The Relevance of Irrelevant perturbations

Genova
10 Marzo 2021

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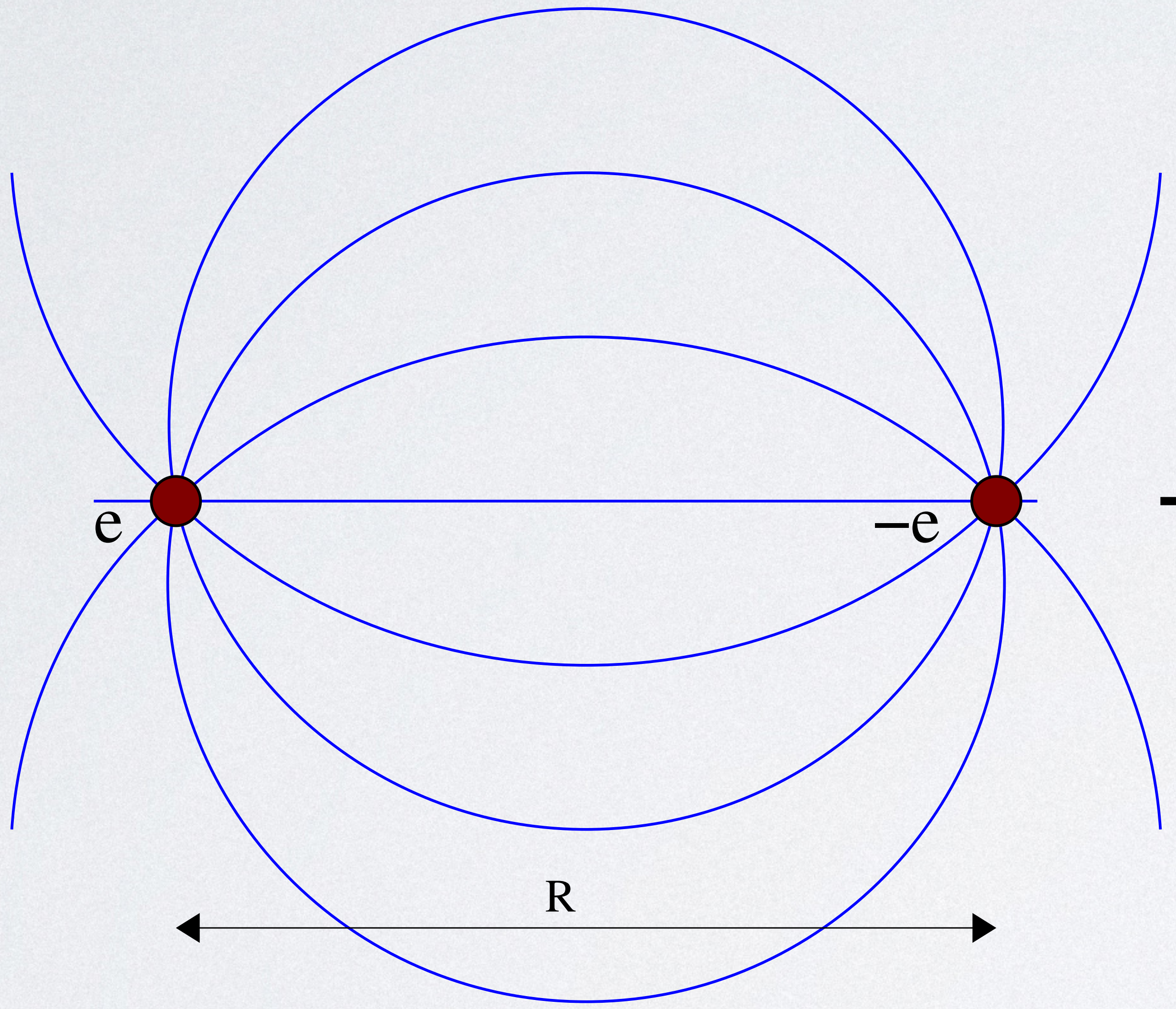
Effective string theory and the quark-antiquark potential

The present understanding of QCD is mainly based on the idea that the confining regime of Yang-Mills theories is described by some kind of effective string model.

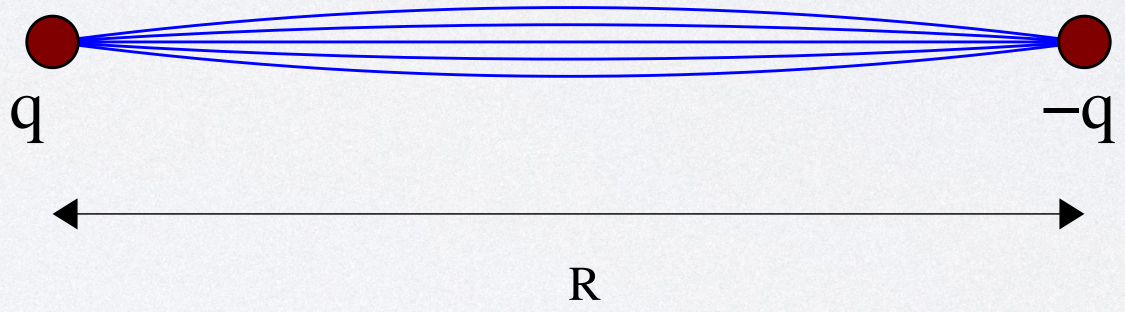
In 4D we may naively guess a hybrid UV/IR inter-quark potential of Cornell's form:



$V(R) = -a/R$

(Coulomb potential)

The chromo-electric flux lines are confined in a thin flux tube.



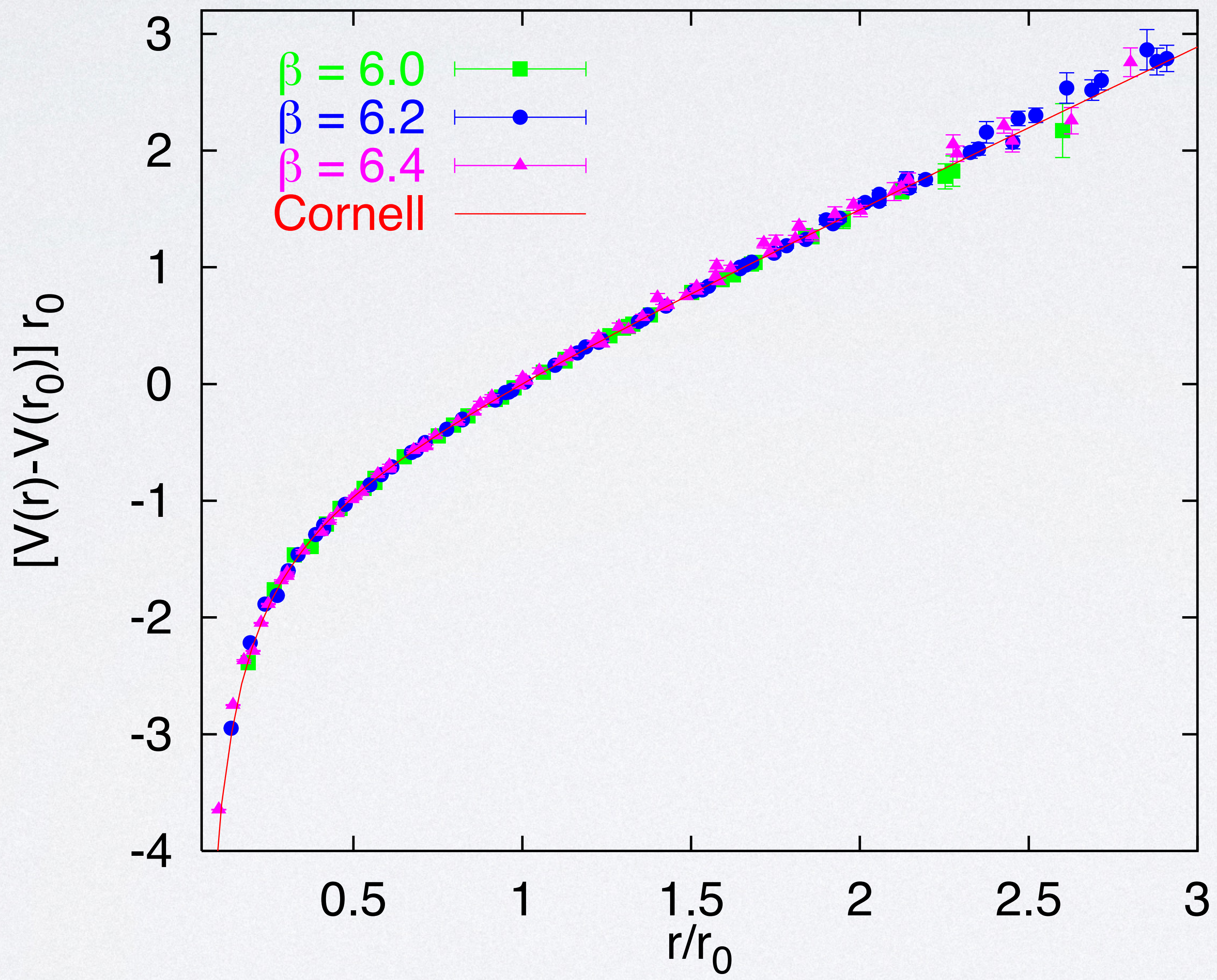
$V(R) = \sigma R - k/R$

(Cornell potential)

where σ is the string tension.

Using lattice regularisation of pure gauge theories one can easily study non perturbative phenomena: the inter-quark potential rises linearly, and the chromo-electric flux lines are indeed confined in a thin string.

Indeed, the Cornell's potential fits very well the numerical results from pure Yang-Mills lattice simulations:



[G.S. Bali, “QCD forces and heavy quark bound states,” hep-ph/0001312]

(The gauge group is $SU(3)$, $\beta = 6/g_{YM}$, and g_{YM} is the Yang-Mills coupling constant.)

