Urbino – Sept. 7, 2021

Hydrodynamics of the Gross–Pitaevskii equation in general Riemannian metric

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Joint work with ALICE ROITBERG *(U. Milano-Bicocca)*

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- *Gross-Pitaevskii (1961) equation:* $\Psi = \Psi(\boldsymbol{x}, t)$

$$\mathrm{i}\,\hbar\,\partial_t \Psi = \left(-\frac{\hbar^2}{2m}\,\nabla^2 + \mathfrak{g}|\Psi|^2 + V\right)\Psi\,,$$

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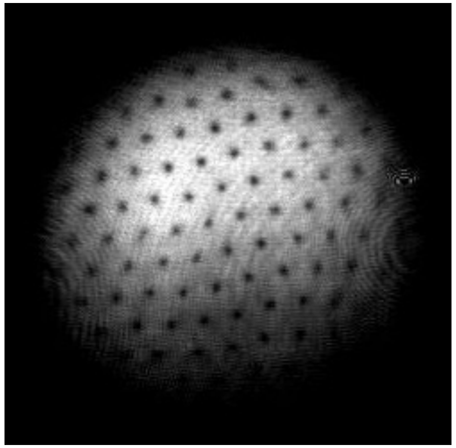

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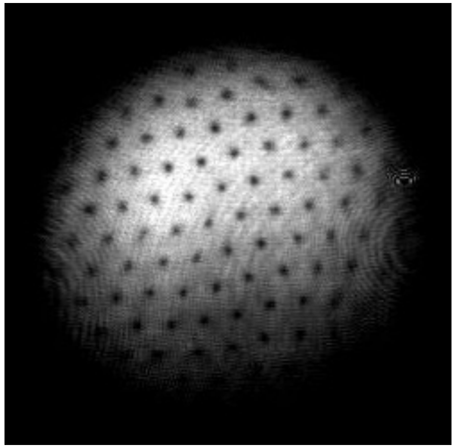
$$\Psi = \sqrt{\rho}\,\exp\left[\mathrm{i}(m/\hbar)\theta\right]\ .$$



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Extension of GPE hydrodynamics to Riemannian manifolds

Hydrodynamic derivation of the Gross–Pitaevskii equation in general Riemannian metric

Alice Roitberg[1] and Renzo L Ricca[1,2,*]

[1] Department of Mathematics & Applications, University of Milano-Bicocca, Via Cozzi 55, 20125 Milano, Italy
[2] BDIC, Beijing University of Technology, 100 Pingleyuan, Beijing 100124, People's Republic of China

E-mail: renzo.ricca@unimib.it

Received 21 March 2021, revised 3 June 2021
Accepted for publication 11 June 2021
Published 1 July 2021

CrossMark

Abstract
Here we show that the Gross–Pitaevskii equation (GPE) for Bose–Einstein condensates (BECs) admits hydrodynamic interpretation in a general Riemannian metric, and show that in this metric the momentum equation has a new term that is associated with local curvature and density distribution profile. In particular conditions of steady state a new Einstein's field equation is determined in presence of negative curvature. Since GPE governs BECs defects that are useful, analogue models in cosmology, a relativistic form of GPE is also considered to show connection with models of analogue gravity, thus providing further grounds for future investigations of black hole dynamics in cosmology.

(Roitberg & Ricca J. Phys. A: Math. Theor. 54, 2021)

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Hydrodynamic form in Riemannian metric

- ***Right-hand side:***

$$i\hbar\partial_t\Psi = \frac{i\hbar}{2\sqrt{\rho}}(\partial_t\rho)\,e^{i\left(\frac{m}{\hbar}\theta\right)} - m\sqrt{\rho}\,e^{i\left(\frac{m}{\hbar}\theta\right)}\partial_t\theta \; .$$

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- ***Left-hand side:***

$$\nabla^2\Psi = \frac{1}{\sqrt{|g|}}\partial_i\left[\sqrt{|g|}\,g^{ij}\partial_j\left(\sqrt{\rho}\,e^{i\left(\frac{m}{\hbar}\theta\right)}\right)\right]$$

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Gross-Pitaevskii equation (GPE) in Euler form

- **Theorem (GPE in Euler form).** GPE admits hydrodynamic description in the form of an Euler equation, given by

$$\partial_t \boldsymbol{u} + \nabla_{\boldsymbol{u}} \boldsymbol{u} = \nabla H \ , \qquad \text{(E)}$$

  where $\nabla_{\boldsymbol{u}} \boldsymbol{u}$ denotes the covariant derivative along itself and $H = U + Q$ is the sum of classical and quantum potentials.

