CANONICAL ANALYSIS OF BRANS-DICKE THEORY AND JORDAN AND EINSTEIN FRAMES

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partially in collaboration with Matteo Galaverni and based on arXiv:2003.04304, Phys. Rev. D 103, 024022 (2021) and some other papers to come…

Outline

- Jordan and Einstein frame. Scalar-Tensor theory with (GHY)-boundary term and Weyl (conformal) transformations from Jordan to Einstein Frame.

- Hamiltonian (canonical) analysis of Brans-Dicke theory with GHY-boundary term.

- Hamiltonian transformations from Jordan to the Einstein Frame. "Vexata Questio": are these transformations canonical? "Anti-Newtonian"transformations as Hamiltonian canonical transformations.

- FLRW flat, k=0, case as finite dimensional example.

- Canonical analysis of BD theory in the case $\omega = -\frac{3}{2}$ in the Jordan Frame

- BD theory for $\omega = -\frac{3}{2}$ in the Einstein Frame

- The inequivalence between the two Dirac constraints' algebras of BD theory for $\omega = -\frac{3}{2}$ in the Jordan and Einstein frames shows the transformations from the Jordan into the Einstein frame are not canonical.

- Conclusions.

Jordan-Einstein Frames

- Old paper: Dicke (Phys. Rev. (1962) **125,** 6 2163-2167)

  Suppose the proton mass is m_p in mass units m_u and, in "natural units", we scale the unit of measurement by a factor λ^{-1} (length)$^{-1}$ $\tilde{m}_u = \lambda^{-1} m_u$. In the new unit the proton mass $\tilde{m}_p = \lambda^{-1} m_p$.

- Confronting the measurement of the proton mass in the two mass units (Faraoni and Nadeau 2006)

$$\frac{\tilde{m}_p}{\tilde{m}_u} = \frac{\lambda^{-1} m_p}{\lambda^{-1} m_u} = \frac{m_p}{m_u}$$

Jordan-Einstein Frames

- Since $d\tilde{s} = \lambda ds$ and $ds = \left(g_{ij}dx^i dx^j\right)^{\frac{1}{2}}$, then the covariant metric functions scales as

$$\tilde{g}_{\mu\nu} = \lambda^2 g_{\mu\nu}$$

- Invariance under rescaling of unit of measurement implies Weyl (conformal invariance) of the metric tensor

- The starting frame is called "Jordan" frame and the conformal transformed the "Einstein Frame. One observable can be computed in both frames. Its measure, obviously different in the two frames, is related by conformal rescaling according to the observable's dimensions.(e.g. $\tilde{m}_p = \lambda^{-1} m_p$).

Jordan-Einstein Frames

- Dimensional analysis shows the following trasformation rules between the Jordan and Einstein frames

$$\tilde{m} = \lambda^{-1} m; \;\; d\tilde{\tau} = \lambda d\tau; d\tilde{s} = \lambda ds;$$
$$\tilde{A}_i = A_i; \quad \tilde{F}_{ij} = F_{ij}; \;\; \tilde{F}^{ij} = \lambda^{-4} F^{ij};$$
$$\tilde{e} = e; \;\; \tilde{c} = c; \;\; \tilde{\hbar} = \hbar; \;\; \tilde{\delta}^4 = \delta^4$$

Scalar-Tensor Theory

- In general, one starts from a scalar-tensor theory, with GHY-like boundary term, in the Jordan Frame

$$S = \int_M d^n x \sqrt{-g} \left(f(\phi) R - \frac{1}{2} \lambda(\phi) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - U(\phi) \right) + 2 \int_{\partial M} d^{n-1} \sqrt{h} f(\phi) K$$

- and passes to the Einstein Frame with the transformation

$$\tilde{g}_{\mu\nu} = \left(16\pi G f(\phi) \right)^{\frac{2}{n-2}} g_{\mu\nu} \ ,$$

- therefore the action becomes

$$S = \int_M d^n x \sqrt{-\tilde{g}} \left(\frac{1}{16\pi G} \tilde{R} - A(\phi) \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right) + \frac{1}{8\pi G} \int_{\partial M} d^{n-1} \sqrt{\tilde{h}} \tilde{K}$$

Scalar-Tensor Theory

where

$$A(\phi) = \frac{1}{16\pi G}\left(\frac{\lambda(\phi)}{2f(\phi)} + \frac{n-1}{n-2}\frac{(f'(\phi))^2}{f^2(\phi)}\right), V(\phi) = \frac{U(\phi)}{[16\pi G f(\phi)]^{\frac{n}{n-2}}}$$

