Scattering Amplitude

Role of Intersection Theory

Manoj Kumar Mandal

INFN & University of Padova

Fellini General Meeting
4th March, 2021

H2020 MSCA COFUND
G.A. 754496

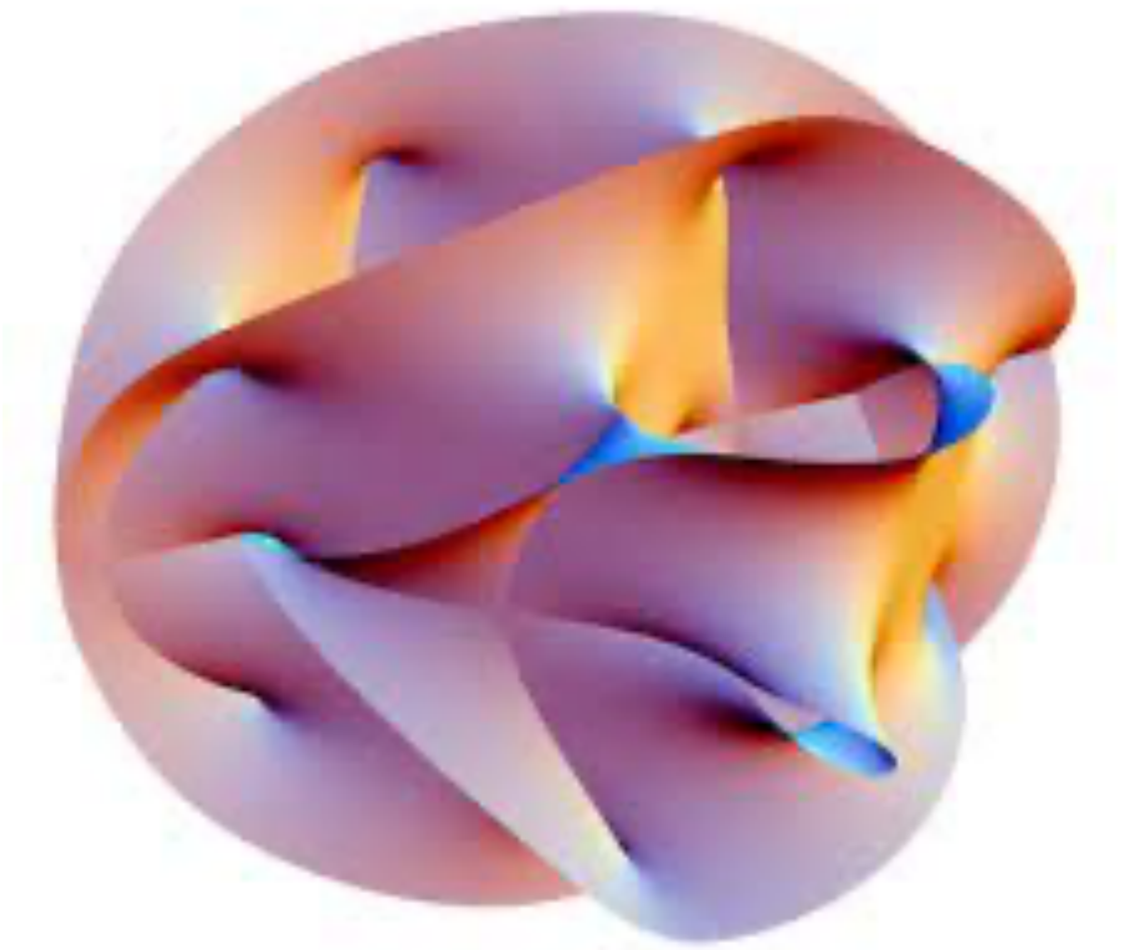
Why Scattering Amplitudes ?

Collider Phenomenology

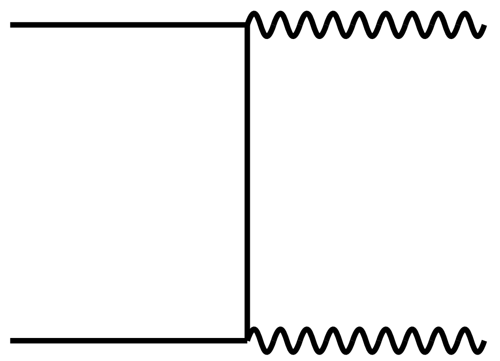
Gravitational Waves

Scattering Amplitudes

Geometry and QFT

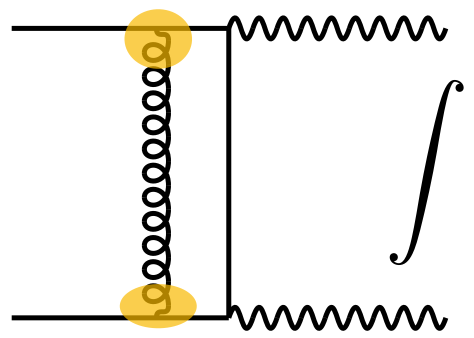


Scattering Amplitude and Cross-Section

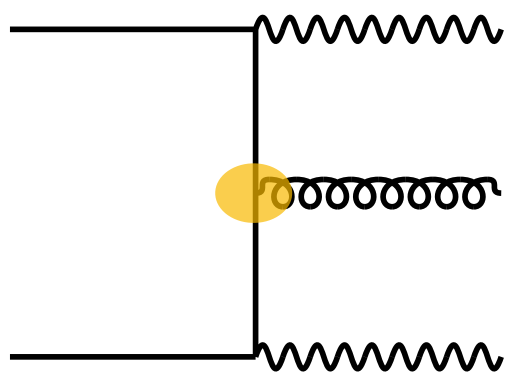


LO

$$\sigma^0 \approx \int |\mathcal{M}_N^{(0)}|^2 d\Phi_N$$



$$\int \left[\frac{V_2}{\epsilon^2} + \frac{V_1}{\epsilon^1} + V_0\right] d\phi_2$$

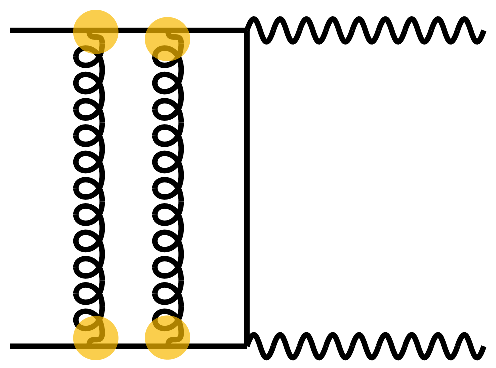


$$\int [R_0]\, d\phi_3$$

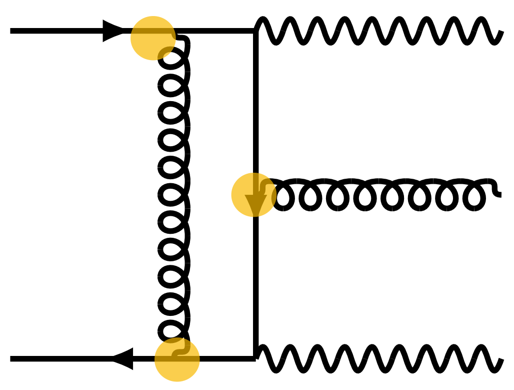
NLO

$$\sigma_N^{(1)} \approx \int 2\mathrm{Re}\left(\mathcal{M}_N^{(0)*}\mathcal{M}_N^{(1)}\right) d\Phi_N + \int |\mathcal{M}_{N+1}^{(0)}|^2 d\Phi_{N+1}$$

