiv:2007.08249
A. Palermo, M. Buzzegoli, F.B., arXiv 2106.08340, to appear in JHEP

$$\widehat{\rho} = \frac{1}{Z} \exp\left[- \int_\Sigma \mathrm{d}\Sigma_\mu \ \left(\widehat{T}^{\mu\nu} \beta_\nu - \zeta \widehat{j}^\mu \right) \right]$$

If the divergence of the integrand vanishes, that is:

$$\partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0$$

$$\partial_\mu \zeta = 0$$

Σ can now be an arbitrary, “time”-independent general spacelike 3D hypersurface

Solution of the Killing equation in Minkowski spacetime:

$$\beta^\nu = b^\nu + \varpi^{\nu\mu} x_\mu$$

constant

$$\varpi_{\nu\mu} = -\frac{1}{2}(\partial_\nu \beta_\mu - \partial_\mu \beta_\nu)$$

Thermal vorticity
Adimensional in natural units

Using the solution into the general covariant expression of the density operator:

$$\widehat{\rho} = \frac{1}{Z} \exp\left[-b_\mu \widehat{P}^\mu + \frac{1}{2} \varpi_{\mu\nu} \widehat{J}^{\mu\nu} + \zeta \widehat{Q} \right]$$

with

$$\widehat{J}^{\mu\nu} = \int_\Sigma \mathrm{d}\Sigma_\lambda \left(x^\mu \widehat{T}^{\lambda\nu} - x^\nu \widehat{T}^{\lambda\mu} \right)$$

Therefore, the most general thermodynamical equilibrium in Minkowski spacetime involves the 10 generators of its maximal symmetry group.

The physical content of ϖ

Decomposition into two spacelike vector fields:

$$\varpi^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} w_\rho u_\sigma + \alpha^\mu u^\nu - \alpha^\nu u^\mu$$

$$u^\mu = \beta^\mu / \sqrt{\beta^2}$$

At global equilibrium, using Killing equation

$$\alpha^\mu = \frac{1}{T} a^\mu \qquad w^\mu = \frac{1}{T} \omega^\mu$$

With:

$$a^\mu = u^\nu \partial_\nu u^\mu \qquad \omega^\mu = -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \partial_\nu u_\rho u_\sigma$$

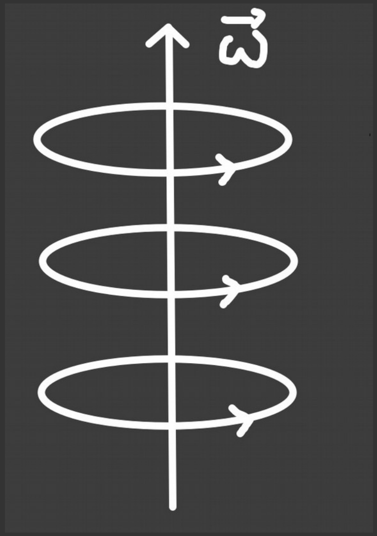
Examples

Pure rotation (Landau *Statistical Physics*)

$$b_\mu = (1/T_0, 0, 0, 0) \qquad \varpi_{\mu\nu} = (\omega/T_0)(g_{1\mu}g_{2\nu} - g_{1\nu}g_{2\mu})$$

$$\beta^\mu = \frac{1}{T_0}(1, \boldsymbol{\omega} \times \mathbf{x})$$

$$\widehat{\rho} = (1/Z)\exp[-\widehat{H}/T_0 + \omega \widehat{J}_z/T_0]$$

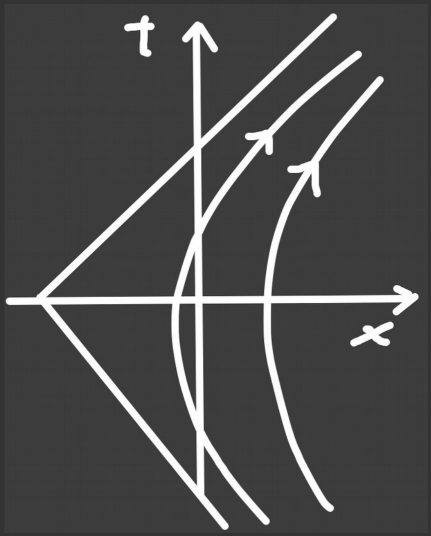


Pure acceleration (new – Unruh-like)

$$b_\mu = (1/T_0, 0, 0, 0) \qquad \varpi_{\mu\nu} = (a/T_0)(g_{0\mu}g_{3\nu} - g_{3\mu}g_{0\nu})$$

$$\beta^\mu = \frac{1}{T_0}(1 + az, 0, 0, at)$$

$$\widehat{\rho} = (1/Z)\exp[-\widehat{H}/T_0 + a\widehat{K}_z/T_0]$$



Method (new)

In the past, similar problems (e.g. rotation) have been solved by choosing suitable curvilinear coordinates and solving the field equations therewith. We keep the plane-wave solution and….