- One is looking for solutions of the equation of motions such that if

$$(g_{\mu\nu}(x), \phi(x))$$

is solution in the Jordan Frame,

$$(\tilde{g}_{\mu\nu}(x,\phi), \phi(x))$$

is solution in the Einstein frame

Brans-Dicke Theory

- Brans-Dicke, with GHY boundary term, is a particular case of Scalar Tensor theory

$$S = \int_M d^4x\sqrt{-g}\left(\phi\ {}^4R - \frac{\omega}{\phi}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - U(\phi)\right) + 2\int_{\partial M} d^3x\sqrt{h}\phi K \quad .$$

- We studied the 3+1 ADM (Hamiltonian) decomposition

$$g = -(N^2 - N_i N^i)dt\otimes dt + N_i(dx^i\otimes dt + dt\otimes dx^i) + h_{ij}dx^i\otimes dx^j \quad ,$$

- The ADM Lagrangian $\mathcal{L}_{ADM}$ is

$$\begin{aligned}\mathcal{L}_{ADM} &= \sqrt{h}\Bigg[N\phi\left({}^{(3)}R + K_{ij}K^{ij} - K^2\right) - \frac{\omega}{N\phi}\left(N^2h^{ij}D_i\phi D_j\phi - (\dot\phi - N^iD_i\phi)^2\right) \\ &+ 2K(\dot\phi - N^iD_i\phi) - NU(\phi) + 2h^{ij}D_iND_j\phi\Bigg]\end{aligned}$$

Brans-Dicke Theory

- We can define the canonical momenta

$$
\begin{aligned}
\pi &= \frac{\partial \mathcal{L}_{ADM}}{\partial \dot{N}} \approx 0 \;, \pi_i = \frac{\partial \mathcal{L}_{ADM}}{\partial \dot{N}^i} \approx 0 \;, \pi^{ij} = \frac{\partial \mathcal{L}_{ADM}}{\partial \dot{h}_{ij}} \\
&= -\sqrt{h}\left[\phi\Big(K^{ij} - K h^{ij}\Big) + \frac{h^{ij}}{N}\Big(\dot{\phi} - N^i D_i \phi\Big)\right] \;, \\
\pi_\phi &= \frac{\partial \mathcal{L}_{ADM}}{\partial \dot{\phi}} = \sqrt{h}\left(2K + \frac{2\omega}{N\phi}(\dot{\phi} - N^i D_i \phi)\right) \;,
\end{aligned}
$$

And the ADM-Hamiltonian $\mathcal{H}_{ADM}$

$$
\mathcal{H}_{ADM} = \pi^{ij}\dot{h}_{ij} + \pi_\phi \dot{\phi} - \mathcal{L}_{ADM}
$$

$$
\begin{aligned}
&\mathcal{H}_{ADM} = \sqrt{h}\Bigg\{ N\left[-\phi\, {}^3R + \frac{1}{\phi h}\left(\pi^{ij}\pi_{ij} - \frac{{\pi_h}^2}{2}\right)\right] \\
+ \quad & \frac{N\omega}{\phi} D_i\phi D^i\phi + N 2 D^i D_i \phi + N V(\phi) \\
+ \quad & \frac{1}{2h\phi}\left(\frac{N}{3+2\omega}\right)(\pi_h - \phi\pi_\phi)^2 \Bigg\} - 2N^i D_j \pi_i^j + N^i D_i \phi \pi_\phi \;,
\end{aligned}
$$

$$
\pi_h \equiv \pi^{ij} h_{ij}
$$

Brans-Dicke Theory

- Therefore $\mathcal{H}_{ADM}$ is the sum of the Hamiltonian constraint $\mathcal{H}$ and the momentum constraint $\mathcal{H}_i$

$$\mathcal{H}_{ADM} = N\mathcal{H} + N^i\mathcal{H}_i$$

$$\begin{aligned}\mathcal{H} &= \sqrt{h}\Bigg\{\left[-\phi\,{}^3R + \frac{1}{\phi h}\left(\pi^{ij}\pi_{ij} - \frac{{\pi_h}^2}{2}\right)\right] \\ &+ \frac{\omega}{\phi}D_i\phi D^i\phi + 2D^iD_i\phi + \frac{1}{2h\phi}\left(\frac{1}{3+2\omega}\right)(\pi_h - \phi\pi_\phi)^2 + V(\phi)\Bigg\}\end{aligned}$$