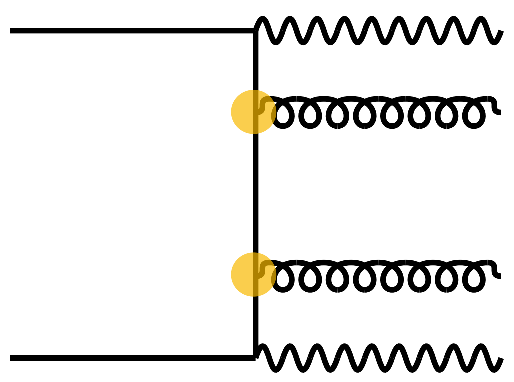
$$\int \left[\frac{VV_4}{\epsilon^4} + \frac{VV_3}{\epsilon^3} + \frac{VV_2}{\epsilon^2} + \frac{VV_1}{\epsilon^1} + VV_0\right] d\phi_2$$



$$\int \left[\frac{RV_2}{\epsilon^2} + \frac{RV_1}{\epsilon^1} + RV_0\right] d\phi_3$$



$$\int [RR_0]\, d\phi_4$$



NNLO

$$\sigma_N^{(2)} \approx \int 2\mathrm{Re}\left(\mathcal{M}_N^{(0)*}\mathcal{M}_N^{(2)}\right) d\Phi_N + \int 2\mathrm{Re}\left(\mathcal{M}_{N+1}^{(0)*}\mathcal{M}_{N+1}^{(1)}\right) d\Phi_{N+1} + \int |\mathcal{M}_{N+2}^{(0)}|^2 d\Phi_{N+2}$$

Feynman Integral

Precision computation of the cross-section in perturbation theory requires the computation of multi-leg / multi loop Feynman Integrals.

Reduction of scalar integrals to Master integrals using IBP

Computation of the MIs

The Main Bottleneck

Integration-By-Parts (IBP) identity

Chetyrkin, Tkachov

Loop momenta

$$\int \prod_{\alpha=1}^{l} d^d k_\alpha \frac{\partial}{\partial k_{j,\mu}} \left(\frac{v^\mu}{D_1^{a_1} \cdots D_N^{a_N}} \right) = 0$$

Loop and external momenta

$$\int_{\alpha=1}^{l} \prod d^d k_\alpha \frac{\partial}{\partial k_{j,\mu}} \left(\frac{v^\mu}{D_1^{a_1} \cdots D_N^{a_N}} \right) = \int_{\alpha=1}^{l} \prod d^d k_\alpha \left[\frac{\partial v^\mu}{\partial k_{j,\mu}} \left(\frac{1}{D_1^{a_1} \cdots D_N^{a_N}} \right) - \sum_{j=1}^{N} \frac{a_j}{D_j} \frac{\partial D_j}{\partial k_{j,\mu}} \left(\frac{v^\mu}{D_1^{a_1} \cdots D_N^{a_N}} \right) \right]$$

$$C_1\, I(a_1, \cdots a_N - 1) + \cdots + C_r\, I(a_1 + 1, \cdots a_N) = 0$$

- Gives relations between different scalar integrals with different exponents
- l(l+E) number of equations
- Solve the system symbolically : Recursion relations LiteRed
- Solve for specific integer value of the exponents : Laporta Algorithm Fire, Reduze, Kira,..

Different New Ideas ?

- Kira - A Feynman Integral Reduction Program
  Maierhoefer, Usovitsch, Uwer (2018)

- FIRE6: Feynman Integral REduction with Modular Arithmetic
  Smirnov, A. V. and Chuharev (2019)

- Two-loop five-point massless QCD amplitudes within the integration-by-parts approach
  Chawdhry,Lim,Mitov (2018)

- Integration-by-parts reductions of Feynman integrals using Singular and GPI-Space
  Bendle, Boehm, Decker,Georgoudis, Pfreundt, Rahn, Wasser, Zhang (2019)

- FiniteFlow: multivariate functional reconstruction using finite fields and dataflow graphs
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$$I = \sum_{i=1}^{\nu} c_i J_i$$

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$$I = \sum_{i=1}^{\nu} c_i J_i$$

$$I \cdot J_i$$

$$J_i \cdot J_j = \delta_{ij}$$

$$I \cdot J_j \left(C^{-1}\right)_{ji}$$

$$J_i \cdot J_j = C_{ij} \neq \delta_{ij}$$

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Goal : To define an Vector space with inner product for the FIs

What is the Vector Space V ?

What is the dual vector space V^*

What is the scalar product $V \times V^* \to \mathbb{C}$

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Mastrolia, Mizera (2018)

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Intersection Theory

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Intersection Theory

Aomoto, Gelfand, Kita, Cho, Matsumoto, Mimachi, Mizera, Yoshida

$\hat{\varphi}(\mathbf{z})d^m\mathbf{z}$

Single-valued differential Form

Twisted Co-cycle

$$I = \int_{\mathcal{C}} u(\mathbf{z})\,\varphi(\mathbf{z})$$

Twisted Cycle

$\langle\varphi|\mathcal{C}]$

Pairing

$u(\mathbf{z})$ is a multi-valued function

$u(\mathbf{z})$ *vanishes on the boundaries* of $\mathcal{C}$, $u(\partial\mathcal{C}) = 0$

Basics of Intersection Theory

$$0 = \int_{\mathcal{C}} d\,(u\,\xi) = \int_{\mathcal{C}} (du \wedge \xi + u\,d\xi) = \int_{\mathcal{C}} u \left(\frac{du}{u} \wedge + d\right) \xi \equiv \int_{\mathcal{C}} u\,\nabla_\omega \xi.$$

$\omega \equiv d \log u$

$\nabla_\omega \equiv d + \omega \wedge$

Equivalence Class $\quad {}_\omega\langle \varphi | \,:\; \varphi \sim \varphi + \nabla_\omega \xi \qquad \int_{\mathcal{C}} u\,\varphi = \int_{\mathcal{C}} u\,(\varphi + \nabla_\omega \xi)$

Vector Space of n-forms $\quad H^n_\omega \equiv \{n\text{-forms } \varphi_n \,|\, \nabla_\omega \varphi_n = 0\} / \{\nabla_\omega \varphi_{n-1}\}$ $\quad$ **Twisted Cohomology Group**

Dual space $\quad H^n_{-\omega} \qquad \nabla_{-\omega} = d - \omega \wedge$

$$\chi(X) = \sum_{k=0}^{2n} (-1)^k \dim H^k_\omega. \qquad H^{k \neq n}_\omega \text{ vanish}$$

Aomoto (1975)

$$\nu = (-1)^n \chi(X)$$
$$= (-1)^n\,(n+1 - \chi(\mathcal{P}_\omega))$$
$$= \{\text{number of solutions of } \omega{=}0\}$$

Integral

$$\langle \varphi_L | \mathcal{C}_L] = \int_{\mathcal{C}_L} u(\mathbf{z}) \varphi_L(\mathbf{z})$$

Intersection Number

$$\langle \varphi_L | \varphi_R \rangle$$

Master Decomposition Formula

$$\langle \varphi | = \sum_{i,j=1}^{\nu} \langle \varphi | h_j \rangle \left(\mathbf{C}^{-1}\right)_{ji} \langle e_i |$$

Univariate Intersection Number

$$\langle \varphi_L | \varphi_R \rangle_\omega = \sum_{p \in \mathcal{P}} \mathrm{Res}_{z=p} \Big(\psi_p\, \varphi_R \Big)$$

Matsumoto, Mizera

$$\nabla_{\omega_p} \psi_p = \varphi_{L,p}$$

First Order Differential Equation

Decomposition of Feynman Integral

Frellesvig, Gasparotto, MKM, Mastrolia, Mattiazzi, Mizera (2019)