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Proof. Define $\boldsymbol{u} = \nabla\theta$:

- re-write covariant derivative using Koszul's formula, so that

$$\nabla_{\nabla\theta}\nabla\theta = \nabla_u \boldsymbol{u} = \nabla\left(\frac{1}{2}|\nabla\theta|^2\right) \ ;$$

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- *take the gradient of $\partial_t \theta + \frac{1}{2} |\nabla\theta|^2 = U + Q$.*

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- *take the gradient of* $\partial_t \theta + \frac{1}{2} |\nabla\theta|^2 = U + Q$ *.*

$\blacksquare$

- ***Corollary.*** *The relation between GPE and hydrodynamics is one-to-one on any manifold with generic metric* g *.*

Gross-Pitaevskii equation (GPE) in Navier-Stokes form

- ***Theorem (GPE in Navier-Stokes form).*** ***GPE admits hydrodynamic description in the form of a Navier-Stokes equation***

$$\rho\,(\partial_t \boldsymbol{u} + \boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{u}) = -\boldsymbol{\nabla} \wp + \boldsymbol{\nabla} \cdot \boldsymbol{\tau} + \boldsymbol{\mathcal{E}}, \qquad \text{(NS)}$$

where $V = 0$, $\wp = (\mathfrak{g}/2m)\rho^2$, $\boldsymbol{\tau} = (\hbar^2/4m^2)\rho\,\mathrm{Hess}(\ln \rho)$, ***and*** $\boldsymbol{\mathcal{E}}$ ***is a density curvature tensor,*** $\mathcal{E}^j = -(\hbar^2/4m^2)R^{jk}\partial_k \rho$.

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Proof. Multiply (E) by ρ and take $V = 0$

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The second term on the r.h.s. can be written as

$$\frac{\hbar^2}{m^2}\rho\,\mathrm{d}\left(\frac{\boldsymbol{\nabla}^2\sqrt{\rho}}{2\sqrt{\rho}}\right) = \boldsymbol{\nabla}\cdot(\rho\,\mathrm{Hess}(\ln\,\rho)) + \tilde{\boldsymbol{\mathcal{E}}}^\flat,$$

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where $\tilde{\boldsymbol{\mathcal{E}}}^\flat = (4m^2/\hbar^2)\mathcal{E}_j\,\mathrm{d}x^j$, $\tilde{\mathcal{E}}_j = g^{ik}R^r_{ikj}\partial_r\rho$ and $\boldsymbol{\mathcal{E}} = \dfrac{\hbar^2}{4m^2}\tilde{\boldsymbol{\mathcal{E}}}$. ■

Conservation law for momentum

- ***Theorem (Momentum conservation law).*** *The momentum* $\rho\boldsymbol{u}$ *associated with the hydrodynamic form of GPE satisfies the following conservation law*

$$\partial_t(\rho\boldsymbol{u}) = -\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{M}} + \chi\,, \qquad \text{(M)}$$

where $\boldsymbol{\mathcal{M}} = \mathcal{M}_{ij}\,\mathrm{d}x^i \wedge \mathrm{d}x^j$, $\mathcal{M}_{ij} = \mathcal{D}_{ij} + \Pi_{ij} - \tau_{ij} - \mathcal{G}_{ij}$ *with* $\mathcal{D}_{ij} = \rho u_i u_j$, $\boldsymbol{u}^\flat = u_j \mathrm{d}x^j$, $\Pi_{ij} = \wp g_{ij}$, $\boldsymbol{\mathcal{G}} = \hbar^2\rho/(4m^2)\boldsymbol{G}$ *(* $\boldsymbol{G}$ *Einstein tensor) and* χ *is explicit function of the geometry of the manifold through the Ricci scalar* R*.*

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Proof. • *Use (C) and (NS) equations;*

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Proof.
- Use (C) and (NS) equations;
- compute divergence of terms;

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- **Theorem (Momentum conservation law).** *The momentum $\rho\boldsymbol{u}$ associated with the hydrodynamic form of GPE satisfies the following conservation law*

$$\partial_t(\rho\boldsymbol{u}) = -\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{M}} + \chi\ , \qquad \text{(M)}$$

where $\boldsymbol{\mathcal{M}} = \mathcal{M}_{ij}\,\mathrm{d}x^i \wedge \mathrm{d}x^j$, $\mathcal{M}_{ij} = \mathcal{D}_{ij} + \Pi_{ij} - \tau_{ij} - \mathcal{G}_{ij}$ *with* $\mathcal{D}_{ij} = \rho u_i u_j$, $\boldsymbol{u}^\flat = u_j\mathrm{d}x^j$, $\Pi_{ij} = \wp g_{ij}$, $\boldsymbol{\mathcal{G}} = \hbar^2\rho/(4m^2)\boldsymbol{G}$ *(* $\boldsymbol{G}$ *Einstein tensor) and* χ *is explicit function of the geometry of the manifold through the Ricci scalar* R.

Proof.
- *Use (C) and (NS) equations;*
- *compute divergence of terms;*
- *use contracted form of Bianchi's identity*

$$\boldsymbol{\nabla}_k R^k_j\,\mathrm{d}x^j = (1/2)\partial_j R\,\mathrm{d}x^j\ .$$

$\blacksquare$

Conservation law for momentum

- **Theorem (Momentum conservation law).** The momentum $\rho\boldsymbol{u}$ associated with the hydrodynamic form of GPE satisfies the following conservation law

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■

The new term $\chi = -\dfrac{\hbar^2}{8m^2}R\,\mathrm{d}\rho$ breaks the standard form of a conservation law.

Thank you!