$$\mathcal{H}_i = -2D_j\pi_i^j + D_i\phi\pi_\phi$$

- The constraint algebra is like Einstein's Geometrodynamics

$$\{\mathcal{H}_i(x), \mathcal{H}_j(x')\} = \mathcal{H}_i(x')\partial_j\delta(x,x') - \mathcal{H}_i(x){\partial_j}'\delta(x,x') \qquad \{\mathcal{H}(x), \mathcal{H}_i(x')\} = -\mathcal{H}(x')\partial_i'\delta(x,x')$$

$$\{\mathcal{H}(x), \mathcal{H}(x')\} = \mathcal{H}^i(x)\partial_i\delta(x,x') - \mathcal{H}^i(x')\partial_i'\delta(x,x')$$

Brans-Dicke Theory-Einstein Frame

- Implementing the Weyl (conformal) transformation, we get the following ADM metric $\tilde{g}$ in the Einstein Frame

$$\tilde{g} = -(\tilde{N}^2 - \tilde{N}_i\tilde{N}^i)dt \otimes dt + \tilde{N}_i(dx^i \otimes dt + dt \otimes dx^i) + \tilde{h}_{ij}dx^i \otimes dx^j$$

$$\tilde{N} = (16\pi G f(\phi))^{\frac{1}{n-2}} N; \tilde{N}_i = (16\pi G f(\phi))^{\frac{2}{n-2}} N_i;$$
$$\tilde{h}_{ij} = (16\pi G f(\phi))^{\frac{2}{n-2}} h_{ij}.$$

- Now we recall that in the Brans-Dicke case $f(\phi) = \phi$

$$S = \frac{1}{16\pi G}\int_M dx^4\sqrt{-\tilde{g}}\left[{}^4\tilde{R} - \frac{(\omega + \frac{3}{2})}{\phi^2}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \tilde{V}(\phi)\right]$$

Brans-Dicke Theory-Einstein Frame

- Canonical momenta ($\tilde{\mathcal{L}}_{\mathrm{ADM}}$ is ADM-Lagrangiam demsiti in the E-F)

$$\tilde{\pi}^{ij} = \frac{\partial \tilde{\mathcal{L}}_{ADM}}{\partial \dot{\tilde{h}}_{ij}} = -\frac{\sqrt{\tilde{h}}}{16\pi G}\left(\tilde{K}^{ij} - \tilde{K}\tilde{h}^{ij}\right) = \frac{\pi^{ij}}{16\pi G\phi}$$

$$\tilde{\pi}_{\phi} = \frac{\partial \tilde{\mathcal{L}}_{ADM}}{\partial \dot{\phi}} = \frac{\sqrt{\tilde{h}}(\omega + \frac{3}{2})}{8\pi G\tilde{N}\phi^2}\left(\dot{\phi} - \tilde{N}^i\partial_i\phi\right)$$

$$= \frac{1}{\phi}(\phi\pi_{\phi} - \pi_h).$$

- The ADM Hamiltonian density $\mathcal{H}_{ADM}$ in the E-F is

$$\mathcal{H}_{ADM} = \frac{\sqrt{\tilde{h}}\tilde{N}}{16\pi G}\left[-\,{}^3\tilde{R} + \frac{(16\pi G)^2}{\tilde{h}}\left(\tilde{\pi}^{ij}\tilde{\pi}_{ij} - \frac{\tilde{\pi}_h^2}{2}\right) + \frac{(\omega + \frac{3}{2})}{\phi^2}\partial_i\phi\partial^i\phi \ + \frac{64(\pi G)^2\phi^2}{h(\omega + \frac{3}{2})}\tilde{\pi}_{\phi}^2 + \tilde{V}(\phi)\right]$$

$$-2\tilde{N}^i\tilde{D}_j\tilde{\pi}_i^j + \tilde{N}^i\partial_i\phi\tilde{\pi}_{\phi}\ .$$