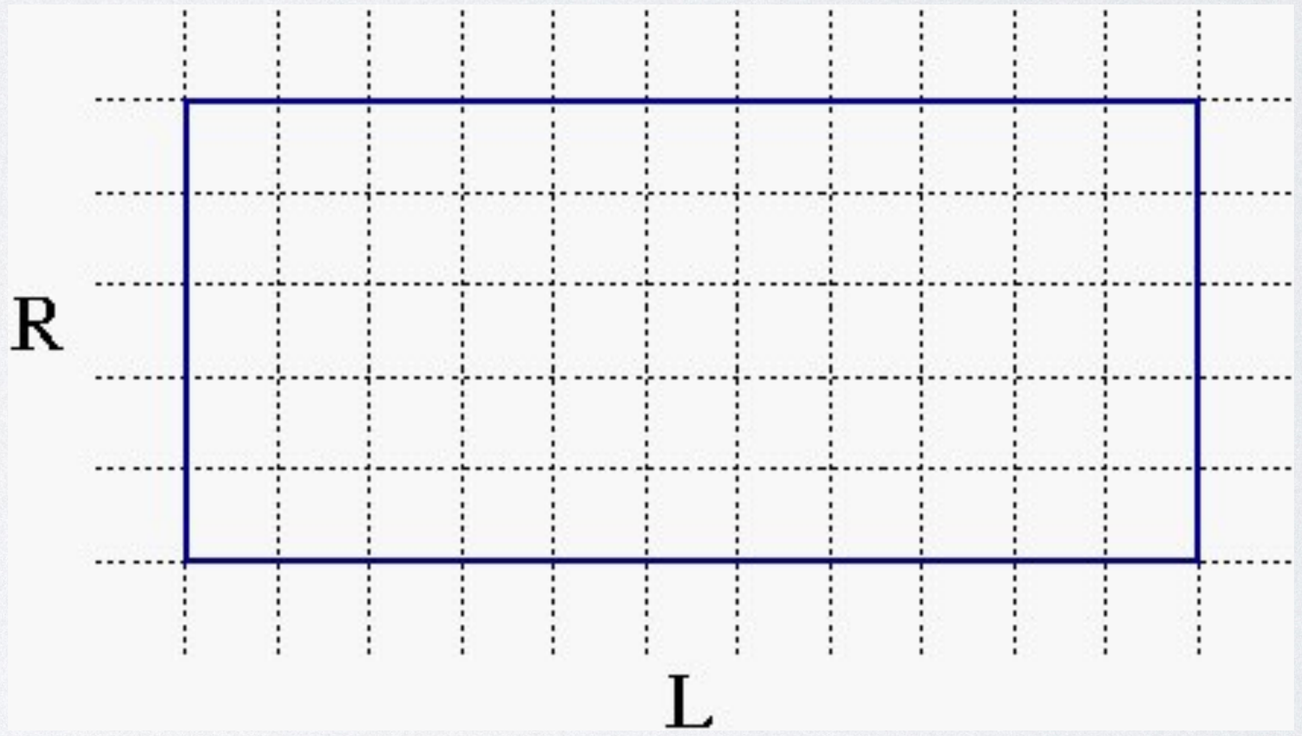
However, a similarly good numerical agreement is obtained in D=3 where the Coulomb potential is logarithmic. In addition, the coefficient **k** depends only on the dimensionality D of the space-time...

$$V(R) = -\lim_{L\to\infty} \frac{1}{L} \log W(R,L) = \sigma R - \frac{(D-2)\pi}{24R} + \dots$$

[**Luscher** et al.]

where

$$W(R,L) = \mathrm{Tr} \prod_{n_\mu \in \gamma_{(R,L)}} e^{A_\mu(n)}$$



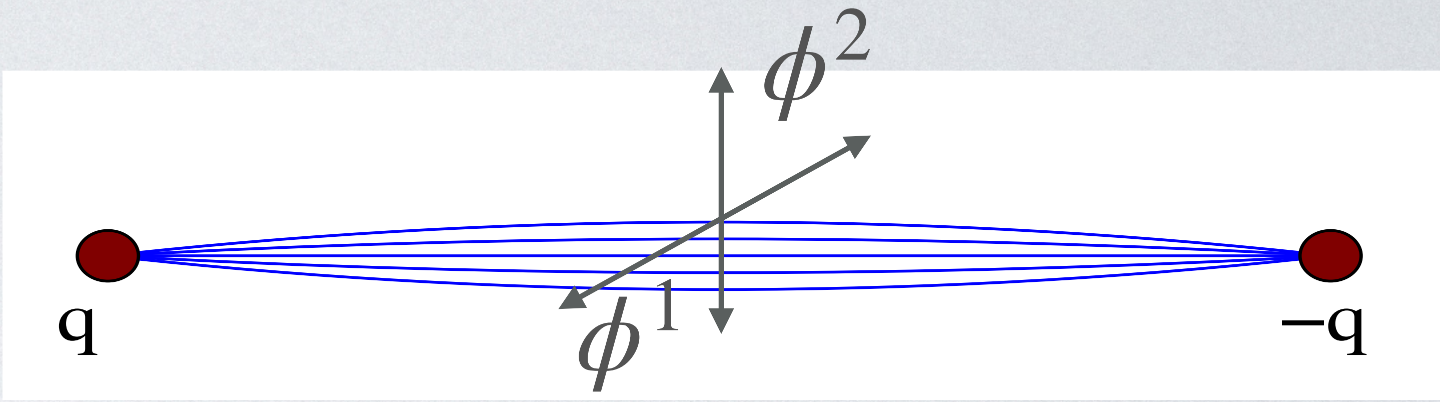
Euclidean time t: ⟶

W is the simplest gauge-invariant observable, the so-called ``Wilson loop''

W ~ partition function of D-2 massless boson fields in two dimensions.

The corresponding action is that of D-2 free massless boson fields ϕ^i, $(i = 1,\ldots,D-2)$:

$$\mathcal{A} = \mathcal{A}_{cl} + \mathcal{A}_0 \qquad \mathcal{A}_0 = \frac{\sigma}{2}\int d^2x \left(\partial_\alpha \vec{\phi} \cdot \partial^\alpha \vec{\phi}\right)$$



But there are observed deviations, and the first possible corrections are

$$\mathcal{A} = \underline{\mathcal{A}_{cl}} + \mathcal{A}_0 + \sigma\int d^2x \left[c_2(\partial_\alpha \vec{\phi} \cdot \partial^\alpha \vec{\phi})^2 + c_3(\partial_\alpha \vec{\phi} \cdot \partial^\beta \vec{\phi})(\partial_\beta \vec{\phi} \cdot \partial^\alpha \vec{\phi})\right] + \mathcal{A}_b + \ldots ,$$

(For the Cornell potential) (Boundary)

where S_b is the boundary action characterising the open effective string.

The full Lorentz invariance of the D-dimensional target space, should be still respected nonlinearly by the expanded action. This gives

$$c_2 = \frac{1}{8}, \; c_3 = -\frac{1}{4},$$

with this extra constraint we have

$$\mathscr{A} = \mathscr{A}_{cl} + \mathscr{A}_0 - \frac{1}{2\pi^2\sigma}\int d^2x \, T\bar{T} + \mathscr{A}_b + \ldots$$

where

$$T\bar{T} = -\pi \, det[T_{\mu\nu}]$$

is the Zamolodchikov $T\bar{T}$ irrelevant composite field.

$T\bar{T}$ Lagrangian and Hamiltonian flow equations:

$$\partial_\tau \mathscr{L}(\tau) = \det[T^{\mu\nu}(\tau)]\,, \qquad T_{\mu\nu}(\tau) = \frac{-2}{\sqrt{g}}\frac{\partial \mathscr{L}(\tau)}{\partial g^{\mu\nu}}\,, \qquad \partial_\tau \mathscr{H}(\tau) = \det[T^{\mu\nu}(\tau)]$$

(Euclidean space-time)

Massless boson field theories

$$\mathscr{L}(0) = \partial\vec{\phi}\cdot\bar{\partial}\vec{\phi}$$

(In complex coordinates: $\partial = \partial_z\,, \bar{\partial} = \partial_{\bar{z}}$)

$(z = x^1 + ix^2, \bar{z} = x^1 - ix^2)$

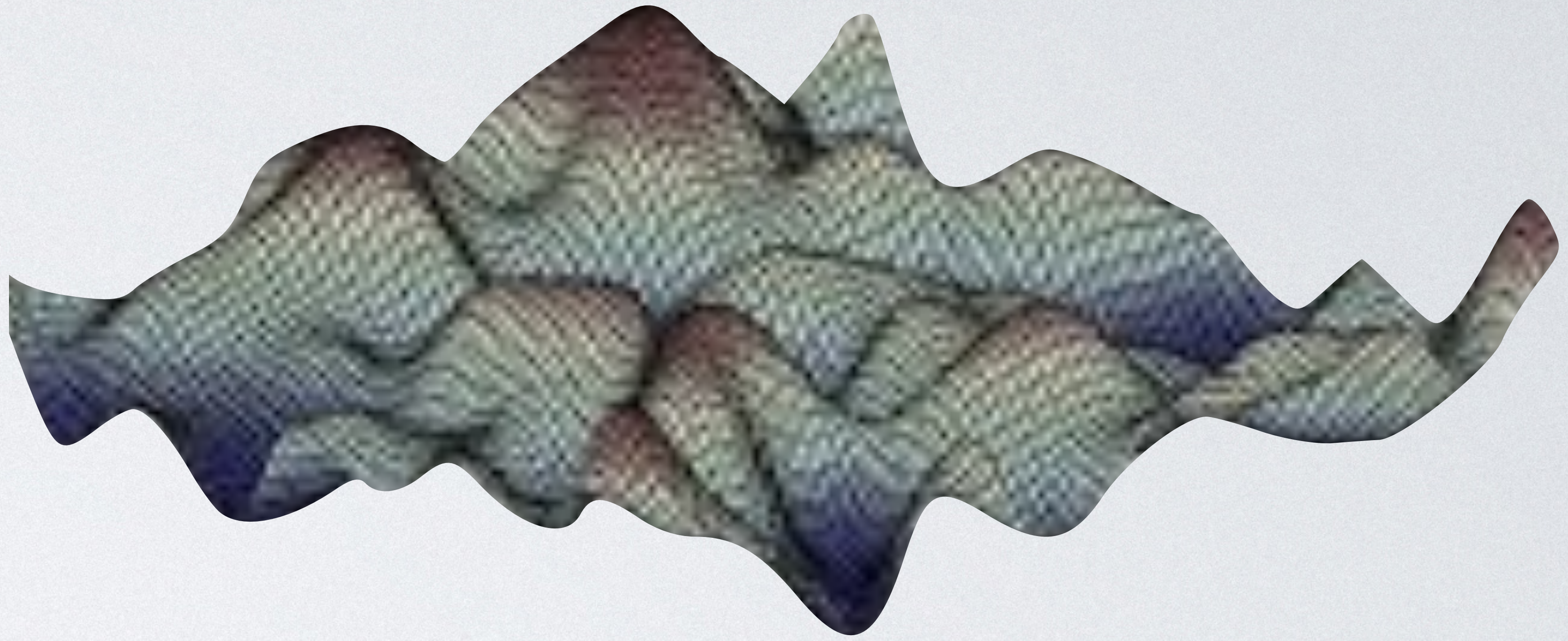
↓

$$\mathscr{L}(\tau) = \frac{1}{2\tau}\left(-1 + \sqrt{1 + 4\tau\partial\vec{\phi}\cdot\bar{\partial}\vec{\phi} - 4\tau^2\,\mathscr{B}}\right) = -\frac{1}{2\tau} + \mathscr{L}^{static}_{NG}(\tau)$$

$$\mathscr{B} = \sum_{i=1}^{N}\left(\partial\phi_i\right)^2\sum_{j=1}^{N}\left(\bar{\partial}\phi_j\right)^2 - \left(\sum_{i=1}^{N}\partial\phi_i\bar{\partial}\phi_i\right)^2$$