Integrals

$$I = \int_{\mathcal{C}} u\,\varphi = \langle \varphi | \mathcal{C}]$$

Number of MIs

$$\omega \equiv d\log u(\mathbf{z}) = \sum_{i=1}^{n} \hat{\omega}_i \, dz_i$$

$\nu =$ Number of solutions of the system of equations

$$\hat{\omega}_i \equiv \partial_{z_i} \log u(\mathbf{z}) = 0\,, \qquad i = 1, \ldots, n$$

$$I = \sum_{i=1}^{\nu} c_i \, J_i \qquad J_i = \langle e_i | \mathcal{C}]$$

Choice of Bases

$$e_i(\mathbf{z}) \qquad h_i(\mathbf{z})$$

$$\mathbf{C}_{ij} = \langle e_i | h_j \rangle$$

Metric Matrix

$$\langle \varphi | = \sum_{i,j=1}^{\nu} \langle \varphi | h_j \rangle \left(\mathbf{C}^{-1}\right)_{ji} \langle e_i |$$

Master Decomposition Formula

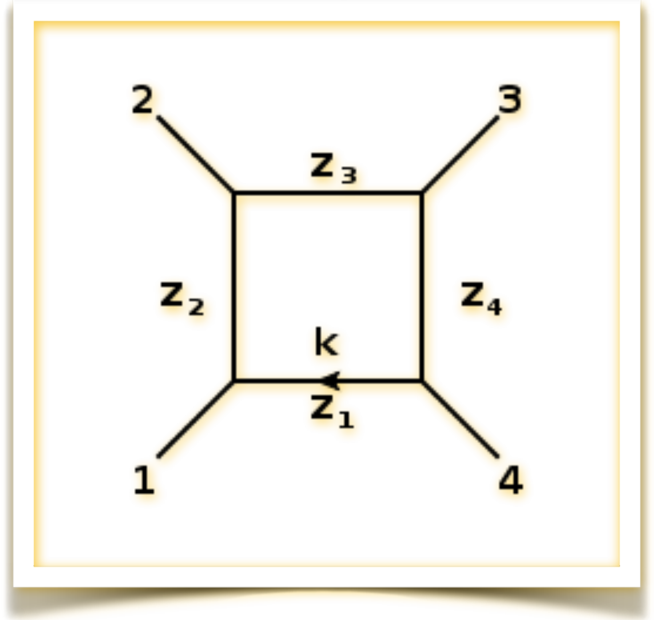
Computation of Multivariate Intersection Number

Mizera

Recursive Formula :

$$_{\mathbf{n}}\langle \varphi_L^{(\mathbf{n})} | \varphi_R^{(\mathbf{n})} \rangle = - \sum_{p \in \mathcal{P}_n} \operatorname*{Res}_{z_n = p} \left({}_{\mathbf{n-1}}\langle \varphi_L^{(\mathbf{n})} | h_i^{(\mathbf{n-1})} \rangle \, \psi_i^{(n)} \right)$$

Intersection Theory: Example

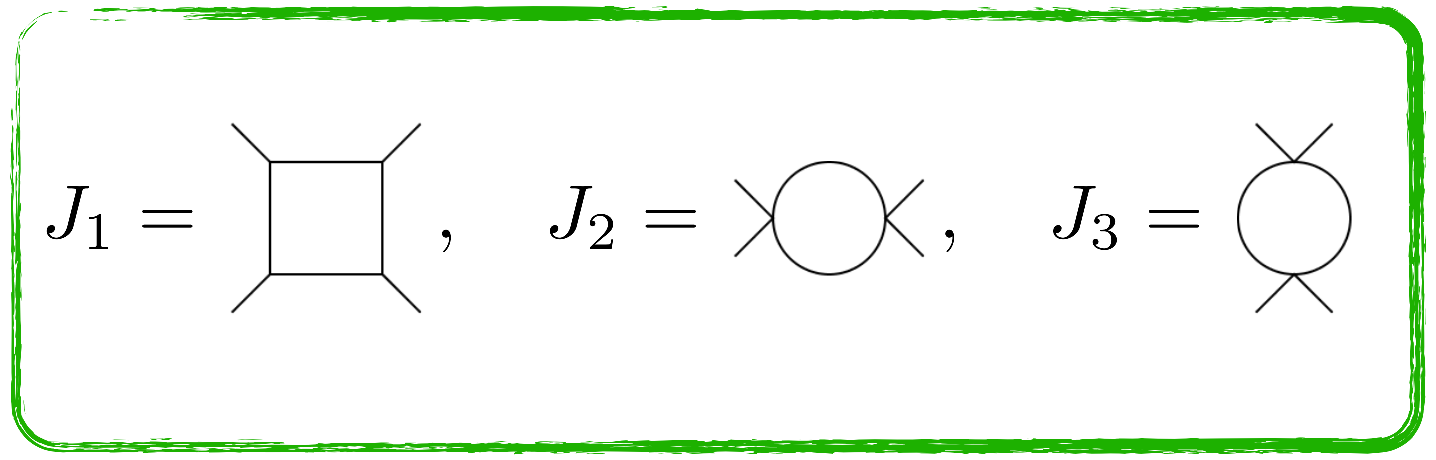


$$u(\mathbf{z}) = \left((st - sz_4 - tz_3)^2 - 2tz_1(s(t+2z_3-z_2-z_4)+tz_3) + s^2z_2^2 + t^2z_1^2 - 2sz_2(t(s-z_3)+z_4(s+2t))\right)^{\frac{d-5}{2}}$$