Hamiltonian Analysis of BD for $\omega \neq -\frac{3}{2}$	
in Jordan Frame	**in Einstein Frame**
constraints	*constraints*
$\pi \approx 0; \pi^i \approx 0; \mathcal{H} \approx 0; \mathcal{H}_i \approx 0;$	$\widetilde{\pi} \approx 0; \widetilde{\pi}_i \approx 0; \widetilde{\mathcal{H}} \approx 0; \widetilde{\mathcal{H}}_i \approx 0;$
constraint algebra	*constraint algebra*
$\{\pi, \pi_i\} = 0; \{\pi, \mathcal{H}\} = 0; \{\pi, \mathcal{H}_i\} = 0; \{\pi_i, \mathcal{H}\} = 0;$	$\{\widetilde{\pi}, \widetilde{\pi}_i\} = 0; \{\widetilde{\pi}, \widetilde{\mathcal{H}}\} = 0; \{\widetilde{\pi}, \widetilde{\mathcal{H}}_i\} = 0; \{\widetilde{\pi}_i, \widetilde{\mathcal{H}}\} = 0;$
$\{\pi_i, \mathcal{H}_j\} = 0;\ \{\mathcal{H}(x), \mathcal{H}_i(x')\} = -\mathcal{H}(x')\partial_i'\delta(x, x');$	$\{\widetilde{\pi}_i, \widetilde{\mathcal{H}}_j\} = 0;\ \left\{\widetilde{\mathcal{H}}(x), \widetilde{\mathcal{H}}_i(x')\right\} = -\widetilde{\mathcal{H}}(x')\partial_i'\delta(x, x');$
$\{\mathcal{H}_i(x), \mathcal{H}_j(x')\} = \mathcal{H}_i(x')\partial_j\delta(x, x') - \mathcal{H}_j(x)\partial_i{}'\delta(x, x');$	$\{\widetilde{\mathcal{H}}_i(x), \widetilde{\mathcal{H}}_j(x')\} = \widetilde{\mathcal{H}}_i(x')\partial_j\delta(x, x') - \widetilde{\mathcal{H}}_i(x)\partial_i{}'\delta(x, x');$
$\{\mathcal{H}(x), \mathcal{H}(x')\} = \mathcal{H}^i(x)\partial_i\delta(x, x') - \mathcal{H}^i(x')\partial_i'\delta(x, x');$	$\{\widetilde{\mathcal{H}}(x), \widetilde{\mathcal{H}}(x')\} = \widetilde{\mathcal{H}}^i(x)\partial_i\delta(x, x') - \widetilde{\mathcal{H}}^i(x')\partial_i'\delta(x, x');$

Canonical Transformations

- In Hamiltonian theory, a transformation of variables $(Q^i(q,p), P_i(q,p))$, from the canonical variables (q^i, p_i) to (Q^i, P_i) is a canonical transformation if the symplectic two form $\omega = dq^i \wedge dp_i$ is invariant. That is $\omega = dQ^i \wedge dP_i$ in the new coordinates. This is equivalent to the following condition on the Poisson Brackets

$$\{Q^i(q,p), P_j(q,p)\}_{q,p} = \delta^i_j$$
$$\{Q^i(q,p), Q^j(q,p)\}_{q,p} = \{P_i(q,p), P_j(q,p)\}_{q,p} = 0.$$

- Transformations from the Jordan to the Einstein frame are the followings

$$(N, N^i, h_{ij}, \phi, \pi, \pi_i, \pi^{ij}, \pi_\phi) \mapsto (\tilde{N}, \tilde{N}^i, \tilde{h}_{ij}, \tilde{\phi}, \tilde{\pi}, \tilde{\pi}_i, \tilde{\pi}^{ij}, \tilde{\pi}_\phi)$$

Canonical Transformations

- Here, for simplicity, we repeat the transformations from the Jordan to the Einstein Frame in Hamiltonian formalism

$$\tilde{N} = N(16\pi G\phi)^{\frac{1}{2}}\,; \tilde{N}_i = N_i(16\pi G\phi)\,; \tilde{h}_{ij} = (16\pi G\phi)\,h_{ij}\,; \tilde{\pi} = \frac{\pi}{(16\pi G\phi)^{\frac{1}{2}}}\,;$$

$$\tilde{\pi}^i = \frac{\pi^i}{(16\pi G\phi)}, ; \tilde{\pi}^{ij} = \frac{\pi^{ij}}{16\pi G\phi}\,; \phi = \phi\,; \tilde{\pi}_\phi = \frac{1}{\phi}(\phi\pi_\phi - \pi_h)$$

- One can check they are not Hamiltonian Canonical Transformations

$$\{\tilde{N}, \tilde{\pi}_\phi\} = \frac{8\pi G N}{\sqrt{16\pi G\phi}} \neq 0, \text{and } \{\tilde{N}_i, \tilde{\pi}_\phi\} = 16\pi G N_i \neq 0$$

- Therefore it is meaningless to perform the Dirac's constraint analysis in the Einstein Frame, where it is easier, and presume it gives the same result in the Jordan frame.