The Nambu-Goto model

$$\mathscr{A} = \int dA = \int d^2x \mathscr{L}_{NG} = \frac{1}{2\tau} \int \sqrt{\det\left(\sum_{\mu=1}^{D} \partial_\alpha X^\mu(x^1, x^2)\, \partial_\beta X^\mu(x^1, x^2) \right)}\, d^2x$$



in the static gauge

$$X^1 \to x^1\,, \quad X^2 \to x^2\,, \quad X^i \to \tau^{\frac{1}{2}}\, \phi^{i-2}\,, \quad (i = 3, \ldots D)$$

we have

$$\mathscr{L}_{NG} \to \mathscr{L}_{NG}^{static}$$

Boson field theories with generic potential

$$\mathscr{L}^V(0) = \mathscr{L}(0) - V \qquad \mathscr{L}(0) = \partial\overrightarrow{\phi} \cdot \bar{\partial}\overrightarrow{\phi}\,,\; V = V(\overrightarrow{\phi})$$

[Conti-Negro-Iannella-RT — Bonelli-Doroud-Zhu 2018]

$$\mathscr{L}^V(\tau) = \frac{-V}{1+\tau V} + \frac{1}{2\bar{\tau}}\left(-1 + \sqrt{1 + 4\bar{\tau}\mathscr{L}(0) - 4\bar{\tau}^2\,\mathscr{B}}\right)$$

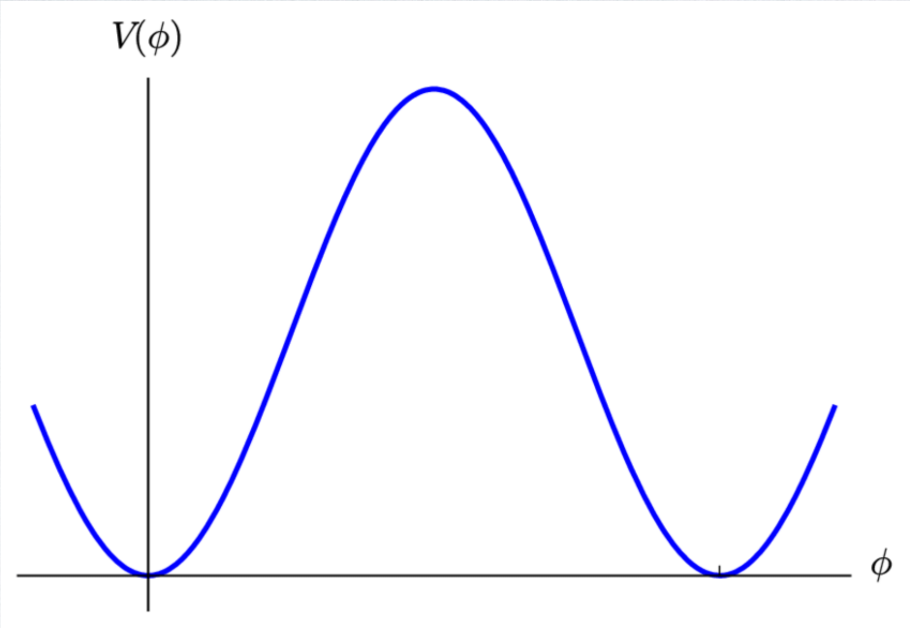
with

$$\bar{\tau} = \tau\,(1 + \tau V)$$

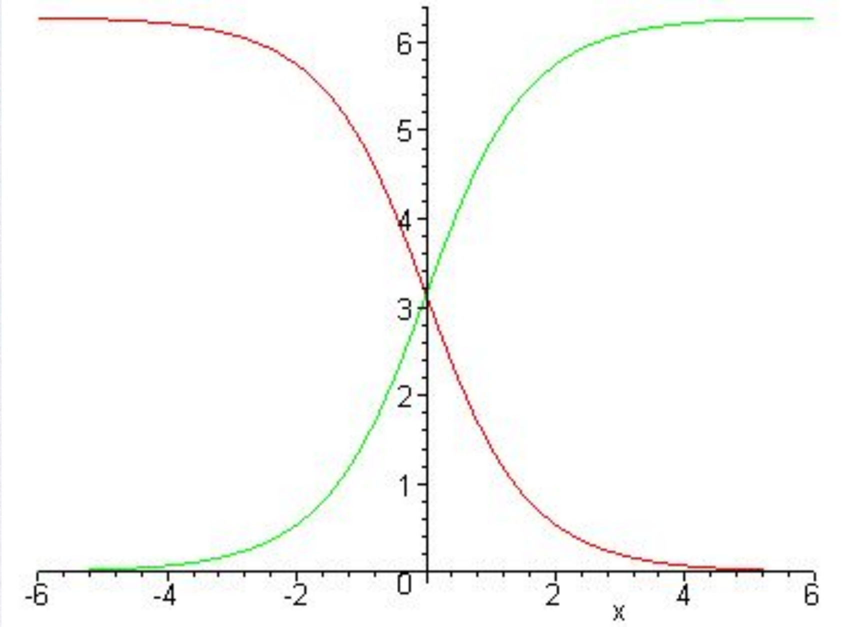
The sine-Gordon model

$$\mathscr{L}_{sG}(\phi,\tau) = \frac{-V}{1+\tau V} + \frac{-1+S(\phi)}{2\tau(1+\tau V)} \qquad S(\phi) = \sqrt{1+4\tau\,(1+\tau V)\,\partial\phi\bar{\partial}\phi}$$

with



$$V = 2\frac{m^2}{\beta^2}\left(1-\cos(\beta\phi)\right)$$



and EoM

$$\partial\left(\frac{\bar{\partial}\phi}{S}\right) + \bar{\partial}\left(\frac{\partial\phi}{S}\right) = -\frac{V'}{4S}\left(\frac{S+1}{1+\tau V}\right)^2 \qquad V' = 2\frac{m^2}{\beta}\sin(\beta\phi)$$

A local change of coordinates

$$\mathcal{J}^{-1} = \begin{pmatrix} \partial_w z & \partial_w \bar{z} \\ \partial_{\bar{w}} z & \partial_{\bar{w}} \bar{z} \end{pmatrix} = \begin{pmatrix} 1+\tau V & -\tau \left(\frac{\partial \phi}{\partial w}\right)^2 \\ -\tau \left(\frac{\partial \phi}{\partial \bar{w}}\right)^2 & 1+\tau V \end{pmatrix}$$

$(z = x^1 + ix^2, \bar{z} = x^1 - ix^2)$

$(w = y^1 + iy^2, \bar{w} = y^1 - iy^2)$

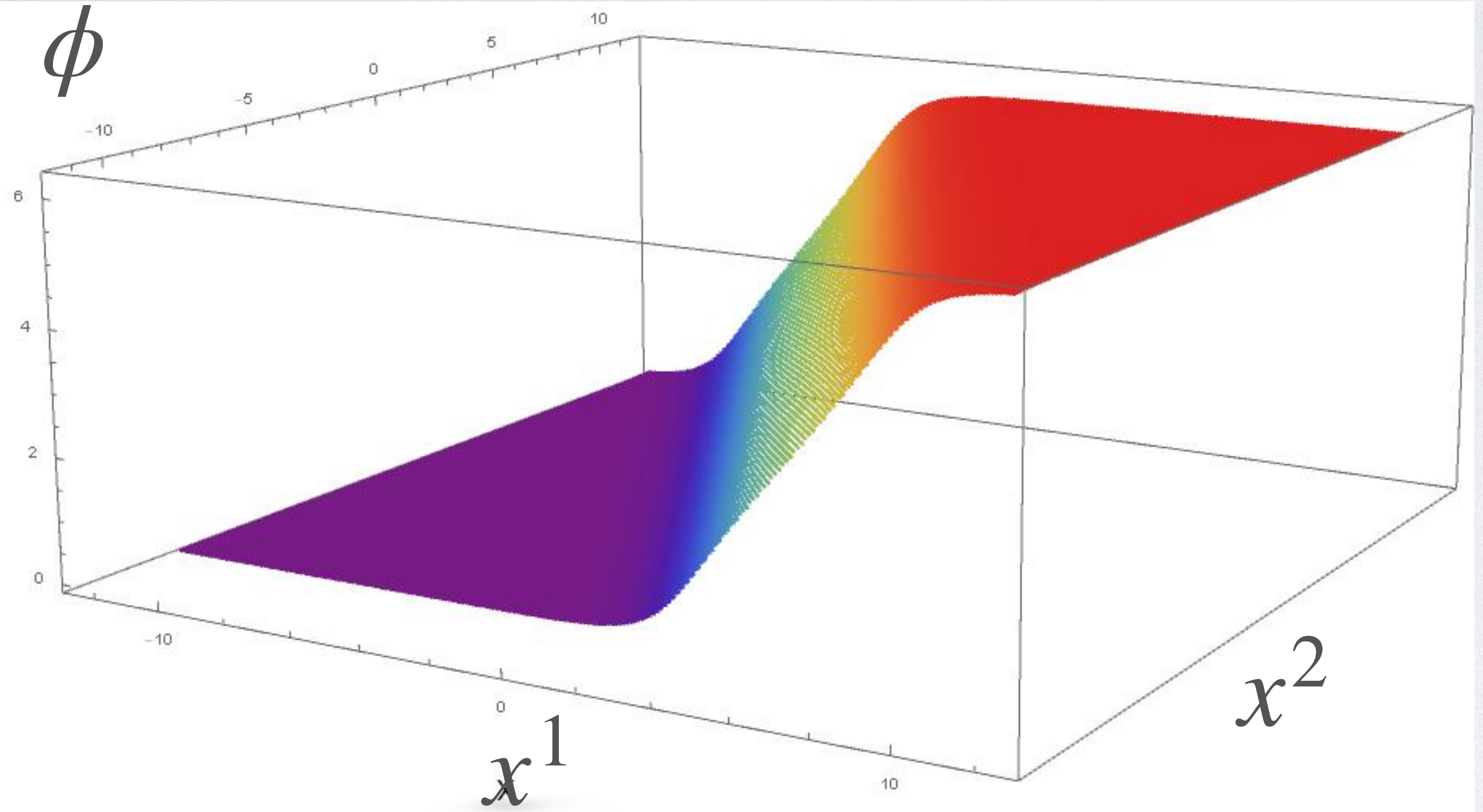
↓

$$\phi^{(\tau)}(\mathbf{z}) = \phi^{(0)}\left(\mathbf{w}(\mathbf{z})\right) \ , \quad \mathbf{z} = (z, \bar{z})\, , \quad \mathbf{w} = (w, \bar{w})$$