- Complex thermal vorticity $\varpi^{\mu\nu} \to -i\phi^{\mu\nu}$

- Factorization of the density Operator

$$\widehat{\rho} = \frac{1}{Z} e^{-b_\mu \hat{P}^\mu - i\frac{1}{2}\phi_{\mu\nu}\hat{J}^{\mu\nu}} = \frac{1}{Z} e^{-\widetilde{b}_\mu(\phi)\hat{P}^\mu} e^{-\frac{i}{2}\phi_{\mu\nu}\hat{J}^{\mu\nu}}$$

$$\widetilde{b}^\mu(\phi) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \underbrace{(\phi^{\mu\nu_1}\phi_{\nu_1\nu_2}\dots\phi^{\nu_{k-1}\nu_k})}_{k-\text{times}} b_{\nu_k}$$

- Algebraic equation for the expectation values:

$$\langle a^\dagger_\sigma(p) a_\tau(p')\rangle = (-1)^{2S} \sum_\alpha W^{(S)}(\Lambda, p)_{\alpha\sigma} e^{-\widetilde{b}\cdot\Lambda p} \langle a^\dagger_\alpha(\Lambda p) a_\tau(p')\rangle + 2\varepsilon e^{-\widetilde{b}\cdot\Lambda p} W^{(S)}(\Lambda, p)_{\tau\sigma} \delta^3(\Lambda p - p')$$

$W^{(S)}(\Lambda, p) = D^{(S)}\left([\Lambda p]^{-1}\Lambda[p]\right)$ is the Wigner rotation

- SOLVED by iteration:

$$\langle a^\dagger_\sigma(p) a_\tau(p')\rangle = 2\epsilon' \sum_{n=1}^{\infty} (-1)^{2S(n+1)} \delta^3(\Lambda^n p - p') W^{(S)}_{\tau\sigma}(\Lambda^n, p) e^{-\widetilde{b}\cdot\sum_{k=1}^n \Lambda^k p}$$

Method - cont'd

For a free field problem, the expectation values of quadratic combinations of creation and annihilation operators is all we need to calculate anything, particularly the stress-energy tensor

The iterative solution introduces non-analytic contributions which must be subtracted away before analytically continue the solution to real thermal vorticity (i.e. real acceleration and vorticity)

ANALYTIC DISTILLATION

Extraction of the unique analytic part (in a point) of a non-analytic complex function

Stress-energy tensor: selected results

$$T^{\mu\nu}(x) \equiv \mathrm{Tr}(\widehat{\rho}\,\widehat{T}^{\mu\nu}(x))_{\mathrm{ren}} = \mathrm{Tr}(\widehat{\rho} : \widehat{T}^{\mu\nu}(x) :)$$

Massless Dirac field with no chemical potential

$$T^{\mu\nu}_C(x) \equiv \langle : \widehat{T}^{\mu\nu}_C(x) : \rangle = \langle : \frac{i}{2}(\overline{\Psi}(x)\gamma^\mu \overleftrightarrow{\partial}^\nu \Psi(x) : \rangle$$

$$\begin{aligned} T^{\mu\nu}_B(x) =& \rho\, u^\mu u^\nu - p\,\Delta^{\mu\nu} + W\, w^\mu w^\nu + A\,\alpha^\mu\alpha^\nu + G^l\, l^\mu l^\nu + G(l^\mu u^\nu + l^\nu u^\mu) + \mathbb{A}(\alpha^\mu u^\nu + \alpha^\nu u^\mu) \\ &+ G^\alpha(l^\mu\alpha^\nu + l^\nu\alpha^\mu) + \mathbb{W}(w^\mu u^\nu + w^\nu u^\mu) + A^w(\alpha^\mu w^\nu + \alpha^\nu w^\mu) + G^w(l^\mu w^\nu + l^\nu w^\mu)\,. \end{aligned}$$

$$\rho = \frac{7\pi^2}{60\beta^4} - \frac{\alpha^2}{24\beta^4} - \frac{w^2}{8\beta^4} - \frac{17\alpha^4}{960\pi^2\beta^4} + \frac{w^4}{64\pi^2\beta^4} + \frac{23\alpha^2 w^2}{1440\pi^2\beta^4} + \frac{11(\alpha\cdot w)^2}{720\pi^2\beta^4},$$

$$p = \frac{7\pi^2}{180\beta^4} - \frac{\alpha^2}{72\beta^4} - \frac{w^2}{24\beta^4} - \frac{17\alpha^4}{2880\pi^2\beta^4} + \frac{w^4}{192\pi^2\beta^4} + \frac{(\alpha\cdot w)^2}{96\pi^2\beta^4},$$

$$G^l = -\frac{11}{160\pi^2\beta^4},$$

$$G = \frac{1}{18\beta^4} - \frac{31\alpha^2}{360\pi^2\beta^4} - \frac{13w^2}{120\pi^2\beta^4},$$

$$W = -\frac{61\alpha^2}{1440\pi^2\beta^4},$$

$$A = -\frac{61w^2}{1440\pi^2\beta^4},$$

$$A^w = \frac{61\alpha\cdot w}{1440\pi^2\beta^4},$$

$$\mathbb{A} = \mathbb{W} = G^\alpha = G^w = 0.$$

Quantum corrections!