Main criticism to this non-canonicity argument

- In litterature, people object N, N^i are mere Lagrangian multipliers and canonicity should be checked on the true physical degrees of freedom.

- This could be misleading. Lapse and Shifts cannot be eliminated "ad hoc", they are still canonical variables

- The only way we can "safely" treat them is by making a gauge fixing (ex. $N \approx c_1$, $N^i \approx c^i$ so that $\pi \approx 0$, $\pi_i \approx 0$ becomes second class constraints).

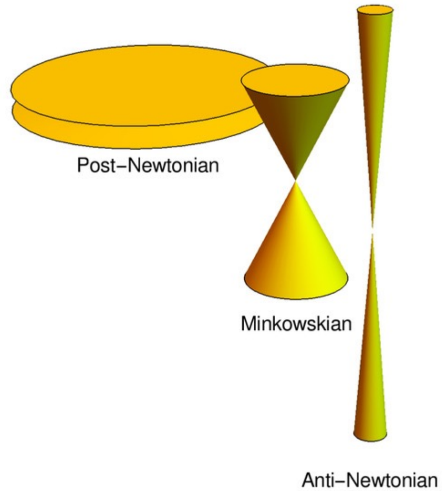
- N, N^i, π , π_i are then eliminated defining Dirac's brackets.

Canonical Transformations

- There exist Hamiltonian Canonical Transformations (Anti-Gravity transformations)

$$N \mapsto N,\ N^i \mapsto N^i,\ h_{ij} \mapsto \lambda^2 h_{ij}$$ (in two dim. $ds^2 = -dt^2 + \lambda^2 dx^2; \lambda > 1$)

$$\tilde{N}^* = N\,;\tilde{N}^*{}_i = N_i\,;\tilde{\pi}^* = \pi\,;\tilde{\pi}^{*i} = \pi^i$$



M. Niedermaier 2019

- In this case the ADM Hamiltonian $\mathcal{H}_{ADM}$

$$\mathcal{H}_{ADM} = \frac{\sqrt{\tilde{h}}\tilde{N}^*(\phi)^{\frac{1}{2}}}{(16\pi G)^{\frac{1}{2}}}\Bigg[-{}^3\tilde{R} + \frac{(16\pi G)^2}{\tilde{h}}\left(\tilde{\pi}^{ij}\tilde{\pi}_{ij} - \frac{\tilde{\pi}_h^2}{2}\right)$$
$$+\frac{(\omega+\frac{3}{2})}{\phi^2}\partial_i\phi\partial^i\phi\ + \frac{64(\pi G)^2\phi^2}{h(\omega+\frac{3}{2})}\tilde{\pi}_\phi^2 + \tilde{V}(\phi)\Bigg] - 2\tilde{N^*}^i\tilde{D}_j\tilde{\pi}_i^j + \tilde{N^*}^i\partial_i\phi\tilde{\pi}_\phi\ .$$

- Since this theory is canonically equivalent to B-D, the constraint algebra of secondary first class constraints ($\mathcal{H}$, $\mathcal{H}_i$) is like B-D's one.

Finite Dimensional Example

- We can apply this considerations on a finite dimensional example: FLRW case with k=0

$$ds^2 = -N^2(t) + a^2(t)\left(dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\varphi^2\right).$$

- If we put this metric in the B-D action, we obtain the following finite dimensional Lagrangian

$$\mathcal{L} = -\frac{6a\dot{a}^2}{N(t)}\phi(t) - \frac{6a^2\dot{a}}{N(t)}\dot{\phi}(t) + \frac{\omega a^3}{N\phi(t)}(\dot{\phi}(t))^2 - Na^3U(\phi(t))$$

- The canonical momenta are

$$\begin{aligned}\pi &= \frac{\partial\mathcal{L}}{\partial\dot{N}} \approx 0\,, \pi_a = \frac{\partial\mathcal{L}}{\partial\dot{a}} = -\frac{12a\dot{a}}{N(t)}\phi(t) - \frac{6a^2\dot{a}}{N(t)}\dot{\phi}(t)\,,\\ \pi_\phi &= \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = -\frac{6a^2\dot{a}}{N(t)} + \frac{2\omega a^3}{N\phi(t)}\dot{\phi}(t)\end{aligned}$$

Finite Dimensional Example

- The relative ADM-Hamiltonian results to be

$$\mathcal{H}_{ADM} = N\Bigg(- \frac{\omega \pi_a^2}{12a\phi(2\omega+3)} - \frac{\pi_a \pi_\phi}{2a^2(2\omega+3)} + \frac{\phi \pi_\phi^2}{2a^3(2\omega+3)} + a^3 U(\phi) \Bigg)$$