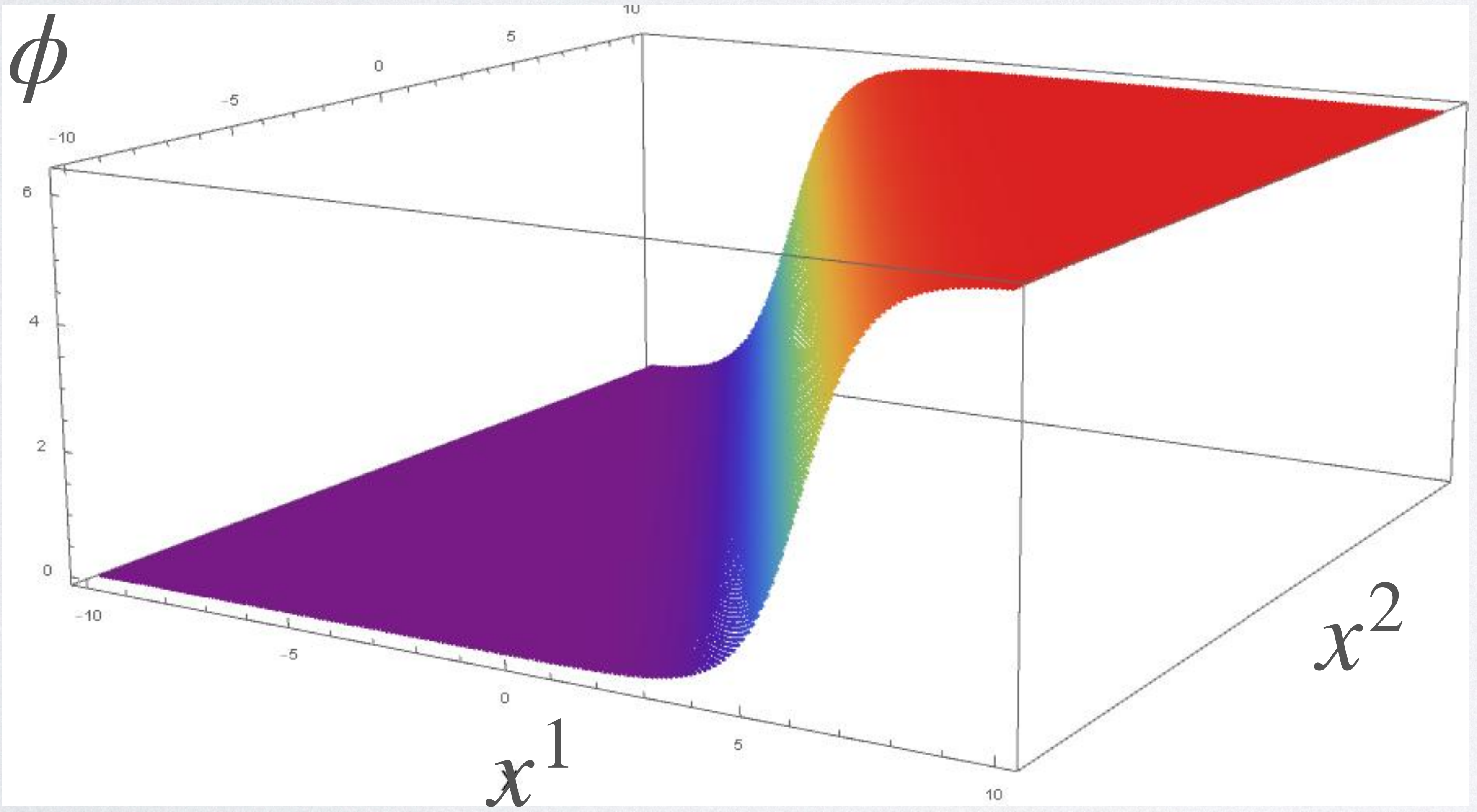
$$\partial \left(\frac{\bar{\partial}\phi}{S}\right) + \bar{\partial}\left(\frac{\partial \phi}{S}\right) = -\frac{V'}{4S}\left(\frac{S+1}{1+\tau V}\right)^2 \quad \Longrightarrow \quad 2\partial_w \partial_{\bar{w}} \phi = -V'$$

The deformed kink solution

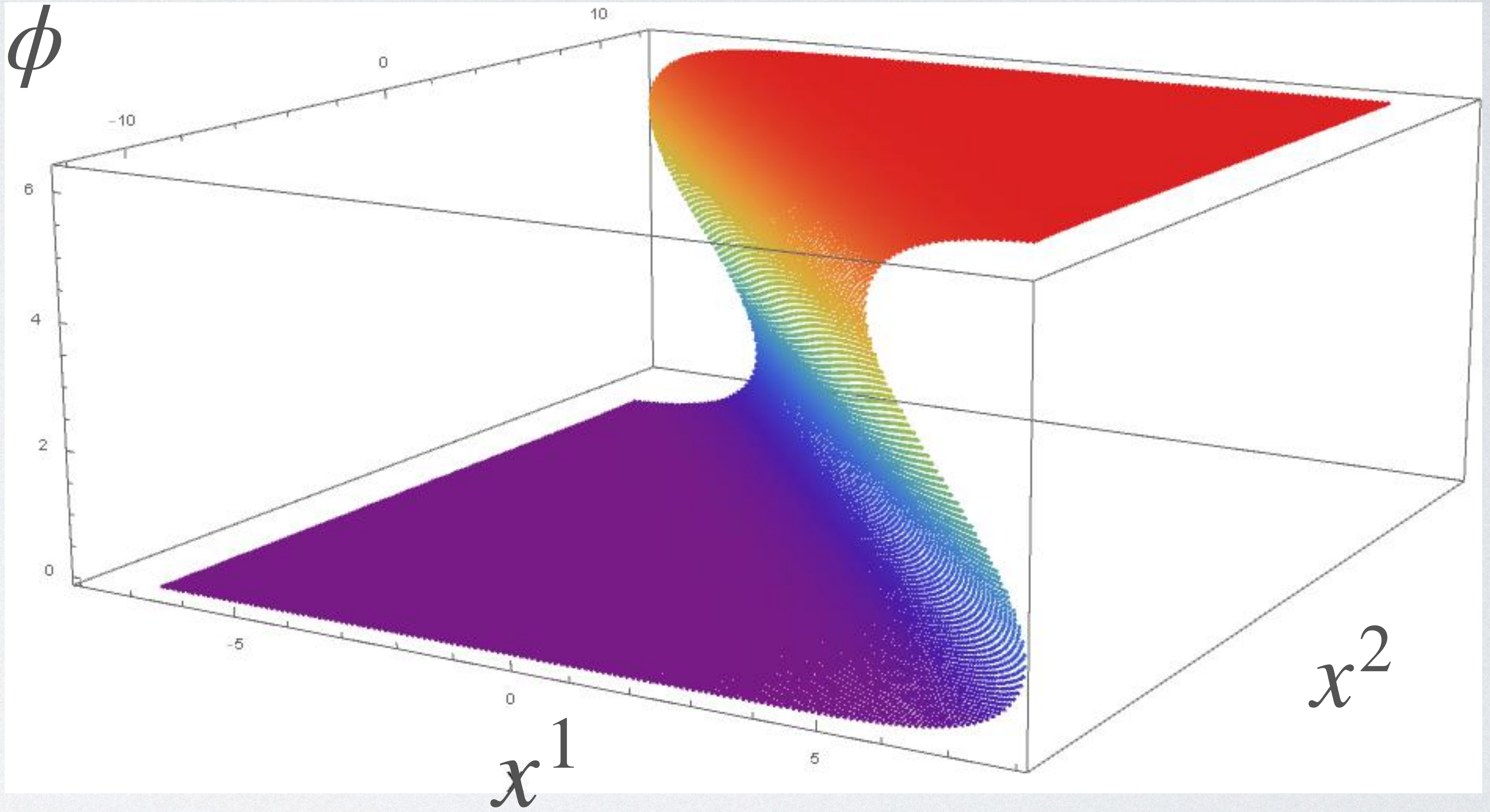
$$\phi^{(0)}_{\text{1-kink}}(\mathbf{w}) = 4\arctan\left(e^{\frac{m}{\beta}\left(aw+\frac{1}{a}\bar{w}\right)}\right)\ ,\ a=\sqrt{\frac{1-v}{1+v}}$$