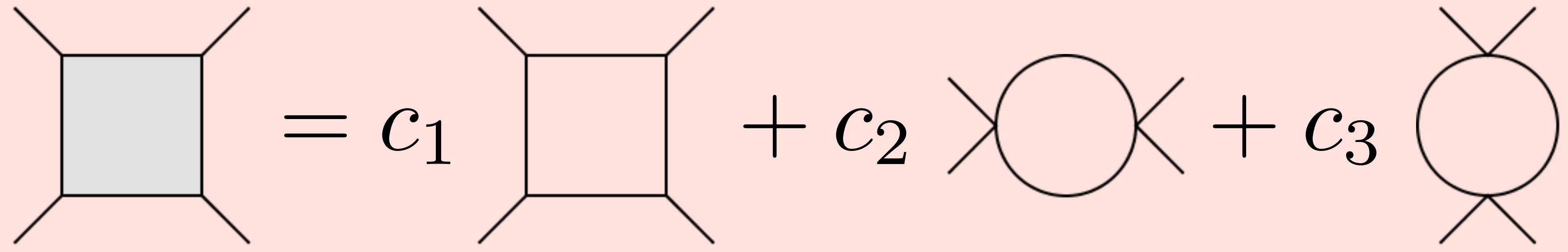
Baikov Polynomial

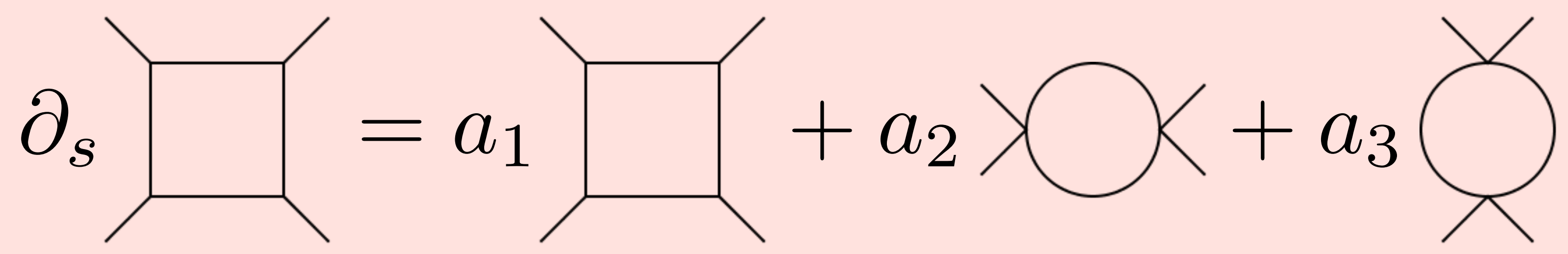
The sectors containing the MIs are

3 MIs
$$\begin{cases} N_{\{1,2,3,4\}} = 1 \\ N_{\{1,3\}} = 1 \\ N_{\{2,3\}} = 1 \end{cases}$$



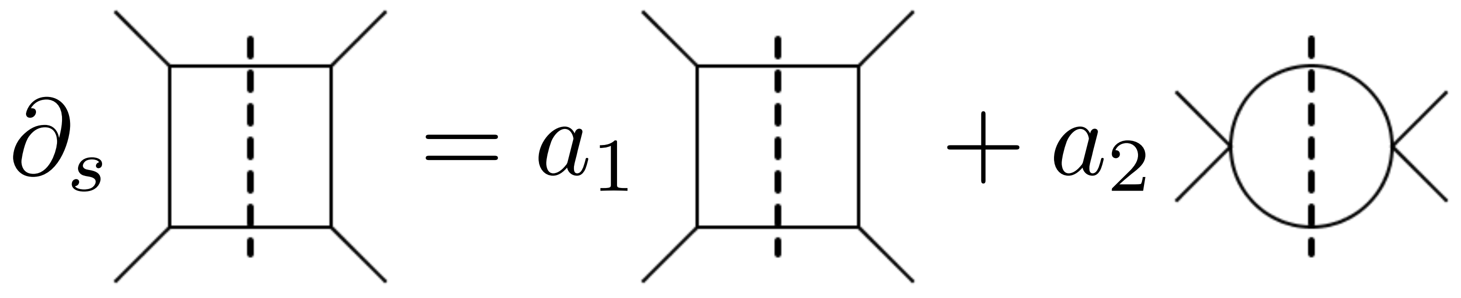
Integral Decomposition

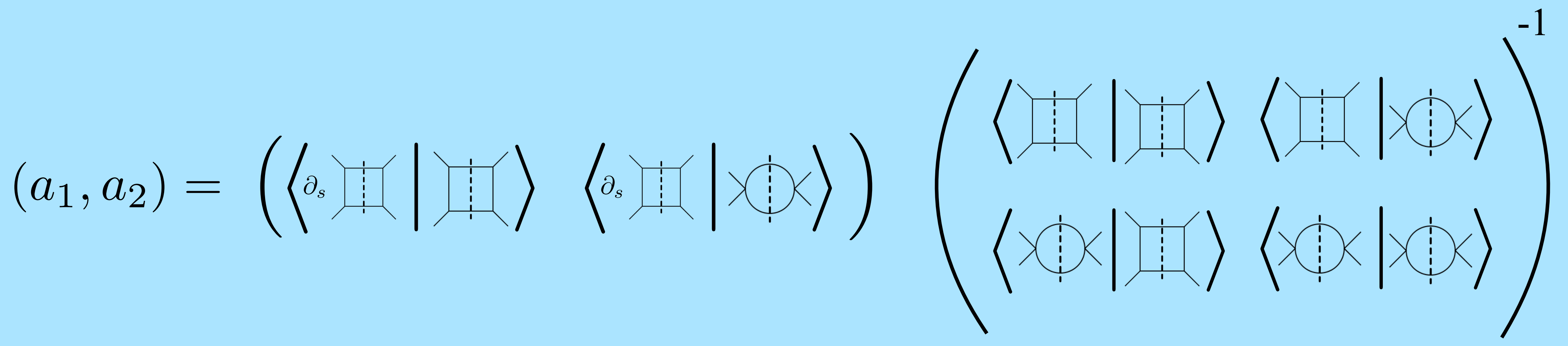




Cut$_{\{1,3\}}$

Differential Equation

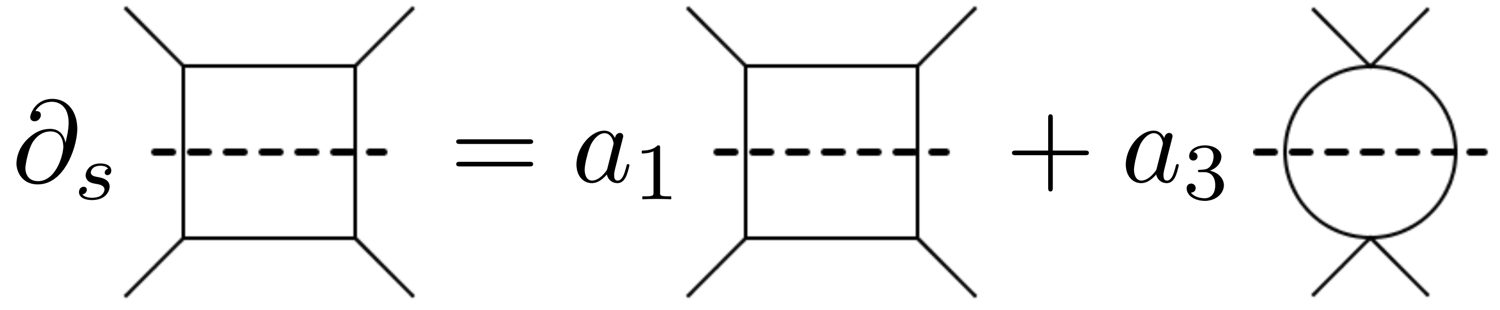


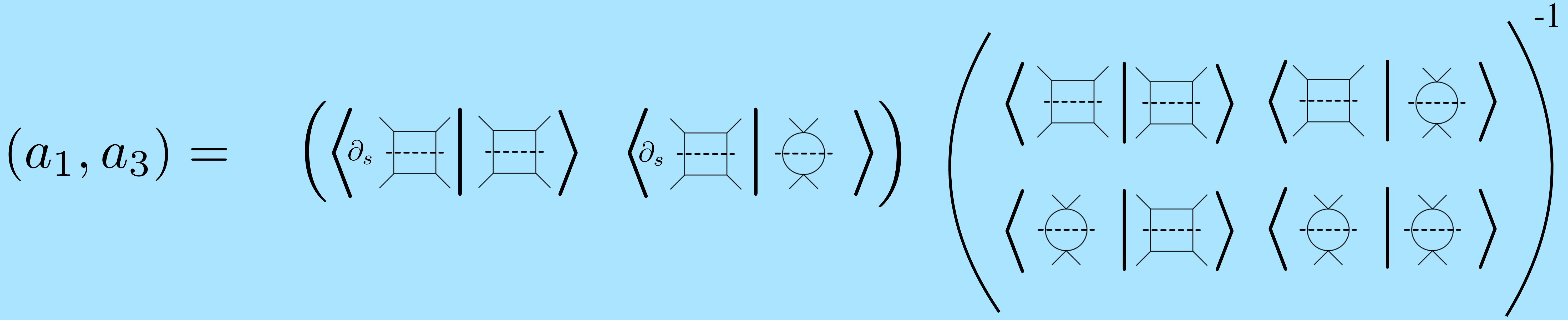


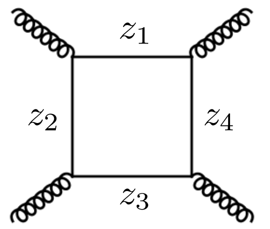
One Loop Box : DE

Cut$_{\{2,4\}}$

Differential Equation





Integral family	Denominators
 $s = (p_1+p_2)^2, \quad t = (p_2+p_3)^2$	$z_1 = k^2 - m_1^2$ $z_2 = (k+p_1)^2 - m_2^2$ $z_3 = (k+p_1+p_2)^2 - m_3^2$ $z_4 = (k+p_1+p_2+p_3)^2 - m_4^2$