Generally very small scale

$$\frac{\hbar\omega}{KT} \qquad \frac{\hbar A}{cKT}$$

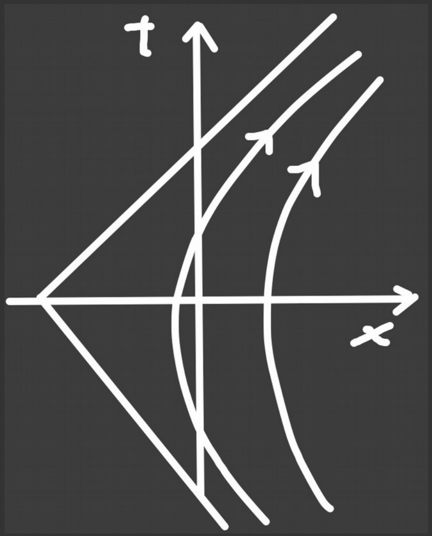
Pure acceleration: Unruh effect

$w = 0$ ⇒

$$\rho = \frac{7\pi^2}{60\beta^4} - \frac{\alpha^2}{24\beta^4} - \frac{17\alpha^4}{960\pi^2\beta^4},$$
$$p = \frac{7\pi^2}{180\beta^4} - \frac{\alpha^2}{72\beta^4} - \frac{17\alpha^4}{2880\pi^2\beta^4},$$
$$\mathcal{A} = 0,$$

Energy density and pressure PRECISELY vanish when

$$\alpha = \frac{A}{T} = 2\pi$$



PROBLEM 2
Longitudinally boost invariant expansion in flat spacetime

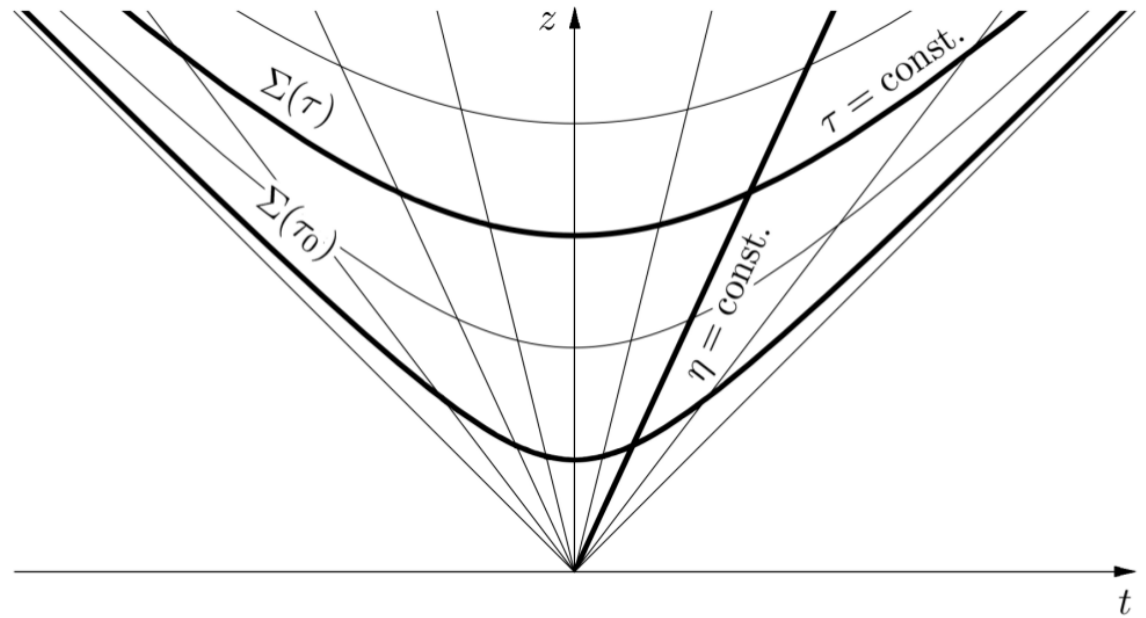
D. Rindori, L. Tinti, F. B., D. Rischke, arxiv:2102.09106

$$\widehat{\rho} = \frac{1}{Z} \exp\left[- \int_{\Sigma(\tau_0)} \mathrm{d}\Sigma_\mu \left(\widehat{T}^{\mu\nu} \beta_\nu - \zeta \widehat{j}^\mu \right) \right] .$$

$$\beta^\mu = \frac{1}{T(\tau_0)} \frac{1}{\tau_0} (t, 0, 0, z) = \frac{1}{T(\tau_0)} u^\mu ,$$