$\tau < 0$

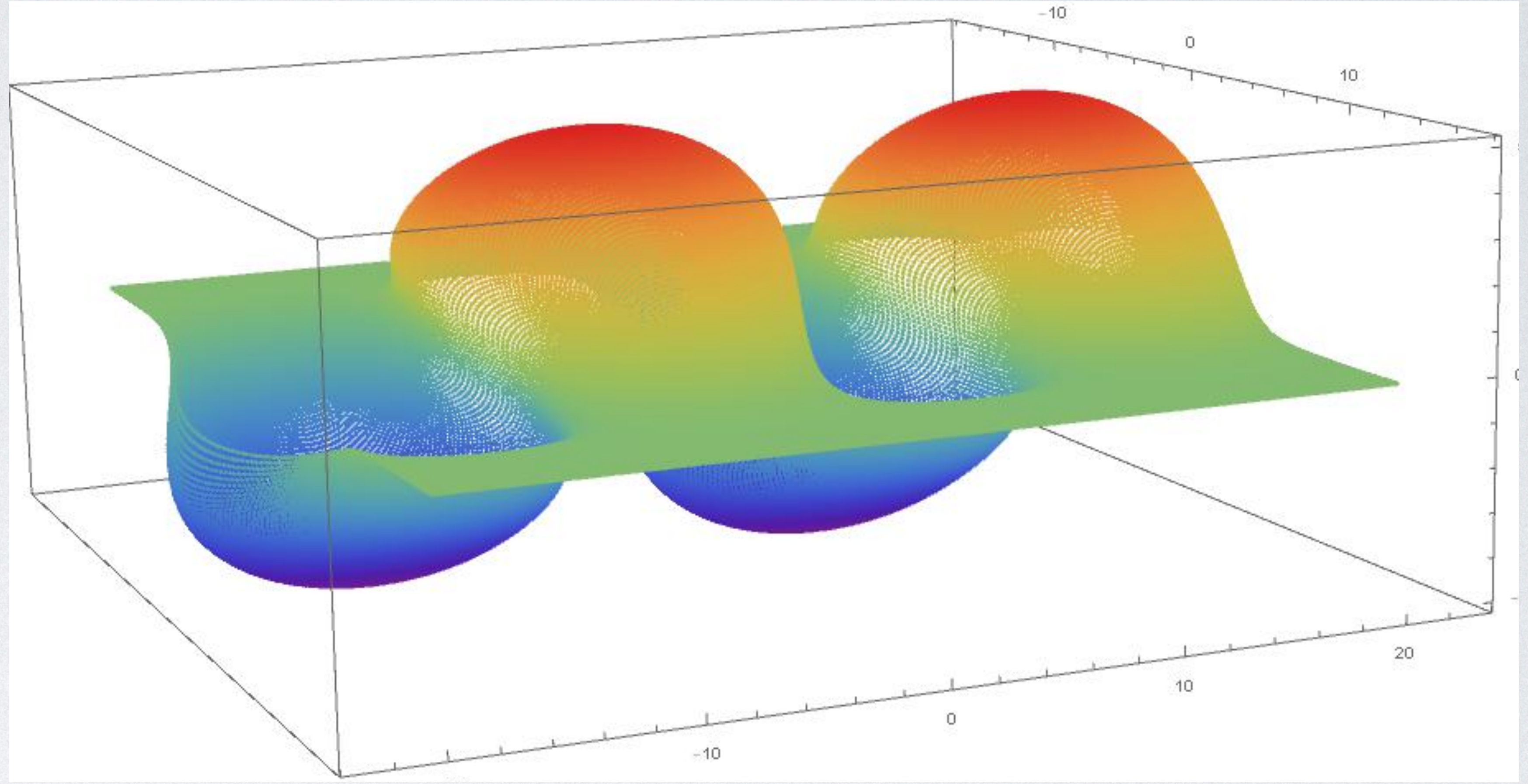


$\tau = 0$

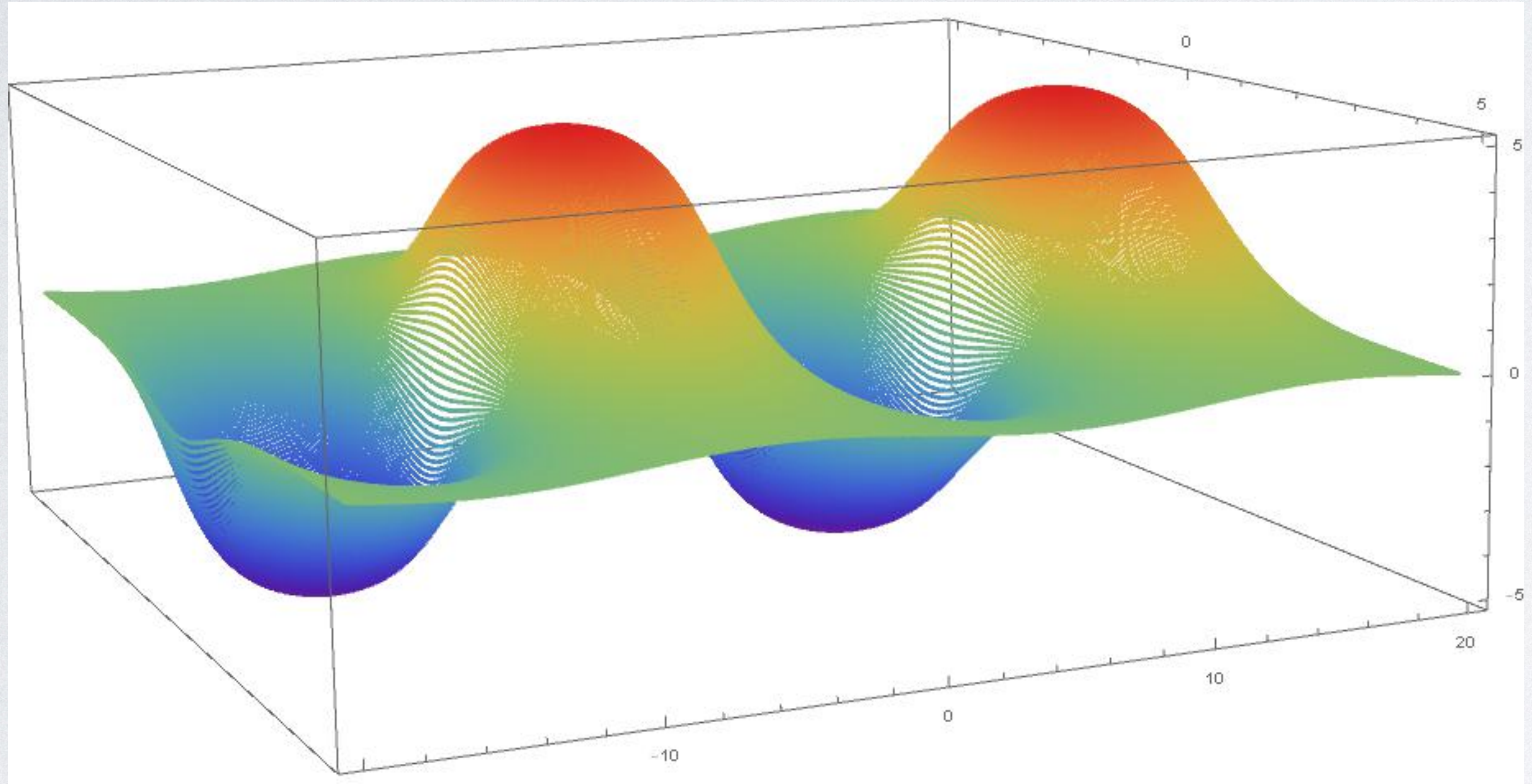


$\tau > 0$

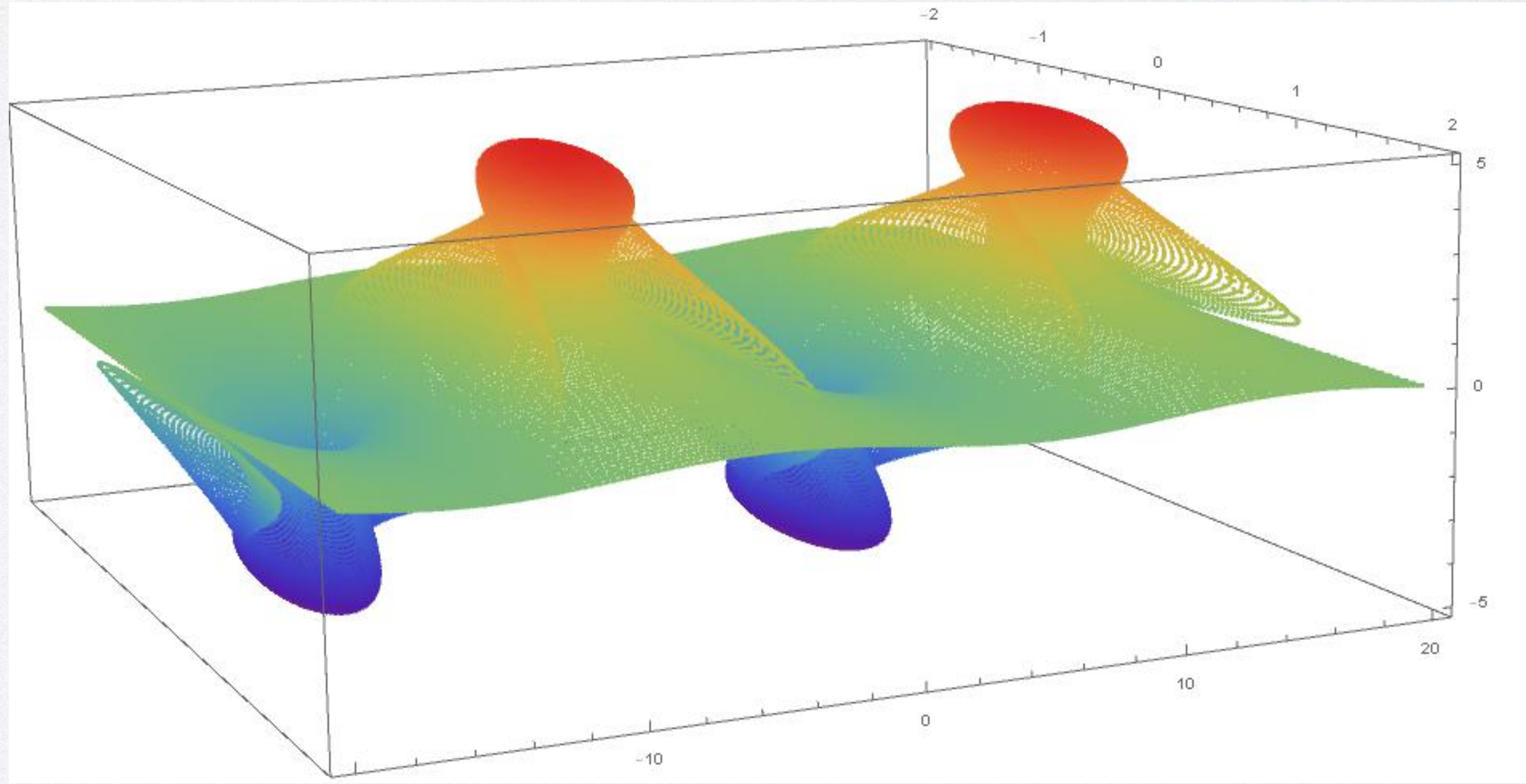
The deformed sine-Gordon breather



$\tau < 0$



$\tau = 0$



$\tau > 0$

Figure and caption from [2010.15733 [hep-th]] by J. Cardy and B. Doyon

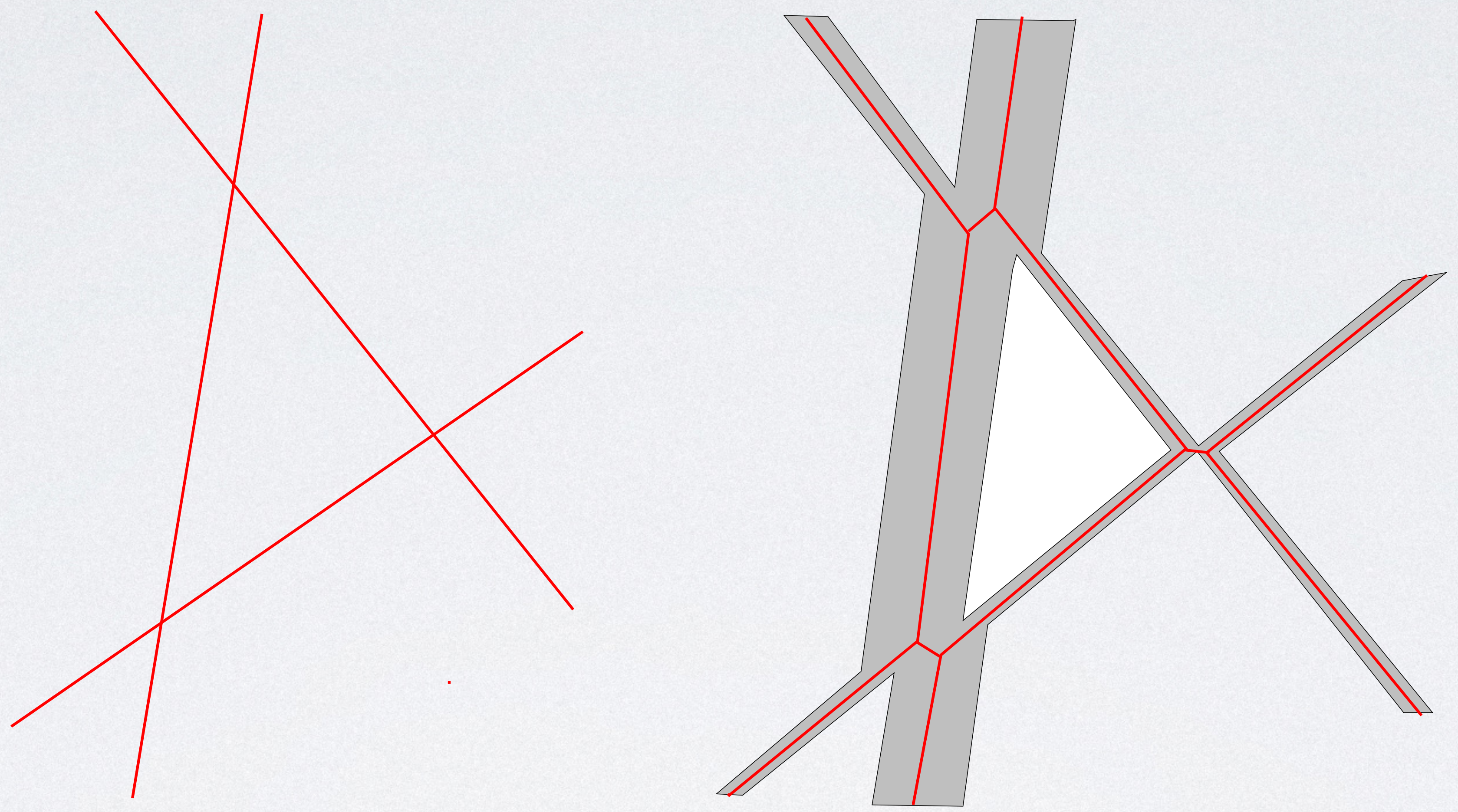


Figure 2: The effect of the λ_{R}-deformation on relativistic scattering processes. The particles gain a width in a consistent fashion in space-time, as if "grout" were added between tiles.

Generic $T\bar{T}$-deformed models

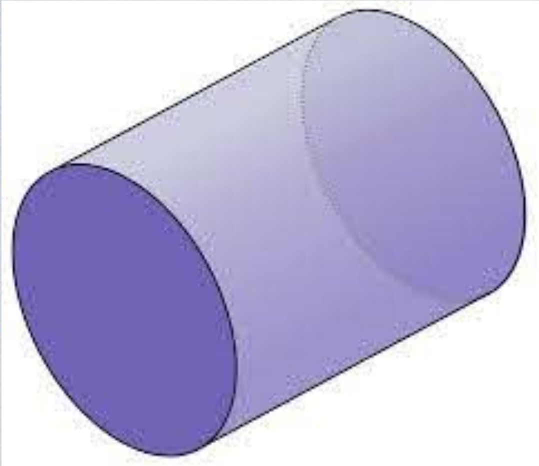
$$\mathcal{J}^{-1} = \begin{pmatrix} \partial_w z & \partial_w \bar{z} \\ \partial_{\bar{w}} z & \partial_{\bar{w}} \bar{z} \end{pmatrix} = \begin{pmatrix} 1 + \tau V & -\tau \left(\frac{\partial \phi}{\partial w}\right)^2 \\ -\tau \left(\frac{\partial \phi}{\partial \bar{w}}\right)^2 & 1 + \tau V \end{pmatrix} \Longleftrightarrow \begin{pmatrix} 1 - \tau \Theta(\mathbf{w}) & -\tau \bar{T}(\mathbf{w}) \\ -\tau T(\mathbf{w}) & 1 - \tau \Theta(\mathbf{w}) \end{pmatrix}$$

Notice that

$$\boxed{\frac{\partial^2 x^\mu}{\partial y^\rho \partial y^\sigma} = \frac{\partial^2 x^\mu}{\partial y^\sigma \partial y^\rho} \quad \Longleftrightarrow \quad \partial_\mu \mathbf{T}^\mu{}_\nu = 0}$$