τ	ν	e
$z_4 = 0$	$\nu_{\{3\}} = 2$	$e^{(3)} = \left\{1, \frac{1}{z_3}\right\}$
	$\nu_{\{32\}} = 3$	$e^{(32)} = \left\{\frac{1}{z_2}, \frac{1}{z_3}, \frac{1}{z_2 z_3}\right\}$
	$\nu_{\{321\}} = 6$	$e^{(321)} = \left\{1, \frac{1}{z_2}, \frac{1}{z_1 z_2}, \frac{1}{z_1 z_3}, \frac{1}{z_2 z_3}, \frac{1}{z_1 z_2 z_3}\right\}$
$z_3 = 0$	$\nu_{\{4\}} = 2$	$e^{(4)} = \left\{1, \frac{1}{z_4}\right\}$
	$\nu_{\{41\}} = 3$	$e^{(41)} = \left\{\frac{1}{z_1}, \frac{1}{z_4}, \frac{1}{z_1 z_4}\right\}$
	$\nu_{\{412\}} = 6$	$e^{(412)} = \left\{1, \frac{1}{z_1}, \frac{1}{z_1 z_2}, \frac{1}{z_1 z_4}, \frac{1}{z_2 z_4}, \frac{1}{z_1 z_2 z_4}\right\}$
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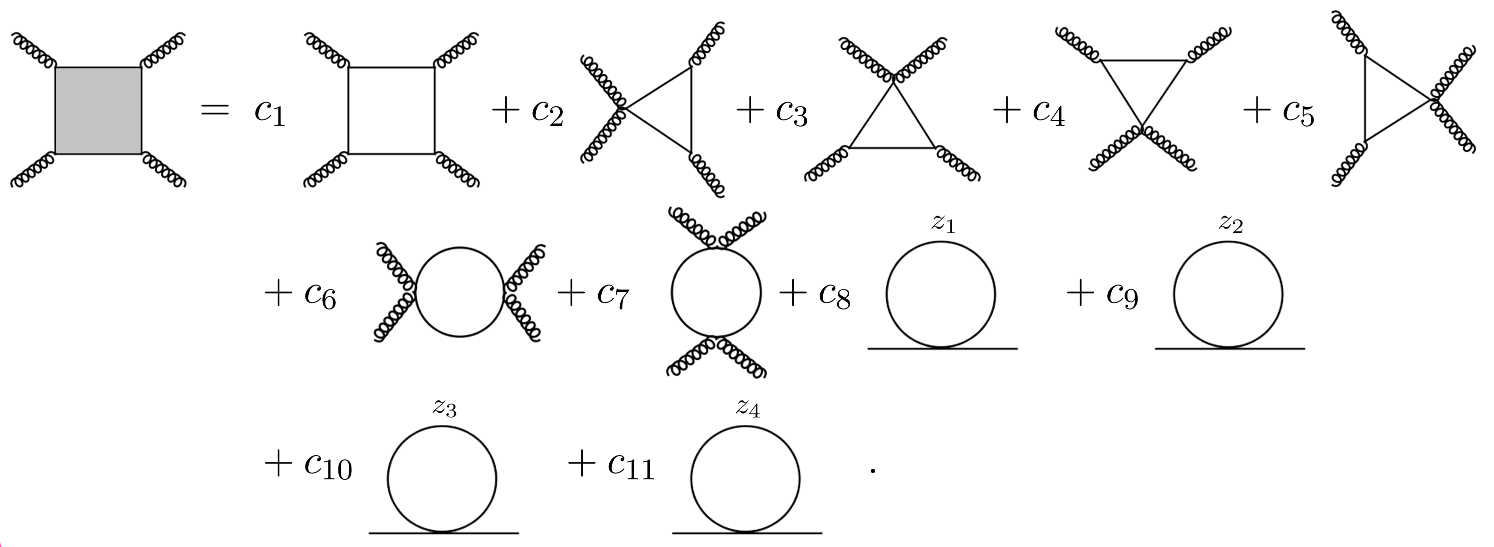
	$\nu_{\{431\}} = 6$	$e^{(431)} = \left\{1, \frac{1}{z_4}, \frac{1}{z_1 z_3}, \frac{1}{z_1 z_4}, \frac{1}{z_3 z_4}, \frac{1}{z_1 z_3 z_4}\right\}$
$z_1 = 0$	$\nu_{\{4\}} = 2$	$e^{(4)} = \left\{1, \frac{1}{z_4}\right\}$
	$\nu_{\{43\}} = 3$	$e^{(43)} = \left\{\frac{1}{z_3}, \frac{1}{z_4}, \frac{1}{z_3 z_4}\right\}$
	$\nu_{\{432\}} = 6$	$e^{(432)} = \left\{1, \frac{1}{z_3}, \frac{1}{z_2 z_3}, \frac{1}{z_2 z_4}, \frac{1}{z_3 z_4}, \frac{1}{z_2 z_3 z_4}\right\}$



H2020 MSCA COFUND G.A. 754496

PREPARED FOR SUBMISSION TO JHEP

arXiv:2008.04823v1 [hep-th] 11 Aug 2020

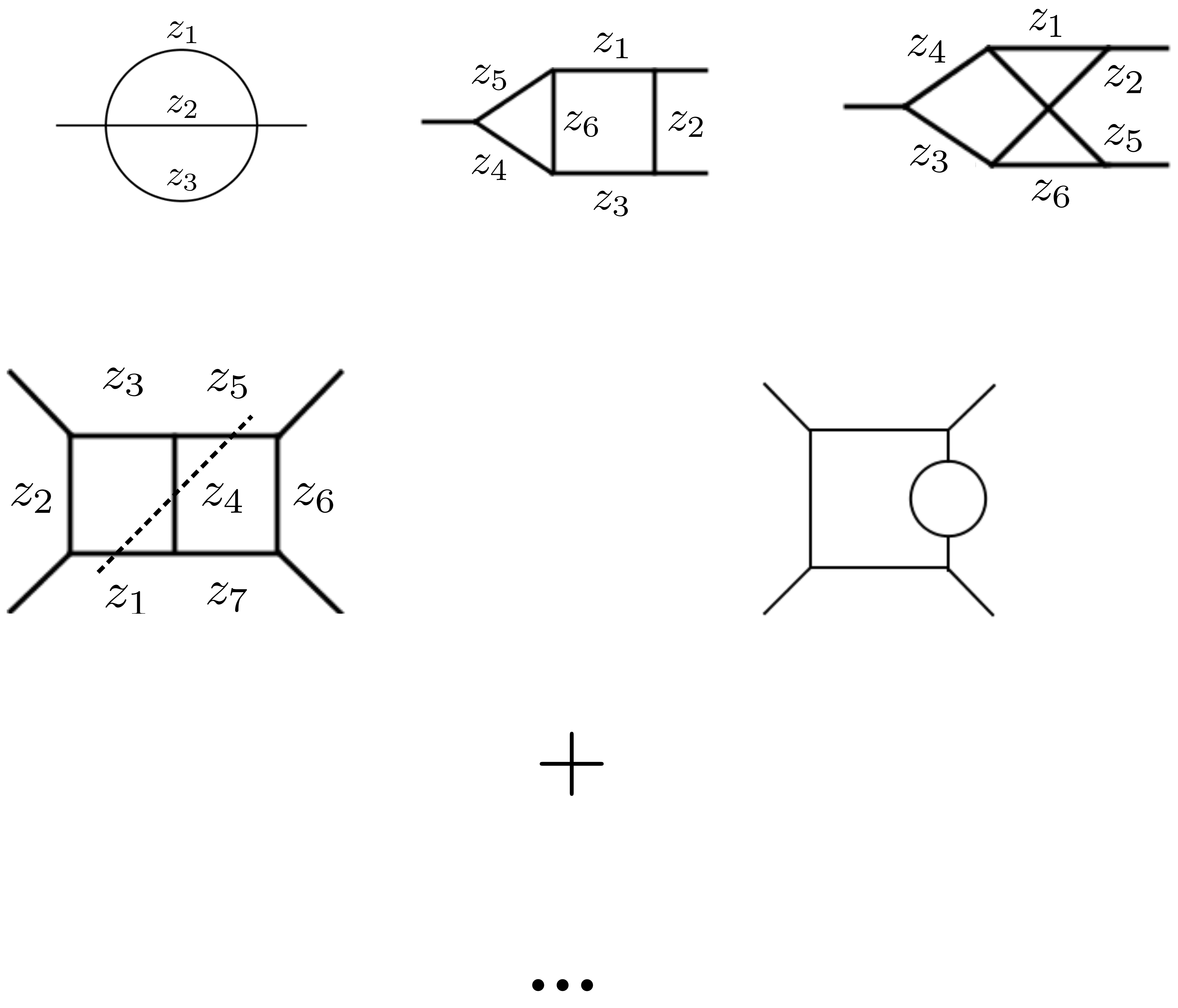
Decomposition of Feynman Integrals by Multivariate Intersection Numbers