The density operator is invariant (commutates with) under a longitudinal boost

$$T^{\mu\nu} = \mathcal{E}(\tau) u^\mu u^\nu + \mathcal{P}_\mathrm{T}(\tau) \left(\hat{i}^\mu \hat{i}^\nu + \hat{j}^\mu \hat{j}^\nu \right) + \mathcal{P}_\mathrm{L}(\tau) \hat{\eta}^\mu \hat{\eta}^\nu .$$



Milne coordinates:

$$t = \tau \cosh \eta , \qquad z = \tau \sinh \eta ,$$

$$\tau = \sqrt{t^2 - z^2} , \qquad \eta = \frac{1}{2} \log \left(\frac{t+z}{t-z} \right) ,$$

Field (real scalar) and density operator

S. Akkelin, Eur. Phys. J 55 (2019) 78;
L. Crispino et al., Rev. Mod. Phys, 80 (2008) 787.

$$\widehat{\psi}(\tau, \mathbf{x}_T, \eta) = \int \frac{d^2 p_T \, d\mu}{4\pi\sqrt{2}} \left[h(\mathsf{p}, \tau) e^{i(\mathbf{p}_T \cdot \mathbf{x}_T + \mu\eta)} \widehat{b}_{\mathsf{p}} + h^*(\mathsf{p}, \tau) e^{-i(\mathbf{p}_T \cdot \mathbf{x}_T + \mu\eta)} \widehat{b}^{\dagger}_{\mathsf{p}} \right] ,$$

$$h(\mathsf{p}, \tau) = -i e^{\frac{\pi}{2}\mu} H^{(2)}_{i\mu}(m_T \tau) , \qquad h^*(\mathsf{p}, \tau) = i e^{-\frac{\pi}{2}\mu} H^{(1)}_{i\mu}(m_T \tau) ,$$

$$\widehat{b}_{\mathsf{p}} |0_M\rangle = 0$$

$$H^{(2)}_{i\mu}(m_T \tau) = -\frac{1}{i\pi} e^{-\frac{\pi}{2}\mu} \int_{-\infty}^{+\infty} d\theta \, e^{-i m_T \tau \cosh\theta + i\mu\theta} ,$$

$$H^{(1)}_{i\mu}(m_T \tau) = \frac{1}{i\pi} e^{\frac{\pi}{2}\mu} \int_{-\infty}^{+\infty} d\theta \, e^{i m_T \tau \cosh\theta - i\mu\theta} ,$$

$$\widehat{T}^{\mu\nu}_C = \frac{1}{2}\left(\partial^\mu \widehat{\psi} \partial^\nu \widehat{\psi} + \partial^\nu \widehat{\psi} \partial^\mu \widehat{\psi} \right) - g^{\mu\nu} \widehat{\mathcal{L}} , \qquad \widehat{\mathcal{L}} = \frac{1}{2}\left(g^{\mu\nu} \partial_\mu \widehat{\psi} \partial_\nu \widehat{\psi} - m^2 \widehat{\psi}^2 \right) ,$$

$$\widehat{\rho} = \frac{1}{Z} \exp\left[-\frac{\widehat{\Pi}(\tau_0)}{T(\tau_0)} \right] ,$$

$$\widehat{\Pi}(\tau_0) = \int_{\Sigma(\tau_0)} d\Sigma \, u_\mu u_\nu \widehat{T}^{\mu\nu} = \tau_0 \int dx \, dy \, d\eta \, u_\mu u_\nu \widehat{T}^{\mu\nu} ,$$

$$\widehat{\Pi}(\tau) = \tau \int dx \, dy \, d\eta \, \widehat{T}^{\mu\nu} u_\mu u_\nu = \int d^2 p_T \, d\mu \, \frac{\omega}{2} \left[K \left(\widehat{b}_{\mathsf{p}} \widehat{b}^{\dagger}_{\mathsf{p}} + \widehat{b}^{\dagger}_{\mathsf{p}} \widehat{b}_{\mathsf{p}} \right) + \Lambda \widehat{b}_{\mathsf{p}} \widehat{b}_{-\mathsf{p}} + \Lambda^* \widehat{b}^{\dagger}_{\mathsf{p}} \widehat{b}^{\dagger}_{-\mathsf{p}} \right] ,$$

$$K(\mathsf{p}, \tau) = \frac{\pi\tau}{4\omega} \left(|\partial_\tau h(\mathsf{p}, \tau)|^2 + \omega^2 |h(\mathsf{p}, \tau)|^2 \right) ,$$

$$\Lambda(\mathsf{p}, \tau) = \frac{\pi\tau}{4\omega} \left\{ [\partial_\tau h(\mathsf{p}, \tau)]^2 + \omega^2 h^2(\mathsf{p}, \tau) \right\} .$$

$$\omega^2 = m_T^2 + \frac{\mu^2}{\tau^2} .$$

Not diagonal!!!