$$\mathcal{A}[\phi] = \int dz\, d\bar{z}\, \mathcal{L}^{(\tau)}(\mathbf{z}) = \int dw\, d\bar{w}\, \left|\det\left(\mathcal{J}^{-1}\right)\right|\, \mathcal{L}^{(\tau)}(\mathbf{z}(\mathbf{w}))$$
$$= \int dw\, d\bar{w}\, \left(\mathcal{L}^{(0)}(\mathbf{w}) + \tau\, \mathrm{T}\bar{\mathrm{T}}^{(0)}(\mathbf{w})\right)$$

$$g_{\mu\nu} = (J^T J)_{\mu\nu}$$



Quantum $T\bar{T}$-deformations on infinite cylinder of circumference R

$$\partial_\tau \mathscr{H}(\tau) = \det[T_{\mu\nu}(\tau)] \rightarrow \partial_\tau \langle n | \mathscr{H}(\tau) | n \rangle = \langle n | \det[T_{\mu\nu}(\tau)] | n \rangle$$

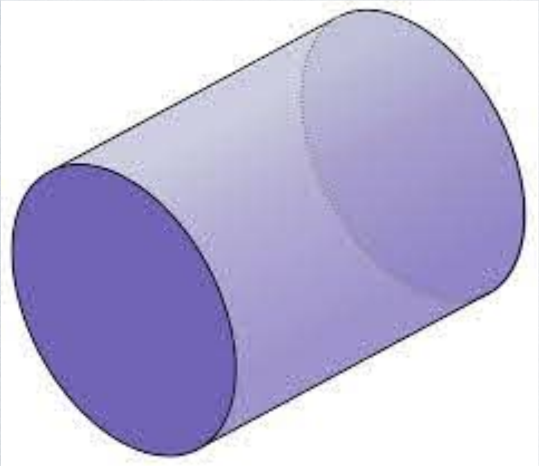
(Exact!)

$$\langle n | \det[T_{\mu\nu}(\tau)] | n \rangle = \langle n | T_{11} | n \rangle \langle n | T_{22} | n \rangle - \langle n | T_{12} | n \rangle \langle n | T_{21} | n \rangle$$

[Zamolodchikov 2004]

$$E_n(R,\tau) = -R \langle n | T_{22} | n \rangle \ , \ \partial_R E_n(R,\tau) = - \langle n | T_{11} | n \rangle \ , \ P_n(R) = -\mathbb{i} R \langle n | T_{12} | n \rangle$$

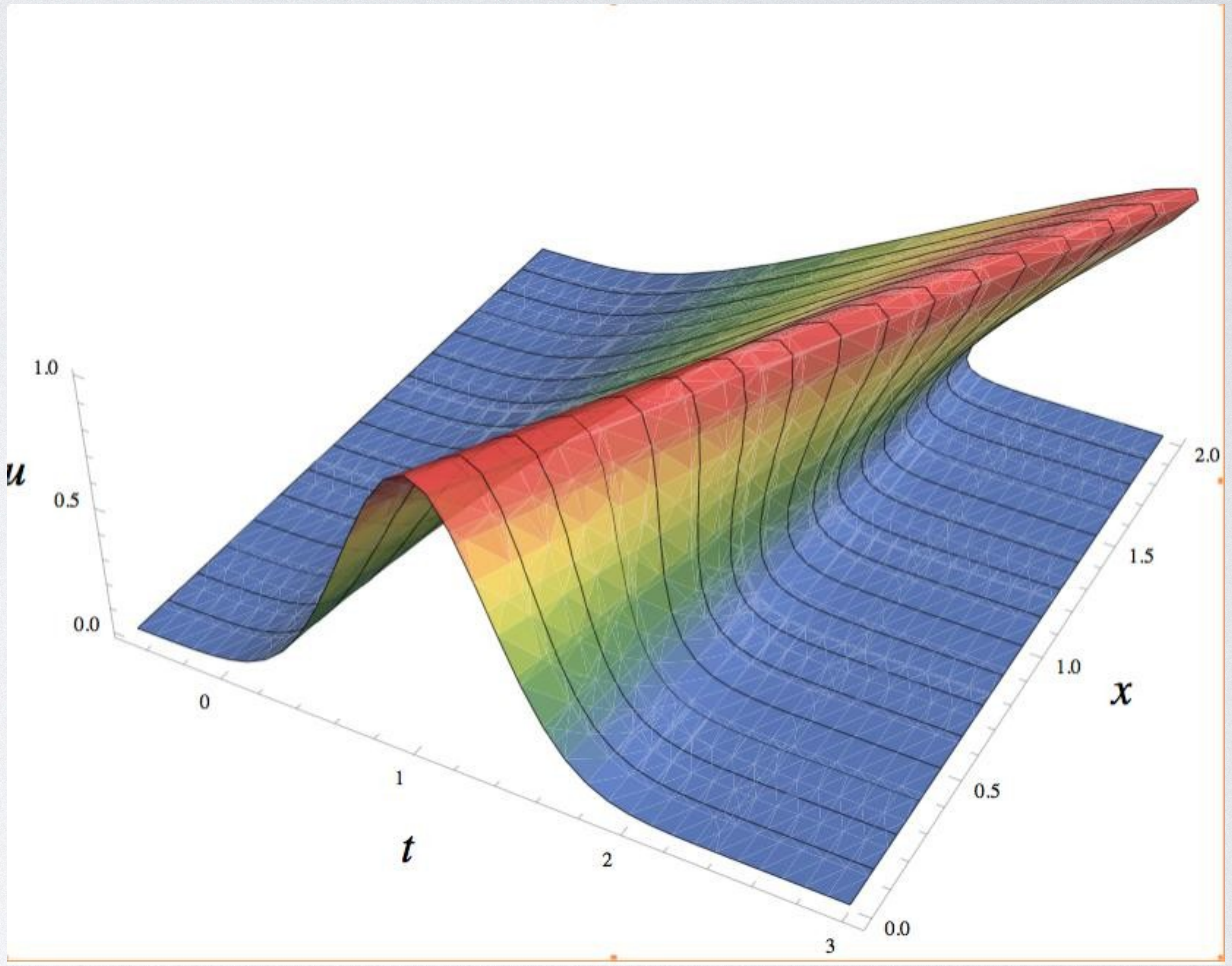
$$P(R,\tau) = P(R) = \frac{2\pi k}{R}, \quad k \in \mathbb{Z}.$$