Hjalte Frellesvig,[a,b] Federico Gasparotto,[a,b] Stefano Laporta,[a,b] Manoj K. Mandal,[b,a] Pierpaolo Mastrolia,[a,b] Luca Mattiazzi,[b,a] Sebastian Mizera[c]

[a] *Dipartimento di Fisica e Astronomia, Università di Padova, Via Marzolo 8, 35131 Padova, Italy*
[b] *INFN, Sezione di Padova, Via Marzolo 8, 35131 Padova, Italy*
[c] *Institute for Advanced Study, Einstein Drive, Princeton, NJ 08540, USA*

E-mail: {hjalte.frellesvig, federico.gasparotto, stefano.laporta, manojkumar.mandal, pierpaolo.mastrolia, luca.mattiazzi}@pd.infn.it, smizera@ias.edu

ABSTRACT: We present a detailed description of the recent idea for a direct decomposition of Feynman integrals onto a basis of master integrals by projections, as well as a direct derivation of the differential equations satisfied by the master integrals, employing multivariate intersection numbers. We discuss a recursive algorithm for the computation of multivariate intersection numbers, and provide three different approaches for a direct decomposition of Feynman integrals, which we dub the *straight decomposition*, the *bottom-up decomposition*, and the *top-down decomposition*. These algorithms exploit the unitarity structure of Feynman integrals by computing intersection numbers supported on cuts, in various orders, thus showing the synthesis of the intersection-theory concepts with unitarity-based methods and integrand decomposition. We perform explicit computations to exemplify all of these approaches applied to Feynman integrals, paving a way towards potential applications to generic multi-loop integrals.



Conclusion

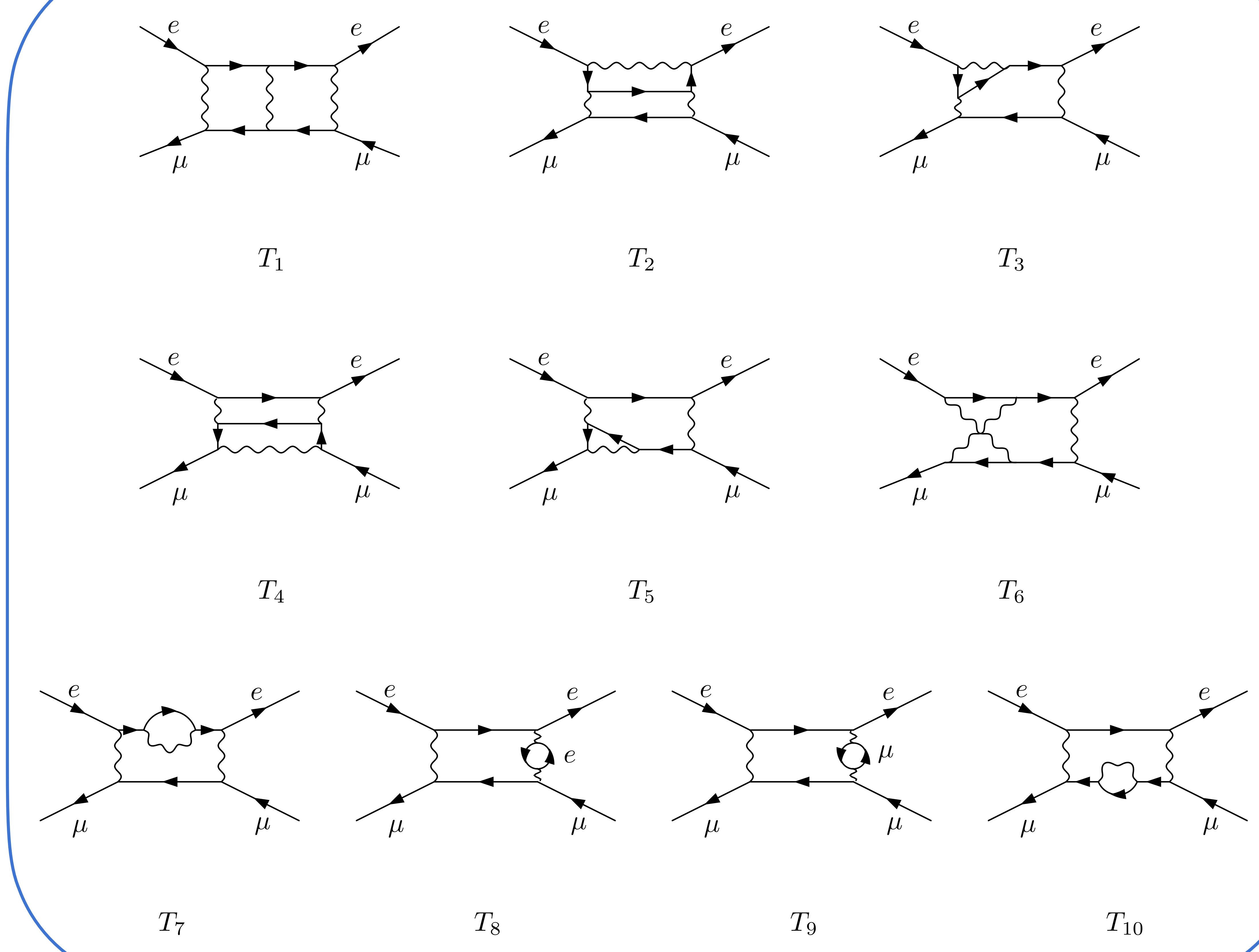
- ☑ Novel Algebraic Property Unveiled
  - ☑ The algebra of Feynman Integrals is controlled by intersection numbers
  - ☑ Intersection Numbers : Scalar Product/Projection between Feynman Integrals
  - ☑ Useful for both Physics and Mathematics

- ☑ Novel decomposition method
  - ☑ Direct decomposition in a Integral Basis
  - ☑ No Intermediate relation required

2-Loop Electron-Muon Amplitude

✦ Computation of 2-loop virtual amplitude for electron-muon scattering, relevant for the MUonE experiment

Bonciani, Di-Vita, Passera, Primo, MKM, Mastrolia, Mattiazzi, Ronca, Schubert , Torres Bobadilla , Tramontano



Other Activities

H2020 MSCA COFUND G.A. 754496

- ✦ Successfully organized the international conference on the EFT methods
- ✦ On the Editorial board of the Proceedings of the MathemAmplitudes Conference, 2019
- ✦ Presented a talk on the Status of the analytical and numerical evaluation of multi-loop integrals in the CEPC meeting, 2020

PROCEEDINGS OF SCIENCE

MA2019 --- MathemAmplitudes 2019: Intersection Theory & Feynman Integrals

Member of the Editorial Board

Analytical and Numerical Evaluation of 2- and 3-Loop Integrals with Massive Lines

Manoj Kumar Mandal

INFN & University of Padova

International workshop on the high energy Circular Electron-Positron Collider (CEPC)
Sanghai (Virtual Meeting)
28 October 2020