Diagonalization

Looking for Bogoliubov transformations to make the density operator of the familiar "Hamiltonian" form

$$\widehat{\xi}^{\dagger}_{\mathsf{p}}(\tau) = A(\mathsf{p},\tau)\widehat{b}^{\dagger}_{\mathsf{p}} - B(\mathsf{p},\tau)\widehat{b}_{-\mathsf{p}} \;,$$
$$\widehat{\xi}_{\mathsf{p}}(\tau) = A^{*}(\mathsf{p},\tau)\widehat{b}_{\mathsf{p}} - B^{*}(\mathsf{p},\tau)\widehat{b}^{\dagger}_{-\mathsf{p}} \;,$$

$$\widehat{\Pi}(\tau) = \int \mathrm{d}^2\mathrm{p}_{\mathrm{T}}\,\mathrm{d}\mu\,\frac{\omega}{2}\left(\widehat{\xi}_{\mathsf{p}}(\tau)\widehat{\xi}^{\dagger}_{\mathsf{p}}(\tau) + \widehat{\xi}^{\dagger}_{\mathsf{p}}(\tau)\widehat{\xi}_{\mathsf{p}}(\tau)\right) = \int \mathrm{d}^2\mathrm{p}_{\mathrm{T}}\,\mathrm{d}\mu\,\omega\left(\widehat{\xi}^{\dagger}_{\mathsf{p}}(\tau)\widehat{\xi}_{\mathsf{p}}(\tau) + \frac{1}{2}\right) \;,$$

Since: $K^2(\mathsf{p},\tau) - |\Lambda(\mathsf{p},\tau)|^2 = 1 \;,$

then $K(\mathsf{p},\tau) = \cosh 2\Theta(\mathsf{p},\tau) \;, \qquad \Lambda(\mathsf{p},\tau) = \sinh 2\Theta(\mathsf{p},\tau)\exp[i\chi(\mathsf{p},\tau)] \;,$

The hyperbolic angle Θ can be determined based on the actual expressions (see previous slide)

SOLUTION: $A = \cosh\Theta \;, \qquad B = -\sinh\Theta\,\mathrm{e}^{i\chi} \;,$

The (unrenormalized) stress-energy tensor

$$\widehat{T}^{\mu\nu}_{C} = \frac{1}{2}\left(\partial^\mu \widehat{\psi}\partial^\nu\widehat{\psi} + \partial^\nu\widehat{\psi}\,\partial^\mu\widehat{\psi}\right) - g^{\mu\nu}\widehat{\mathcal{L}}\,, \qquad \widehat{\mathcal{L}} = \frac{1}{2}\left(g^{\mu\nu}\partial_\mu \widehat{\psi}\partial_\nu\widehat{\psi} - m^2\widehat{\psi}^2\right) ,$$

With the diagonal form of Π it is straightforward to calculate thermal expectation values

$$\begin{aligned}
\langle\widehat{\xi}^{\dagger}_{\mathsf{p}}(\tau)\widehat{\xi}_{\mathsf{p}'}(\tau)\rangle_{\rm LE} &= n_{\rm B}(\mathsf{p},\tau)\,\delta^2(\mathbf{p}_{\rm T}-\mathbf{p}'_{\rm T})\delta(\mu-\mu')\ ,\\
\langle\widehat{\xi}_{\mathsf{p}}(\tau)\widehat{\xi}^{\dagger}_{\mathsf{p}'}(\tau)\rangle_{\rm LE} &= [n_{\rm B}(\mathsf{p},\tau)+1]\,\delta^2(\mathbf{p}_{\rm T}-\mathbf{p}'_{\rm T})\delta(\mu-\mu')\ ,\\
\langle\widehat{\xi}_{\mathsf{p}}(\tau)\widehat{\xi}_{\mathsf{p}'}(\tau)\rangle_{\rm LE} &= 0 = \langle\widehat{\xi}^{\dagger}_{\mathsf{p}}(\tau)\widehat{\xi}^{\dagger}_{\mathsf{p}'}(\tau)\rangle_{\rm LE}\ ,
\end{aligned}$$

$$n_{\rm B}(\mathsf{p},\tau) = \frac{1}{{\rm e}^{\omega(\tau)/T(\tau)}-1}\ ,$$

and the unrenormalized stress-energy tensor:

$$T^{\mu\nu} = \mathcal{E}(\tau)u^\mu u^\nu + \mathcal{P}_{\rm T}(\tau)\left(\hat{i}^\mu\hat{i}^\nu + \hat{j}^\mu\hat{j}^\nu\right) + \mathcal{P}_{\rm L}(\tau)\hat{\eta}^\mu\hat{\eta}^\nu .$$