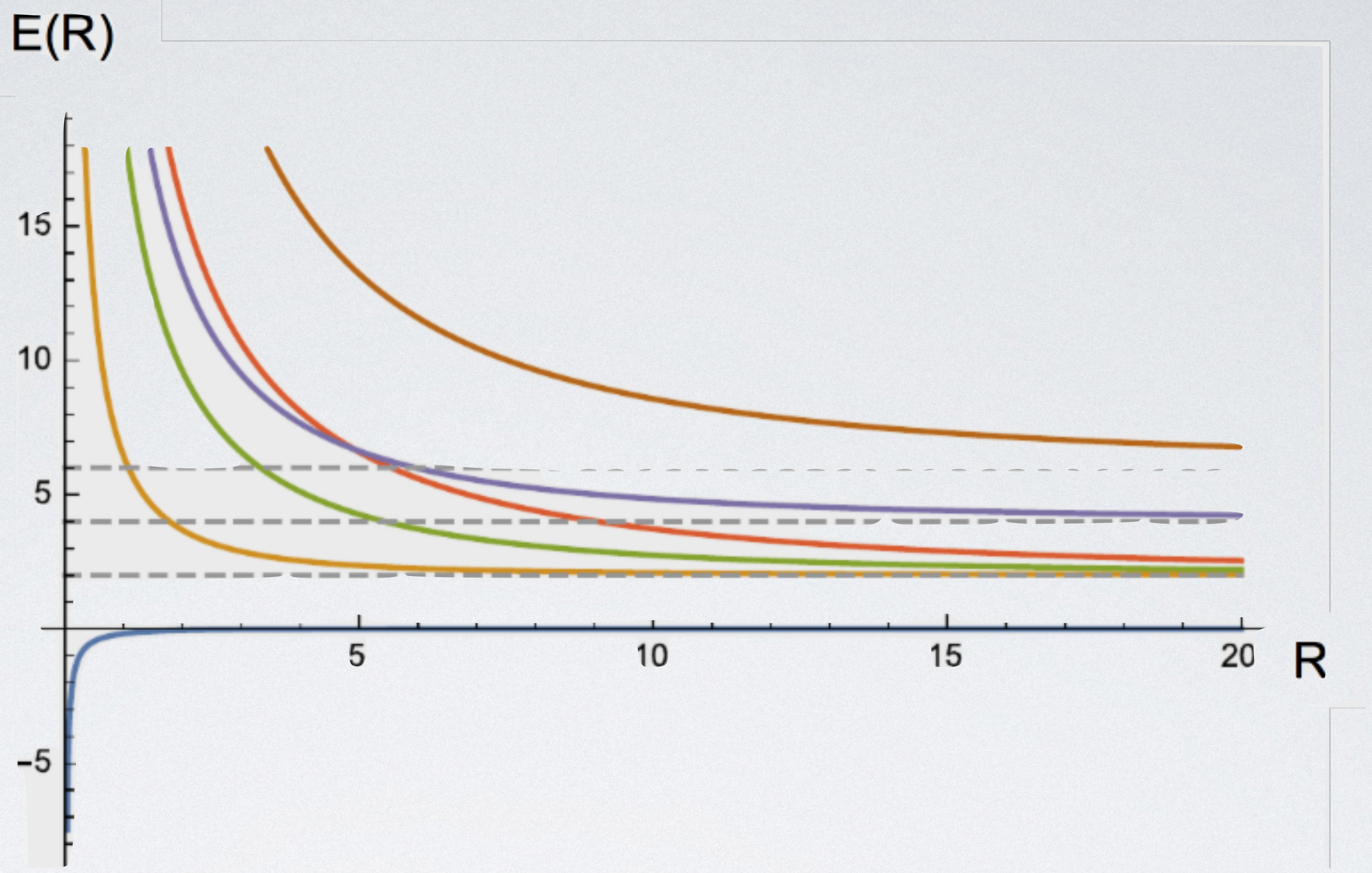
The inviscid Burgers equation for the quantum spectrum

$$\partial_\tau E_n(R,\tau) = E_n(R,\tau)\partial_R E_n(R,\tau) + \frac{P_n^2(R)}{R}$$

source term



$$P_n = 0 \rightarrow E_n(R,\tau) = E_n(R + \tau E_n(R,\tau),0)$$

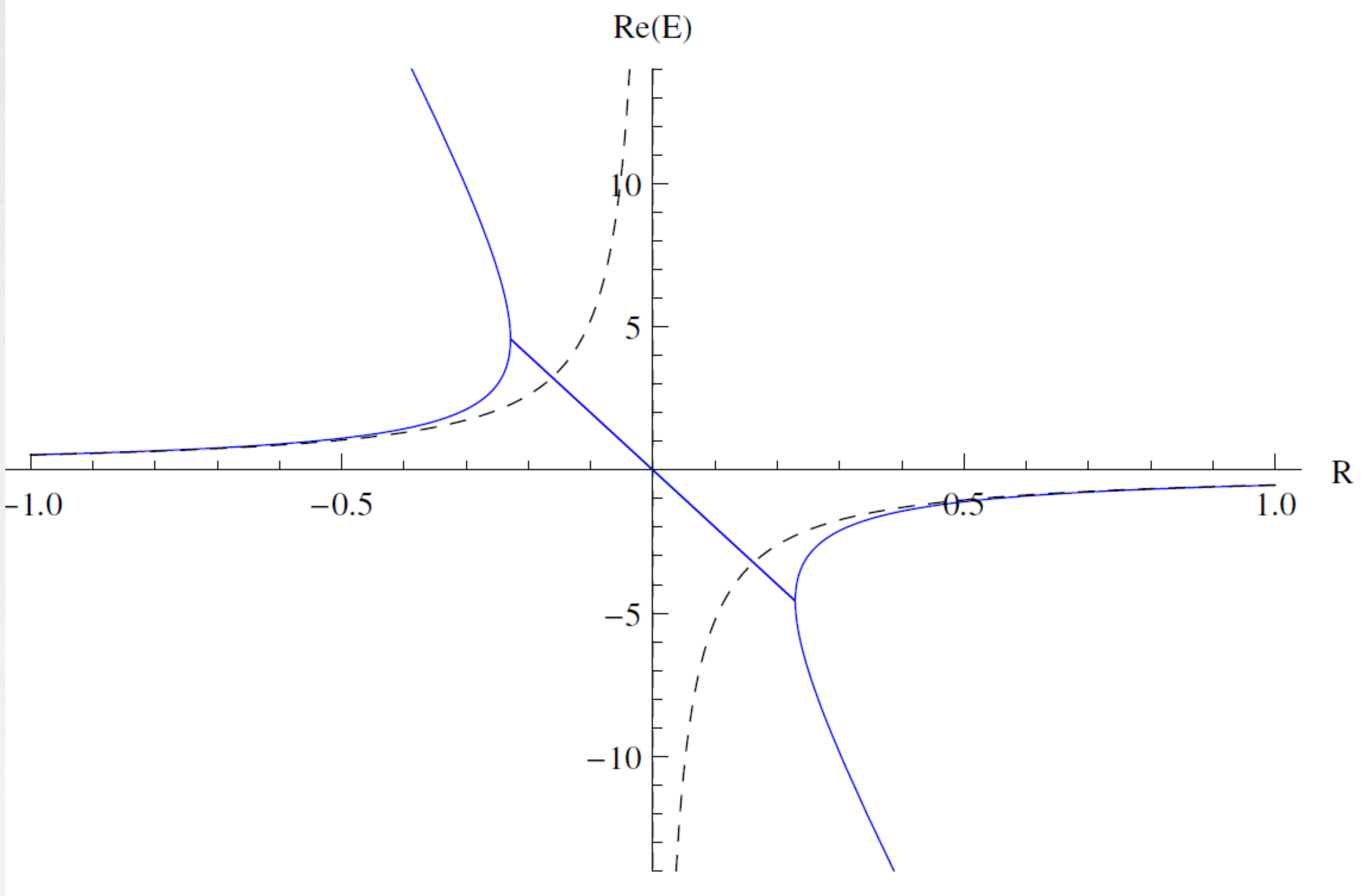


(Typical $\tau = 0$ finite-volume spectrum)