- Non-Canonical and Canonical Transformations

$$\tilde{N} = N(16\pi G\phi)^{\frac{1}{2}} \ ; \ \tilde{\pi} = \frac{\pi}{(16\pi G\phi)^{\frac{1}{2}}} \ ; \ \tilde{a} = (16\pi G\phi)^{\frac{1}{2}} a \ ;$$

$$\tilde{\pi}_a = \frac{\pi_a}{16\pi G\phi} \ ; \ \phi = \phi \ ; \ \tilde{\pi}_\phi = \frac{1}{\phi}\,(\phi\pi_\phi - \frac{1}{2} a \pi_a)$$

$$\{\tilde{N}, \tilde{\pi}_\phi\} = \frac{8\pi G N}{\sqrt{16\pi G\phi}} \neq 0$$

$$\tilde{N}^* = N \ ; \ \tilde{\pi}^* = \pi \ ; \ \tilde{a} = (16\pi G\phi)^{\frac{1}{2}} a \ ;$$

$$\tilde{\pi}_a = \frac{\pi_a}{16\pi G\phi} \ ; \ \phi = \phi \ ; \ \tilde{\pi}_\phi = \frac{1}{\phi}\,(\phi\pi_\phi - \frac{1}{2} a \pi_a)$$

- The ADM-Hamiltonian in the right canonical transformatiom is

$$\mathcal{H}_{ADM} = \frac{\tilde{a}^3 \tilde{N}^* \phi^{\frac{1}{2}}}{(16\pi G)^{\frac{1}{2}}} \Bigg(- \frac{(16\pi G)^3 \phi \tilde{\pi}_a}{24\tilde{a}^4} + \frac{(16\pi G)^2 \phi^2 \pi_\phi^2}{2\tilde{a}^6(2\omega+3)} + \tilde{V}(\phi) \Bigg).$$

BRANS-DICKE PARTICULAR CASE $\omega = -\frac{3}{2}$

- The B-D action for $\omega = -\frac{3}{2}$ is (for consistency reasons here U(ϕ)=$\alpha\phi^2$ α is a constant)

$$S^{(-3/2)} = \int_M d^4x\sqrt{-g}\left(\phi R + \frac{3}{2}\frac{g^{\mu\nu}}{\phi}\partial_\mu\phi\partial_\nu\phi - \alpha\phi^2\right) + 2\int_{\partial M} d^3x\sqrt{h}\phi K \ .$$

- It is invariant under this conformal transformations

$$\widetilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \quad \widetilde{\phi} = \frac{\phi}{\Omega^2}$$

- The ADM Hamiltonian in this particular case is

$$\mathcal{H}^{(-3/2)}_{ADM} = \sqrt{h}\Bigg\{N\left[-\phi\,{}^3R + \frac{1}{\phi h}\left(\pi^{ij}\pi_{ij} - \frac{{\pi_h}^2}{2}\right)\right]$$
$$- \quad \frac{3N}{2\phi}D_i\phi D^i\phi + N2D^iD_i\phi + NU(\phi)\Bigg\} - 2N^iD_j\pi^j_i + N^iD_i\phi\pi_\phi$$

BRANS-DICKE PARTICULAR CASE $\omega = -\frac{3}{2}$

- Clearly the Hamiltonian and momenta constraints are

$$\mathcal{H}^{(-3/2)} = \sqrt{h}\left\{ \left[-\phi\ {}^3R + \frac{1}{\phi h}\left(\pi^{ij}\pi_{ij} - \frac{{\pi_h}^2}{2}\right)\right] - \quad \frac{3}{2\phi}D_i\phi D^i\phi + 2D^i D_i\phi + U(\phi)\right\}$$

$$\mathcal{H}_i^{(-3/2)} = -2D_j\pi_i^j + D_i\phi\pi_\phi$$

- We also have a further primary constraint due to conformal invariance

$$C_\phi \equiv \pi_h - \phi\pi_\phi \approx 0$$

- All the constraints (shown through lengthy and technically complicated calculations) are first class .

BRANS-DICKE PARTICULAR CASE $\omega = -\frac{3}{2}$

- Momentum and Hamiltonian constraints have the same algebra as Einstein geometro-dynamics.