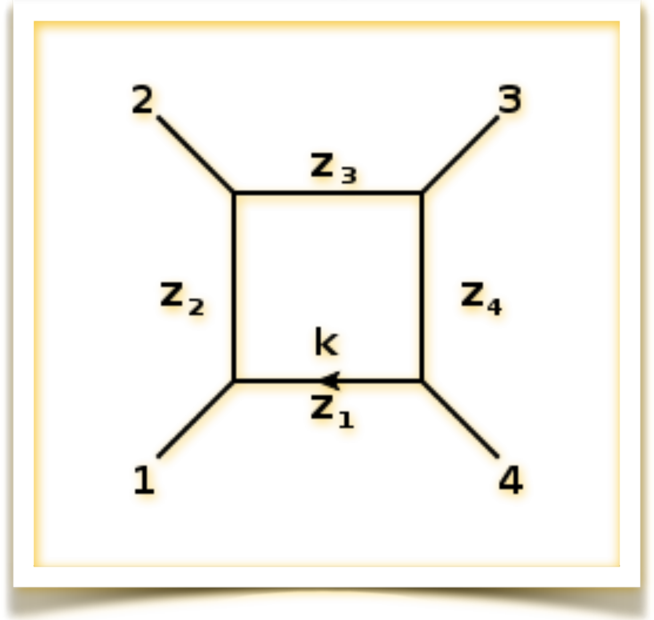
MathemAmplitudes 2021

October, 2021

Organizers

- H. Frellesvig
- S. Laporta
- M. K. Mandal
- P. Mastrolia
- S. Mizera

Thank you



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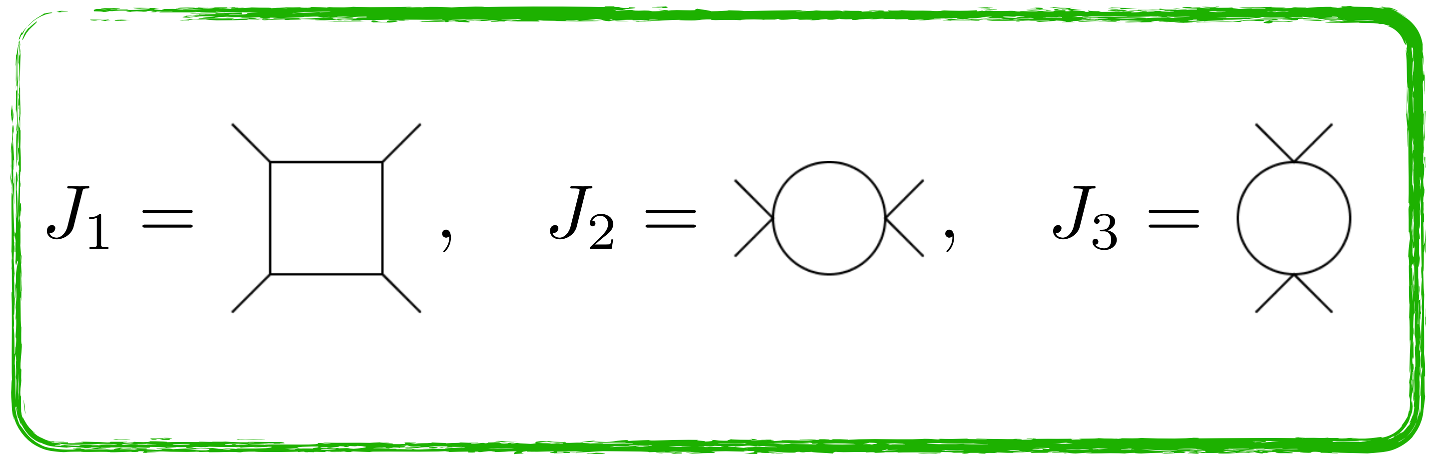
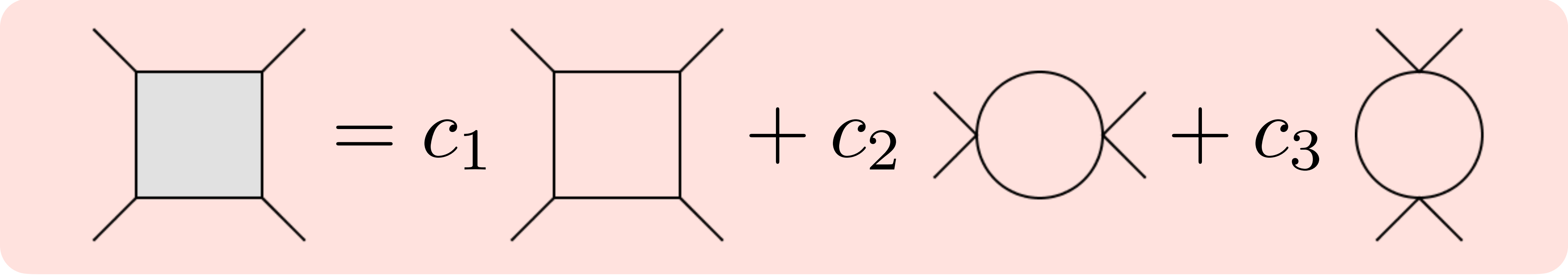
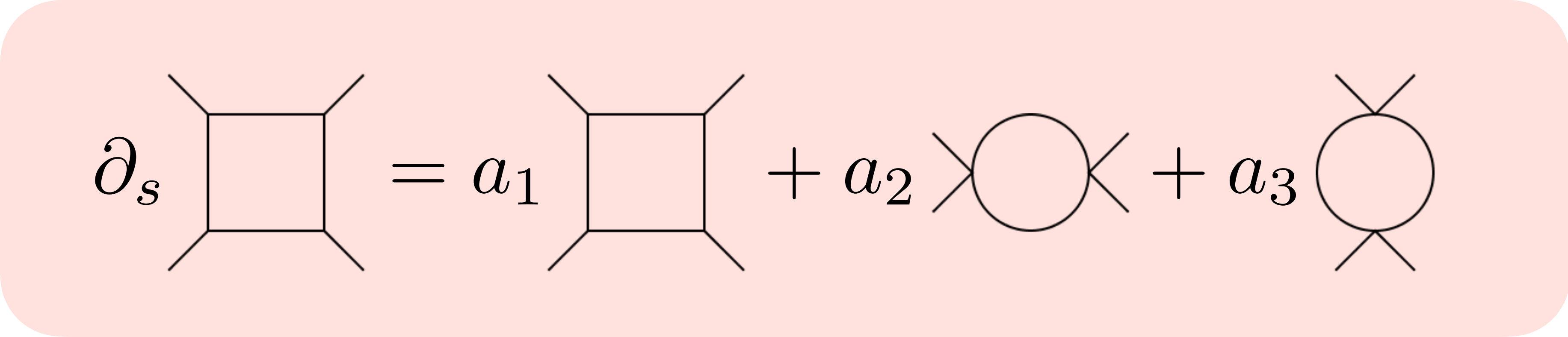
The sectors containing the MIs are