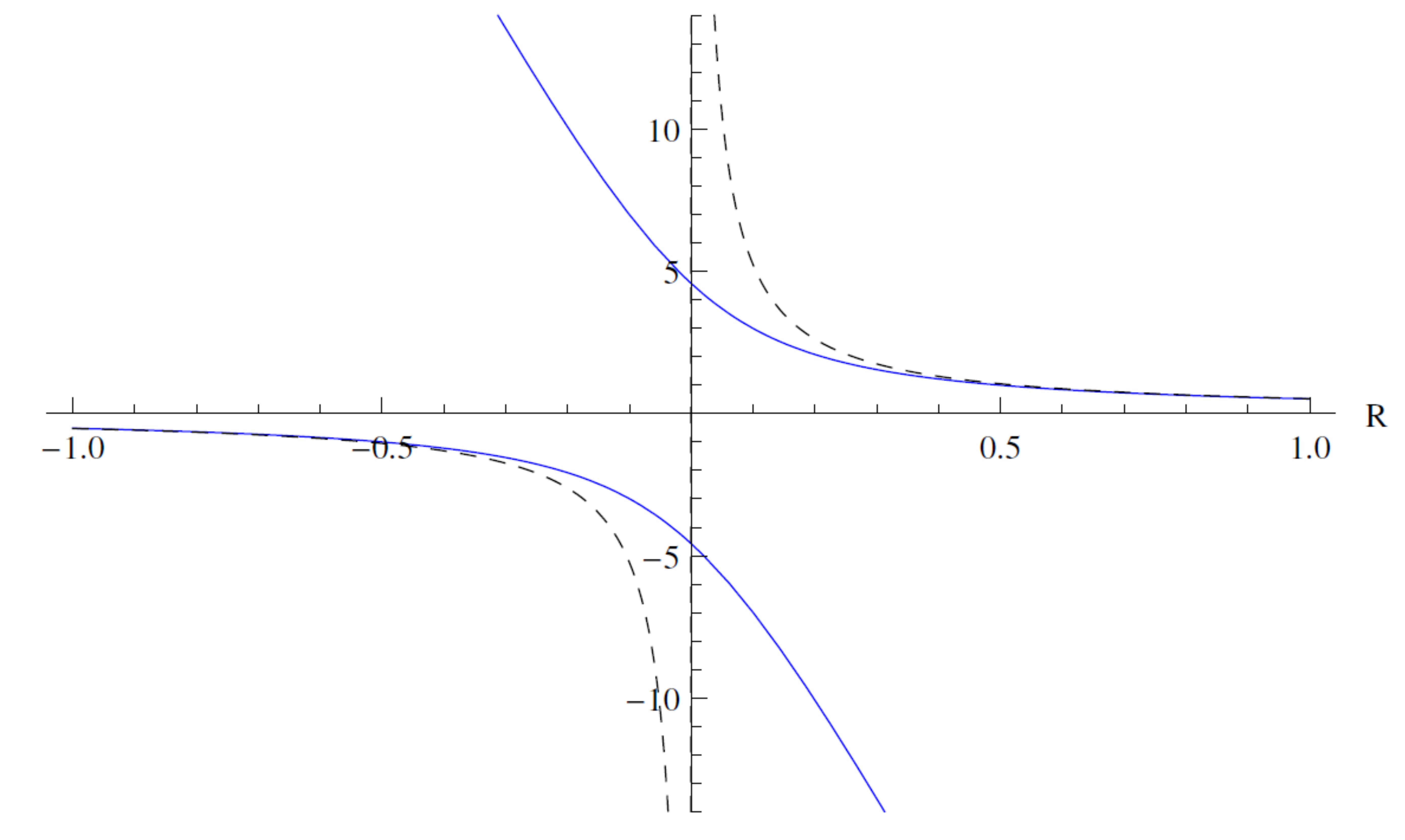
$$E(R,0) \sim -\pi \frac{c_{\text{eff}}}{6\,R}, \quad R \sim 0,$$

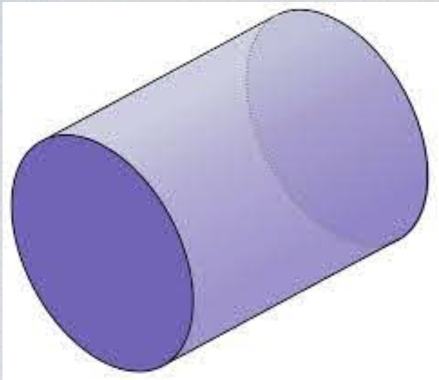
where, $c_{eff} = c - 24\Delta$ is the "effective central charge" of the UV CFT state.

For $c_{eff} > 0$ (i.e. the ground-state energy) we have a "wave-breaking" phenomena



For $c_{eff} < 0$ (i.e. generic excited state) the branch points move off, along the imaginary axis





The CFT case

$$E^{(+)}(R,\tau) = 2\pi\left(\frac{n_0 - c_{\rm eff}/24}{R + 2\tau E^{(-)}(R,\tau)}\right), \qquad E^{(-)}(R,\tau) = 2\pi\left(\frac{\bar{n}_0 - c_{\rm eff}/24}{R + 2\tau E^{(+)}(R,\tau)}\right)$$

$c_{eff} = c - 25\Delta$(primary), obtained by an energy-dependent shift:

$$R \to R + 2\tau E^{(\pm)}(R,\tau)$$

Dubovsky-Flauger-Gorbenko 2012

Caselle-Gliozzi-Fioravanti-Tateo 2013

The total energy:

$$\begin{aligned} E(R,\tau) &= E^{(+)}(R,\tau) + E^{(-)}(R,\tau) \\ &= -\frac{R}{2\tau} + \sqrt{\frac{R^2}{4\tau^2} + \frac{2\pi}{\tau}\left(n_0 + \bar{n}_0 - \frac{c_{\rm eff}}{12}\right) + \left(\frac{2\pi(n_0 - \bar{n}_0)}{R}\right)^2} \end{aligned}$$

which matches the form of the (D=26, $c_{eff} = 24$) Nambu-Goto spectrum, for a generic CFT.

D=4: Born-Infeld nonlinear electrodynamics

To circumvent the problem of divergent fields in classical electrodynamics, Born and Infeld followed a non-relativistic/relativistic analogy:

$$\mathscr{L}_{non-rel} = m\frac{v^2}{2} \Longrightarrow \mathscr{L}_{rel} = -mc^2\left(1-\sqrt{1-\frac{v^2}{c^2}}\right)$$

and proposed

$$\mathscr{L}_M = \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right) \Longrightarrow \mathscr{L}_{BI} = \beta^2\left(1-\sqrt{1-\frac{1}{\beta^2}\left(\vec{E}^2-\vec{B}^2\right)-\frac{1}{\beta^4}\left(\vec{E}\cdot\vec{B}\right)^2}\right)$$

$$\vec{E}_M = \frac{q}{4\pi r^2}\hat{r} \Longrightarrow \vec{E}_{BI} = \frac{\frac{q}{4\pi r^2}}{\sqrt{1+\left(\frac{q}{4\pi r^2\beta}\right)^2}}\hat{r}$$

$$L_{BI} = \frac{-\sqrt{g} + \sqrt{\det(g_{\mu\nu} + \sqrt{2\tau} F_{\mu\nu})}}{2\tau}$$

The BI Lagrangian fulfils the flow equation

$$\partial_\tau L = \frac{\sqrt{g}}{4} \left(\frac{1}{2} g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma} \right) T^{\mu\nu} T^{\rho\sigma}$$

[Conti-Jannella-Negro-Romano-RT]

where we can set

$$\widetilde{T}_{\rho\sigma} = f_{\mu\rho\nu\sigma} T^{\mu\nu}$$

$$f_{\mu\nu\rho\sigma} = \frac{1}{2} g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}$$

and write

$$\partial_\tau L = \frac{\sqrt{g}}{4} T^{\mu\nu} \widetilde{T}_{\mu\nu}$$

The infinitesimally deformed actions

$$\mathscr{A}' = \int d^4x \, L(g_{\mu\nu}, x, \tau + \delta\tau) = \int d^4x \left(L(g_{\mu\nu}, x, \tau) + \delta\tau \partial_\tau L \right) = \int d^4x \left(L(g_{\mu\nu}, x, \tau) + \frac{\sqrt{g}\delta\tau}{4} T^{\mu\nu} \widetilde{T}_{\mu\nu} \right)$$

and

$$\mathscr{A} = \int d^4x \left(L(x, g_{\mu\nu}, \tau) + \delta\tau \sqrt{g} \left(-\frac{1}{4} f^{\mu\nu\rho\sigma} h_{\mu\rho} h_{\nu\sigma} - \frac{1}{2} h_{\mu\nu} T^{\mu\nu} \right) \right)$$

Lead to the same (infinitesimally-deformed) equations of motion for the fields. For **h**, the EoM are

$$h_{\mu\nu} = -\widetilde{T}_{\mu\nu} = -f_{\mu\rho\nu\sigma} T^{\rho\sigma} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\rho_\rho \, .$$

$$\mathscr{A} = \int d^4x \left(L(x, g_{\mu\nu} + \delta\tau\, h_{\mu\nu}, \tau) + \delta\tau\sqrt{g}\left(-\frac{1}{4} f^{\mu\nu\rho\sigma} h_{\mu\rho} h_{\nu\sigma} \right) \right)$$

and the perturbation has been moved from the parameter to the metric!

The flow equation for the metric is:

$$\partial_{\tau_1} g_{\mu\nu} = -\, h_{\mu\nu} = \widetilde{T}_{\mu\nu}(g, x, \tau_1)$$

The infinitesimal change in the Ricci tensor is:

$$\delta R_{\mu\nu} = -\frac{1}{2} \Box\, T_{\mu\nu}$$

Therefore, the metric deformation does not come from a change of coordinates.

To solve the flow equation for $\mathbf{g}$ we can use a perturbative method: in general there is no truncation!

However, for BI: $$g_{\mu\nu} = \delta_{\mu\nu} + (\tau_1 - \tau_0)\widetilde{T}_{\mu\nu}(\delta, \tau_0, x)$$

as a consequence to the degeneracy (in pairs) of the eigenvalues of $T_{\mu\nu}$.

In conclusion, if we set $\tau_0 = 0$ and $\tau_1 = \tau$ we have:

$$g'_{\mu\nu} = \delta_{\mu\nu} - \tau(T_M)_{\mu\nu}$$

and the BI theory with this metric has the same equations of motion of Maxwell theory in flat space.

Vice versa we can set $\tau_0 = \tau$ and $\tau_1 = 0$ and find: $$g_{\mu\nu} = \delta_{\mu\nu} - \tau(\widetilde{T}_{BI})_{\mu\nu}$$

in this case we have that the EoMs of the BI in flat space are equal to the Maxwell ones in this metric.

Thank you for your attention!