- The constraint algebra of the primary constraint C_ϕ the momentum and Hamiltonian constraint is

$$\left\{C_\phi(x), \mathcal{H}_i^{(-3/2)}(x')\right\} = -\partial_i\delta(x,x')C_\phi(x')$$

- The Poisson Bracket of the Hamiltonian-Hamiltonian constraint results to be

$$\left\{C_\phi(x), \mathcal{H}^{(-3/2)}(x')\right\} = \frac{1}{2}\mathcal{H}^{(-3/2)}(x)\delta(x,x')$$

- Notice the extra term is generated by the diffeomorphisms for conformal invariance

$$\{\mathcal{H}^{(-3/2)}(x), \mathcal{H}^{(-3/2)}(x')\} = \mathcal{H}_i^{(-3/2)}(x)\partial^i\delta(x,x') - \mathcal{H}_i^{(-3/2)}(x')\partial'^i\delta(x,x')$$
$$+ \left((D^i \log\phi)C_\phi\right)(x)\partial_i\delta(x,x') - \left((D^i \log\phi)C_\phi\right)(x')\partial_i'\delta(x,x')$$

BD PARTICULAR CASE EINSTEIN FRAME

- The B-D action for $\omega = -\frac{3}{2}$ in the Einstein Frame is nothing else but (the potential is now $\widetilde{V}(\phi)$ a constant function)

$$S^{(-3/2)} = \frac{1}{16\pi G}\int_M dx^4\sqrt{-\widetilde{g}}\left({}^4\widetilde{R} - \widetilde{V}(\phi)\right) + \frac{1}{8\pi G}\int_{\partial M} d^3x\sqrt{\widetilde{h}}\widetilde{K}$$

- Notice that the theory is just Einstein GR and is not (Weyl)-Conformal invariant

- The relative ADM Hamiltonian

$$\begin{aligned}\widetilde{\mathcal{H}}^{(-3/2)}_{ADM} &= \frac{\sqrt{\widetilde{h}}\widetilde{N}}{16\pi G}\left[-{}^3\widetilde{R} + \frac{(16\pi G)^2}{\widetilde{h}}\left(\widetilde{\pi}^{ij}\widetilde{\pi}_{ij} - \frac{\widetilde{\pi}_h^2}{2}\right) + \widetilde{V}(\phi)\right] - 2\widetilde{N}^i\widetilde{D}_j\widetilde{\pi}_i^j \\ &= \widetilde{N}\widetilde{\mathcal{H}}^{(-3/2)} + \widetilde{N}^i\widetilde{\mathcal{H}}_i^{(-3/2)}, \qquad (1)\end{aligned}$$

- The Dirac primary constraint $C_\phi \approx 0$ becomes $\widetilde{\pi}_\phi \approx 0$. The other Dirac's constraints are the same as Einstein's GR.

BD PARTICULAR CASE IN EINSTEIN FRAME

- The constraint algebra among momentum and Hamiltonian constraints are like Einstein geometro-dynamics

- The remaining constraint algebra is

$$\left\{\widetilde{C}_\phi(x), \widetilde{\mathcal{H}}_i^{(-3/2)}(x')\right\} = \left\{\widetilde{C}_\phi(x), \widetilde{\mathcal{H}}^{(-3/2)}(x')\right\} = 0$$

$$\{\widetilde{\mathcal{H}}^{(-3/2)}(x), \widetilde{\mathcal{H}}^{(-3/2)}(x')\} = \widetilde{\mathcal{H}}^{(-3/2)i}(x)\partial_i\delta(x, x') - \widetilde{\mathcal{H}}^{(-3/2)i}(x')\partial'_i\delta(x, x')$$

- Very clearly the algebra of the Dirac's constraints of the BD theory in the Einstein Frame is completely different respect to the Jordan frame. Therefore, we continue to remark, the transformation between the two frames results not to be canonical.