3 MIs $\begin{cases} N_{\{1,2,3,4\}} = 1 \\ N_{\{1,3\}} = 1 \\ N_{\{2,3\}} = 1 \end{cases}$

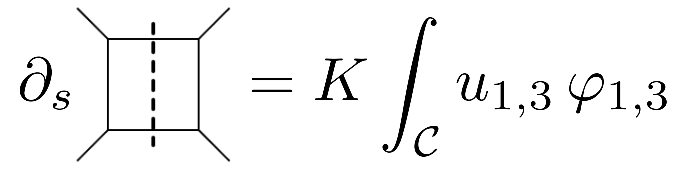


Integral Decomposition





Cut$_{\{1,3\}}$

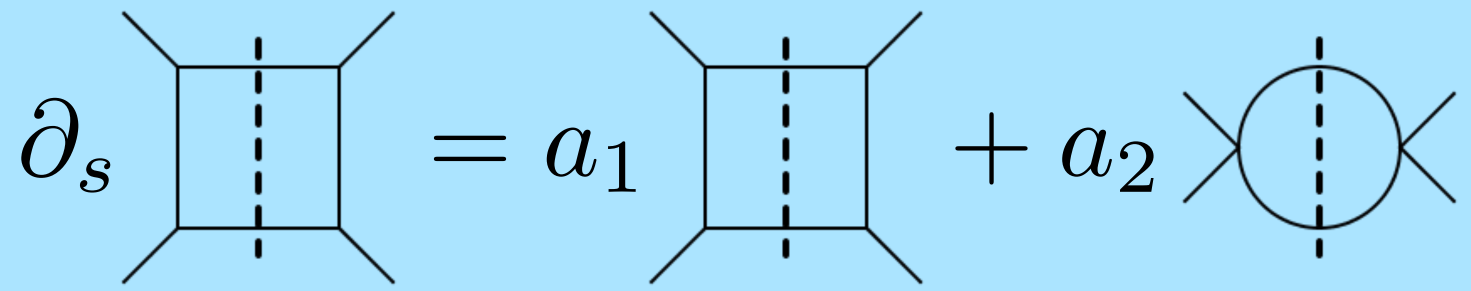


$\varphi_{1,3} = \hat{\varphi}_{1,3}\, dz_4 \wedge dz_2$

$$u_{1,3} = z_2^{\rho_2} z_4^{\rho_4}\, u(0, z_2, 0, z_4)$$

$$\hat{\varphi}_{1,3} = \frac{f}{z_2 z_4} \qquad f = \frac{1}{Ku}\frac{\partial(Ku)}{\partial s}$$

Differential Equation

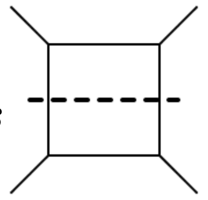


$$a_1 = \sum_{j=1}^{2} \langle \varphi_{1,3} | e_j^{(42)} \rangle \left(\mathbf{C}_{(42)}^{-1}\right)_{j1} = \frac{(d-6)t - 2s}{2s(s+t)}$$

$$a_2 = \sum_{j=1}^{2} \langle \varphi_{1,3} | e_j^{(42)} \rangle \left(\mathbf{C}_{(42)}^{-1}\right)_{j2} = \frac{2(d-3)}{s^2(s+t)}$$

One Loop Box : DE

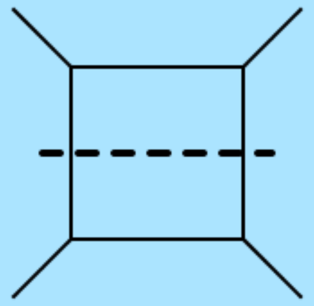
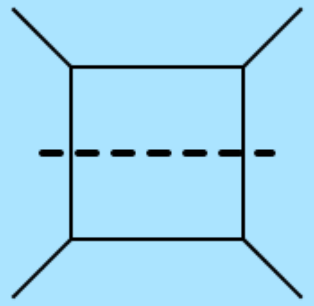
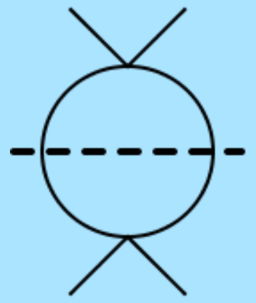
Cut$_{\{2,4\}}$

∂_s  $= K \int_{\mathcal{C}} u_{2,4}\, \varphi_{2,4}$ $\qquad \varphi_{2,4} = \hat{\varphi}_{2,4}\, dz_3 \wedge dz_1$

$$u_{2,4} = z_1^{\rho_1} z_3^{\rho_3}\, u(z_1, 0, z_3, 0)$$

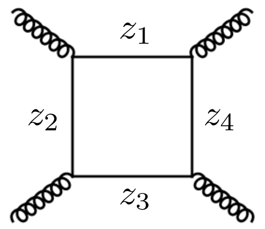
$$\hat{\varphi}_{2,4} = \frac{f}{z_1 z_3} \qquad f = \frac{1}{Ku} \frac{\partial (Ku)}{\partial s}$$

Differential Equation

∂_s  $= a_1$  $+ a_3$ 

$$a_1 = \sum_{j=1}^{2} \langle \varphi_{1,3} | e_j^{(42)} \rangle \left(\mathbf{C}_{(42)}^{-1} \right)_{j1} = \frac{(d-6)t - 2s}{2s(s+t)}$$

$$a_3 = \sum_{j=1}^{2} \langle \varphi_{2,4} | e_j^{(31)} \rangle \left(\mathbf{C}_{(31)}^{-1} \right)_{j2} = -\frac{2(d-3)}{st(s+t)}$$

Integral family	Denominators
 $s = (p_1 + p_2)^2, \quad t = (p_2 + p_3)^2$	$z_1 = k^2 - m_1^2$ $z_2 = (k + p_1)^2 - m_2^2$ $z_3 = (k + p_1 + p_2)^2 - m_3^2$ $z_4 = (k + p_1 + p_2 + p_3)^2 - m_4^2$

τ	ν	e
$z_4 = 0$	$\nu_{\{3\}} = 2$	$e^{(3)} = \left\{1, \frac{1}{z_3}\right\}$
	$\nu_{\{32\}} = 3$	$e^{(32)} = \left\{\frac{1}{z_2}, \frac{1}{z_3}, \frac{1}{z_2 z_3}\right\}$
	$\nu_{\{321\}} = 6$	$e^{(321)} = \left\{1, \frac{1}{z_2}, \frac{1}{z_1 z_2}, \frac{1}{z_1 z_3}, \frac{1}{z_2 z_3}, \frac{1}{z_1 z_2 z_3}\right\}$
$z_3 = 0$	$\nu_{\{4\}} = 2$	$e^{(4)} = \left\{1, \frac{1}{z_4}\right\}$
	$\nu_{\{41\}} = 3$	$e^{(41)} = \left\{\frac{1}{z_1}, \frac{1}{z_4}, \frac{1}{z_1 z_4}\right\}$
	$\nu_{\{412\}} = 6$	$e^{(412)} = \left\{1, \frac{1}{z_1}, \frac{1}{z_1 z_2}, \frac{1}{z_1 z_4}, \frac{1}{z_2 z_4}, \frac{1}{z_1 z_2 z_4}\right\}$
$z_2 = 0$	$\nu_{\{4\}} = 2$	$e^{(4)} = \left\{1, \frac{1}{z_4}\right\}$
	$\nu_{\{43\}} = 3$	$e^{(43)} = \left\{\frac{1}{z_3}, \frac{1}{z_4}, \frac{1}{z_3 z_4}\right\}$
	$\nu_{\{431\}} = 6$	$e^{(431)} = \left\{1, \frac{1}{z_4}, \frac{1}{z_1 z_3}, \frac{1}{z_1 z_4}, \frac{1}{z_3 z_4}, \frac{1}{z_1 z_3 z_4}\right\}$
$z_1 = 0$	$\nu_{\{4\}} = 2$	$e^{(4)} = \left\{1, \frac{1}{z_4}\right\}$
	$\nu_{\{43\}} = 3$	$e^{(43)} = \left\{\frac{1}{z_3}, \frac{1}{z_4}, \frac{1}{z_3 z_4}\right\}$
	$\nu_{\{432\}} = 6$	$e^{(432)} = \left\{1, \frac{1}{z_3}, \frac{1}{z_2 z_3}, \frac{1}{z_2 z_4}, \frac{1}{z_3 z_4}, \frac{1}{z_2 z_3 z_4}\right\}$