Each coefficient reads:

$$\frac{1}{(2\pi)^3\tau}\int {\rm d}^2{\rm p}_{\rm T}\,{\rm d}\mu\;\omega(\tau)\left\{K_\gamma(\tau)K(\tau_0) - {\rm Re}\left[\Lambda_\gamma(\tau)\Lambda^*(\tau_0)\right]\right\}\left[n_{\rm B}(\tau_0)+\frac{1}{2}\right] .$$

$$K_\gamma(\mathsf{p},\tau) = \frac{\pi\tau}{4\omega}\left[|\partial_\tau h(\mathsf{p},\tau)|^2 + \gamma(\mathsf{p},\tau)|h(\mathsf{p},\tau)|^2\right]\ ,$$

$$\Lambda_\gamma(\mathsf{p},\tau) = \frac{\pi\tau}{4\omega}\left\{[\partial_\tau h(\mathsf{p},\tau)]^2 + \gamma(\mathsf{p},\tau)[h(\mathsf{p},\tau)]^2\right\}\ ,$$

$$\gamma(\mathsf{p},\tau) = \begin{cases} \omega^2(\mathsf{p},\tau) = m_{\rm T}^2 + \dfrac{\mu^2}{\tau^2}\,, & \text{for}\quad \mathcal{E}(\tau)_{\rm LE}\ ,\\ -m_{\rm L}^2 \equiv -\dfrac{\mu^2}{\tau^2} - m^2\,, & \text{for}\quad \mathcal{P}_{\rm T}(\tau)_{\rm LE}\ ,\\ -m_{\rm T}^2 + \dfrac{\mu^2}{\tau^2}\,, & \text{for}\quad \mathcal{P}_{\rm L}(\tau)_{\rm LE}\ . \end{cases}$$

Renormalization

Vacuum subtraction: not a trivial problem. Which vacuum?

The vacuum of Π (ξ operators) is not the same vacuum of ψ (b operators)

$$\widehat{b}_{\mathsf{p}}|0_M\rangle = 0 \qquad \widehat{\xi}_{\mathsf{p},\tau}|0_\tau\rangle = 0$$

$$|0_\tau\rangle = \prod_{\mathsf{p}} \frac{1}{|\cosh\Theta(\mathsf{p},\tau)|^{1/2}} \exp\left[-\frac{1}{2}\tanh\Theta(\mathsf{p},\tau)\mathrm{e}^{-i\chi(\mathsf{p},\tau)}\widehat{b}^\dagger_{\mathsf{p}}\widehat{b}^\dagger_{-\mathsf{p}}\right]|0_M\rangle\ .$$

Most familiar option: subtract Minkowski vacuum of the field expansion:

$$T^{\mu\nu} \equiv \mathrm{Tr}(\widehat{\rho}\,\widehat{T}^{\mu\nu}) - \langle 0_M|\widehat{T}^{\mu\nu}|0_M\rangle$$

$$\mathcal{E}(\tau_0)_{\rm ren} = \frac{1}{(2\pi)^3\tau_0}\int \mathrm{d}^2\mathrm{p_T}\,\mathrm{d}\mu\,\omega(\tau_0)n_{\rm B}(\tau_0) - \frac{1}{2(2\pi)^3\tau_0}\int \mathrm{d}^2\mathrm{p_T}\,\mathrm{d}\mu\,\omega(\tau_0)\left[K(\tau_0)-1\right]\ .$$

DIVERGES!

Renormalization - 2

The only reasonable option, giving rise to a finite value, is to subtract the $|0_{\tau_0}\rangle$ VEV

$$T^{\mu\nu} \equiv \mathrm{Tr}(\widehat{\rho}\widehat{T}^{\mu\nu}) - \langle 0_{\tau_0}|\widehat{T}^{\mu\nu}|0_{\tau_0}\rangle$$

$$T^{\mu\nu} = \mathcal{E}(\tau)u^\mu u^\nu + \mathcal{P}_\mathrm{T}(\tau)\left(\hat{i}^\mu\hat{i}^\nu + \hat{j}^\mu\hat{j}^\nu\right) + \mathcal{P}_\mathrm{L}(\tau)\hat{\eta}^\mu\hat{\eta}^\nu .$$

$$\Gamma_\gamma(\tau)_\mathrm{ren} = \frac{1}{(2\pi)^3\tau}\int \mathrm{d}^2\mathrm{p}_\mathrm{T}\,\mathrm{d}\mu\;\omega(\tau)\left\{K_\gamma(\tau)K(\tau_0) - \mathrm{Re}\left[\Lambda_\gamma(\tau)\Lambda^*(\tau_0)\right]\right\}n_\mathrm{B}(\tau_0)\;,$$

It is interesting to go to late times ($m\tau$ >>1)

$$\mathcal{E}(\tau)_\mathrm{ren} \underset{\tau\to\infty}{\simeq} \frac{1}{(2\pi)^3\tau}\int \mathrm{d}^2\mathrm{p}_\mathrm{T}\,\mathrm{d}\mu\;\omega(\tau)n_\mathrm{B}(\tau_0) + \frac{2}{(2\pi)^3\tau}\int \mathrm{d}^2\mathrm{p}_\mathrm{T}\,\mathrm{d}\mu\;\omega(\tau)\sinh^2\Theta(\tau_0)n_\mathrm{B}(\tau_0)\;.$$