Hamiltonian Analysis of BD for $\omega = -\frac{3}{2}$	
in Jordan Frame	**in Einstein Frame**
constraints $\pi \approx 0; \pi^i \approx 0; C_\phi \approx 0; \mathcal{H}^{(-3/2)} \approx 0; \mathcal{H}_i^{(-3/2)} \approx 0;$	*constraints* $\widetilde{\pi} \approx 0; \widetilde{\pi}_i \approx 0; \widetilde{C}_\phi = \widetilde{\pi}_\phi \approx 0; \widetilde{\mathcal{H}}^{(-3/2)} \approx 0; \widetilde{\mathcal{H}}_i^{(-3/2)} \approx 0;$
constraint algebra $\{\pi, \pi_i\} = \{\pi, \mathcal{H}^{(-3/2)}\} = \{\pi, \mathcal{H}_i^{(-3/2)}\} = 0;$ $\{\pi_i, \mathcal{H}^{(-3/2)}\} = \{\pi_i, \mathcal{H}_j^{(-3/2)}\} = 0;$ $\left\{C_\phi(x), \mathcal{H}_i^{(-3/2)}(x')\right\} = -\partial_i \delta(x, x') C_\phi(x');$ $\left\{C_\phi(x), \mathcal{H}^{(-3/2)}(x')\right\} = \frac{1}{2}\mathcal{H}^{(-3/2)}(x)\delta(x, x');$ $\left\{\mathcal{H}^{(-3/2)}(x), \mathcal{H}_i^{(-3/2)}(x')\right\} = -\mathcal{H}^{(-3/2)}(x')\partial_i'\delta(x, x');$ $\left\{\mathcal{H}_i^{(-3/2)}(x), \mathcal{H}_j^{(-3/2)}(x')\right\} = \mathcal{H}_i^{(-3/2)}(x')\partial_j\delta(x, x')$ $-\mathcal{H}_j^{(-3/2)}(x)\partial_i{}'\delta(x, x');$ $\{\mathcal{H}^{(-3/2)}(x), \mathcal{H}^{(-3/2)}(x')\} =$ $\mathcal{H}_i^{(-3/2)}(x)\partial^i\delta(x, x') - \mathcal{H}_i^{(-3/2)}(x')\partial'^i\delta(x, x') +$ $\left((D^i \log\phi)C_\phi\right)(x)\partial_i\delta(x, x')$ $-\left((D^i \log\phi)C_\phi\right)(x')\partial_i'\delta(x, x');$	*constraint algebra* $\{\widetilde{\pi}, \widetilde{\pi}_i\} = \{\widetilde{\pi}, \widetilde{\mathcal{H}}^{(-3/2)}\} = 0; \{\widetilde{\pi}, \widetilde{\mathcal{H}}_i^{(-3/2)}\} = 0;$ $\{\widetilde{\pi}_i, \widetilde{\mathcal{H}}^{(-3/2)}\} = \{\widetilde{\pi}_i, \widetilde{\mathcal{H}}_j^{(-3/2)}\} = 0;$ $\left\{\widetilde{C}_\phi(x), \widetilde{\mathcal{H}}_i^{(-3/2)}(x')\right\} = 0;$ $\left\{\widetilde{C}_\phi(x), \widetilde{\mathcal{H}}^{(-3/2)}(x')\right\} = 0;$ $\left\{\widetilde{\mathcal{H}}^{(-3/2)}(x), \widetilde{\mathcal{H}}_i^{(-3/2)}(x')\right\} = -\widetilde{\mathcal{H}}^{(-3/2)}(x')\partial_i'\delta(x, x');$ $\{\widetilde{\mathcal{H}}_i^{(-3/2)}(x), \widetilde{\mathcal{H}}_j^{(-3/2)}(x')\} = \widetilde{\mathcal{H}}_i^{(-3/2)}(x')\partial_j\delta(x, x')$ $-\widetilde{\mathcal{H}}_i^{(-3/2)}(x)\partial_i{}'\delta(x, x');$ $\{\widetilde{\mathcal{H}}^{(-3/2)}(x), \widetilde{\mathcal{H}}^{(-3/2)}(x')\} =$ $\widetilde{\mathcal{H}}_i^{(-3/2)}(x)\partial^i\delta(x, x') - \widetilde{\mathcal{H}}_i^{(-3/2)}(x')\partial_i'\delta(x, x');$

Conclusions

- We have analyzed the Hamiltonian formalism of Brans-Dicke theory in the Jordan Frame with GHY boundary term both in the general case $\omega \neq -\frac{3}{2}$, and $\omega = -\frac{3}{2}$

- We have shown the Weyl (conformal) transformations, in the Hamiltonian formalism, from Jordan to Einstein frame are not canonical transformations. For $\omega \neq -\frac{3}{2}$, a set of true canonical transformations have been found (Anti-Gravity or Anti-Newtonian transformations). In the case $\omega = -\frac{3}{2}$, the inequivalence is straightforward by confronting and contrasting the Dirac's constraint algebra in the two frames

- Some studies have already singled out the inequivalence between Jordan and Einstein Frames at Quantum Level (Benedetti & Gualtieri (2014), Falls and Herrero Valea(2019))