$$\begin{aligned}\mathcal{P}_\gamma(\tau)_\mathrm{ren} &\underset{\tau\to\infty}{\simeq} \frac{1}{(2\pi)^3\tau}\int \mathrm{d}^2\mathrm{p}_\mathrm{T}\,\mathrm{d}\mu\;\frac{m_\mathrm{T}^2+\gamma}{2m_\mathrm{T}}K(\tau_0)n_\mathrm{B}(\tau_0)\\ &= \frac{1}{(2\pi)^3\tau}\int \mathrm{d}^2\mathrm{p}_\mathrm{T}\,\mathrm{d}\mu\;\frac{m_\mathrm{T}^2+\gamma}{2m_\mathrm{T}}n_\mathrm{B}(\tau_0) + \frac{2}{(2\pi)^3\tau}\int \mathrm{d}^2\mathrm{p}_\mathrm{T}\,\mathrm{d}\mu\;\frac{m_\mathrm{T}^2+\gamma}{2m_\mathrm{T}}\sinh^2\Theta(\tau_0)n_\mathrm{B}(\tau_0)\;,\end{aligned}$$

$$K(\tau_0) - 1 = \cosh\Theta(\tau_0) - 1 \simeq \frac{\Theta^2}{2} \simeq \frac{1}{8m_\mathrm{T}^2\tau_0^2}\;,$$

Quantum vacuum corrections to the classical free-streaming relations whose magnitude ultimately depends on $m\tau_0$

Summary and outlook

- Study of selected problems of quantum field theory with non-trivial density operators, describing local and global thermodynamic equilibrium.

- We have devised a general method to obtain global thermodynamic equilibrium solutions in flat space-time and calculated all quantum corrections for the free scalar and Dirac field: the stress-energy tensor has quantum corrections to the classical equations of state with relevant scales ha/cKT and $h\omega/KT$

- We have solved a non-equilibrium problem, the longitudinally boost invariant expansion in flat space-time (a.k.a. Kasner or Milne universe) for the free scalar field. Non-trivial renormalization of the stress-energy tensor is implied. Quantum corrections to energy density and pressure may be relevant even at late time.

- Next step: attack the problem of FRW metric.

The most fundamental definition

In a quantum statistical framework, the stress-energy tensor is defined as:

$$T^{\mu\nu}(x) = \mathrm{tr}(\widehat{\rho}\widehat{T}^{\mu\nu}(x))_{\mathrm{ren}}$$

The density operator of the familiar global thermodynamical equilibrium in flat spacetime (in covariant form):

$$\widehat{\rho} = (1/Z)\exp[-\beta \cdot \widehat{P} + \zeta \widehat{Q}]$$

$$T^{\mu\nu}(x) = (\rho + p)u^\mu u^\nu - p g^{\mu\nu}$$

$$\rho = \rho(T, \mu) = \rho(\beta^2, \zeta)$$ energy density

Local Taylor expansion

$$\langle \widehat{O}(x) \rangle = \frac{1}{Z} \mathrm{tr} \left(\exp \left[- \int_\Sigma \mathrm{d}\Sigma \; n_\mu (\widehat{T}^{\mu\nu} \beta_\nu - \zeta \widehat{j}^\mu) \right] \widehat{O}(x) \right)_{\mathrm{ren}}$$

Taylor expansion from the point *x* where a local operator is to be calculated (leading terms):

$$\int_\Sigma \mathrm{d}\Sigma_\mu(y) \widehat{T}^{\mu\nu}(y) \beta_\nu(y) \simeq \beta_\nu(x) \widehat{P}^\nu - \frac{1}{2} \varpi_{\lambda\nu}(x) \widehat{J}_x^{\lambda\nu}$$

$$\varpi_{\nu\mu} = \frac{1}{2} (\partial_\mu \beta_\nu - \partial_\nu \beta_\mu)$$

Example: local gravity acceleration in a rotationally symmetric field (Schwarzschild)

Tolman's law

$$\beta^\mu = \frac{1}{T_0}(1,0,0,0) \qquad T = \frac{1}{\sqrt{\beta^2}} = \frac{T_0}{\sqrt{g_{00}(r)}} \qquad \varpi_{tr} = \frac{1}{2T_0} \partial_r g_{00}(r)$$

$$\alpha_r = -\frac{1}{2T_0} \partial_r g_{00}(r) \qquad w = 0$$

The corrections are quantum

$$T^{\mu\nu}(x) = \left[\rho + \left(\frac{\hbar|a|}{cKT}\right)^2 U_\alpha + \left(\frac{\hbar|\omega|}{KT}\right)^2 U_w\right] u^\mu u^\nu - \left[p + \left(\frac{\hbar|a|}{cKT}\right)^2 D_\alpha + \left(\frac{\hbar|\omega|}{KT}\right)^2 D_w\right] \Delta^{\mu\nu}$$
$$+A\left(\frac{\hbar|a|}{cKT}\right)^2 \hat{a}^\mu \hat{a}^\nu + W\left(\frac{\hbar|\omega|}{KT}\right)^2 \hat{\omega}^\mu \hat{\omega}^\nu + G\frac{\hbar^2|\omega||a|}{c(KT)^2}(u^\mu \hat{\gamma}^\nu + \hat{\gamma}^\mu u^\nu) + o(\varpi^2)$$

$$|a| = \sqrt{-a_\mu a^\mu} \qquad |\omega| = \sqrt{-\omega_\mu \omega^\mu}.$$

In the free scalar field, the coefficients *U, D, A, W, G* have a classical limit, whereas the adimensional scales vanish in the $h \to 0$ limit. The reason is that with *m, T* as scales no non-quantum correction can exist.

Acceleration and vorticity play the role of two new scales in the quantum field problem

The magnitude of these corrections depends on the coefficients *U, D,* …. which are new thermodynamical equilibrium functions.

Back to local thermodynamic equilibrium

$$\widehat{\rho} = \frac{1}{Z} \exp\left[- \int_\Sigma \mathrm{d}\Sigma_\mu \; \left(\widehat{T}^{\mu\nu} \beta_\nu - \zeta \widehat{j}^\mu \right) \right]$$

$$\langle \widehat{O}(x) \rangle = \frac{1}{Z} \mathrm{tr} \left(\exp\left[- \int_\Sigma \mathrm{d}\Sigma \; n_\mu (\widehat{T}^{\mu\nu} \beta_\nu - \zeta \widehat{j}^\mu) \right] \widehat{O}(x) \right)_{\rm ren}$$

Idea: Taylor expand the β vector field (and ζ) around the point where an operator is to be evaluated under the assumption that they vary slowly in comparison with microscopic scales (hydrodynamic limit)

$$\widehat{\rho}_{\rm LE} \simeq \frac{1}{Z_{\rm LE}} \exp\left[-\beta_\nu(x) \widehat{P}^\nu + \xi(x) \widehat{Q} - \frac{1}{4} (\partial_\nu \beta_\lambda(x) - \partial_\lambda \beta_\nu(x)) \widehat{J}_x^{\lambda\nu} + \frac{1}{2} (\partial_\nu \beta_\lambda(x) + \partial_\lambda \beta_\nu(x)) \widehat{L}_x^{\lambda\nu} + \nabla_\lambda \xi(x) \, \widehat{d}_x^\lambda \right] .$$

$$\varpi_{\nu\mu} = \frac{1}{2} (\partial_\mu \beta_\nu - \partial_\nu \beta_\mu)$$

This is how the polarization in relativistic heavy ion collisions arises

Killing β frame vs Landau frame

$$u^\mu = \beta^\mu / \sqrt{\beta^2}$$

$$T^{\mu\nu}(x) = (\rho - \alpha^2 U_\alpha - w^2 U_w) u^\mu u^\nu - (p - \alpha^2 D_\alpha - w^2 D_w)\Delta^{\mu\nu} + A\alpha^\mu\alpha^\nu + W w^\mu w^\nu + G(u^\mu\gamma^\nu + \gamma^\mu u^\nu) + o(\varpi^2)$$

$$T^{\mu\nu} u_\nu = (\rho - \alpha^2 U_\alpha - w^2 U_w) u^\mu + G\gamma^\mu$$

$$\gamma^\mu = (\alpha \cdot \varpi)_\lambda \Delta^{\lambda\mu} = \epsilon^{\mu\nu\rho\sigma} w_\nu \alpha_\rho u_\sigma$$

In other words, the Killing vector field, which *defines equilibrium*, is NOT an eigenvector of the stress-energy tensor.

An observer moving along the eigenvector of the stress-energy tensor, in general relativity, will see the metric tensor changing, which is not desirable in a proper definition of equilibrium

Other consequences

- As the stress-energy tensor has non-dissipative thermal quantum corrections beyond its ideal form if the fluid is rotating or accelerated. they will also be present in gravitational fields (see later).

- Second-order, non-dissipative corrections depend on the specific form of the quantum stress-energy tensor operator. Thermodynamics with rotation or acceleration makes a distinction between,
e.g. the canonical and Belinfante symmetrized tensors. (F. B., L. Tinti, Phys. Rev. D 84 (2011) 025013

- Dependence of the effective equation of state on the acceleration, hence on local gravitational acceleration

In the non-relativistic limit

$$\rho_{\rm eff} \simeq \rho + \frac{1}{24}\frac{mc^2}{KT}\rho\bar{a}^2 = \left(1 + \frac{1}{24}\frac{m\hbar^2|a|^2}{(KT)^3}\right)\rho$$

$$p_{\rm eff} \simeq p + \left(\frac{2}{3}\xi - \frac{1}{8}\right) mc^2\bar{a}^2 n = p\left[1 + \left(\frac{2}{3}\xi - \frac{1}{8}\right)\frac{m\hbar^2|a|^2}{(KT)^3}\right]$$

$$p_{eff} \simeq \rho_{eff}\frac{KT}{m}\left[1 + \left(\frac{2}{3}\xi - \frac{1}{6}\right)\frac{m\hbar^2|a|^2}{(KT)^3}\right]$$