Panofsky-Wenzel theorem

April 8, 2010, Alex Chao

Wakefields

- Gauss' law => A charged particle is always connected with its field lines. You can distort the field lines by perturbing the particle motion and by wall boundaries, but you can't get rid of them. Field lines are indestructible.
- It follows that a beam necessarily carries with it a lot of field lines, bouncing back and forth in the vacuum chamber.
- It then follows that the vacuum chamber is FAR FROM in vacuum. It is filled with E&M fields.
- These E&M field lines are referred to as wakefields. Vacuum chamber is filled with strong wakefields, literally like a microwave oven.

Beam-structure interaction

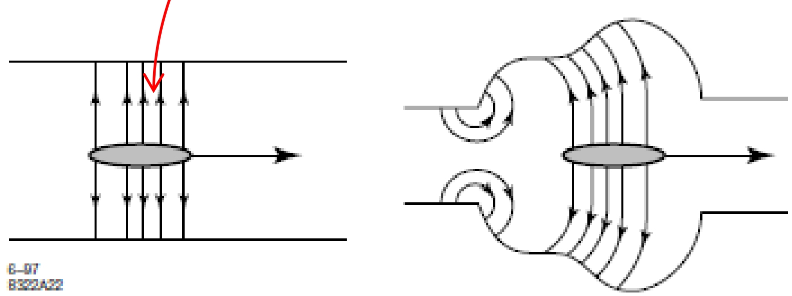
- Accelerator designers are fortunate to have the following very special case that works in their great favor. Theorem: when the beam is relativistic and when the vacuum chamber is longitudinally smooth perfectly-conducting pipe, then there is no wake force.
- In this very special case, there are still E&M field lines – called "pancake fields" because they shape like a pancake. Pancake fields are the strongest beam-generated E&M fields in a vacuum chamber. However, due to a magical exact cancellation between electric force and magnetic force, there is no net wake force in this special case!
- All accelerators are based on this theorem. That is why, to zero-th order, all accelerators are perfectly conducting smooth pipes!
- Wakefields are generated by beam-structure interaction. Whenever there is a discontinuity in the vacuum chamber (even as small as 1 mm!), or when the pipe is slightly resistive, wakefields are generated, and that is still a serious matter.
- This theorem helps greatly. But we need more help.

Smooth pipe

➔ No wake force

Discontinuity

➔ Wakefileds



Instabilities

- Most of these wakefields, fortunately, are harmless. The relativistic beam will typically penetrate them only with minor perturbations even with high beam intensities.

- However if the wakefields systematically perturb the beam motion, then the minor perturbations can accumulate up to a large perturbation, causing the beam to become unstable. Only the systematic part of the wakefields causes instability.

- Note that we are speaking of a relativistic beam. For a nonrelativistic beam, much of our discussions will not apply.

Panofsky-Wenzel theorem

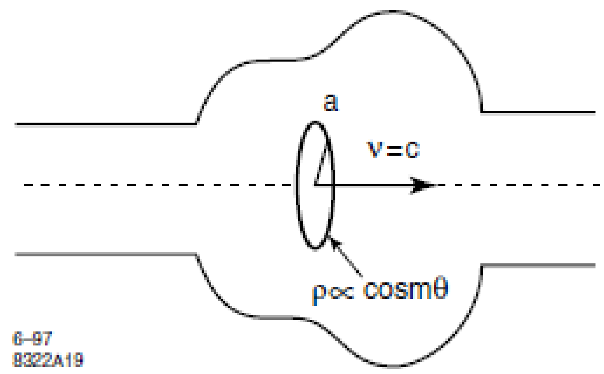
- Beam-structure interaction is a difficult calculational problem. Applying PIC codes is reasonable for small devices such as electron guns and klystrons, but becomes impractical for large accelerators.

- So, what can we do for large accelerators? The answer is we must find a clever way to simplify, and there is a very clever way! ➔ We do not need to do extensive 3-D PIC simulations element by element in order to study instability effects.

- For high energy accelerators, this is achieved by making two approximations. These two approximations, together with the Panofsky-Wenzel theorem that follows from them, lay the foundation for the concepts of ``wake function'' and ``impedance'' of the modern accelerators.

First approximation: the Rigid-beam approximation

- At high energies, beam motion is little affected during the passage of a structure.
- One can calculate the wakefields assuming the beam shape is rigid and its motion is ultrarelativistic with *v=c* in a straight line.
- In fact, we only need to calculate the wakefields generated by a rigid $\cos m\theta$ ring beam (*m=0* monopole (net charge), *m=1* dipole, etc.)
- Wakefield of a general beam can be obtained by superposition of wakefields due to the ring beams with different m's and different ring radii.



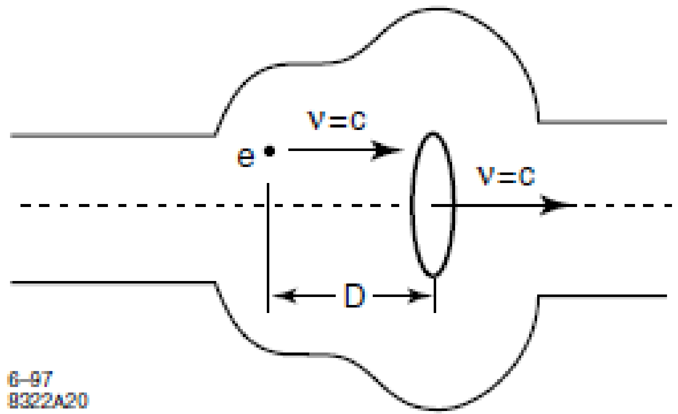
Second approximation: the Impulse Approximation

First note that we don't need to know the instantaneous electric field $\vec{E}$ or magnetic field $\vec{B}$ separately. We need only to know the force *f = e(E + v x B).*

Second, for high energies, we don't even need the instantaneous f. We only need the integrated impulse

$$\Delta\vec{p} = \int_{-\infty}^{\infty} dt\ \vec{f}$$

where the integration over t is performed along the <u>unperturbed</u> straightline trajectory of the test charge e, holding *D* fixed.



The instantaneous wakefields are complicated, but the integrated impulse $\Delta\vec{p}$ is much simpler and it is $\Delta\vec{p}$ that we need!

The quantity $\Delta\vec{p}$ is sometimes called the ``wake potential''.

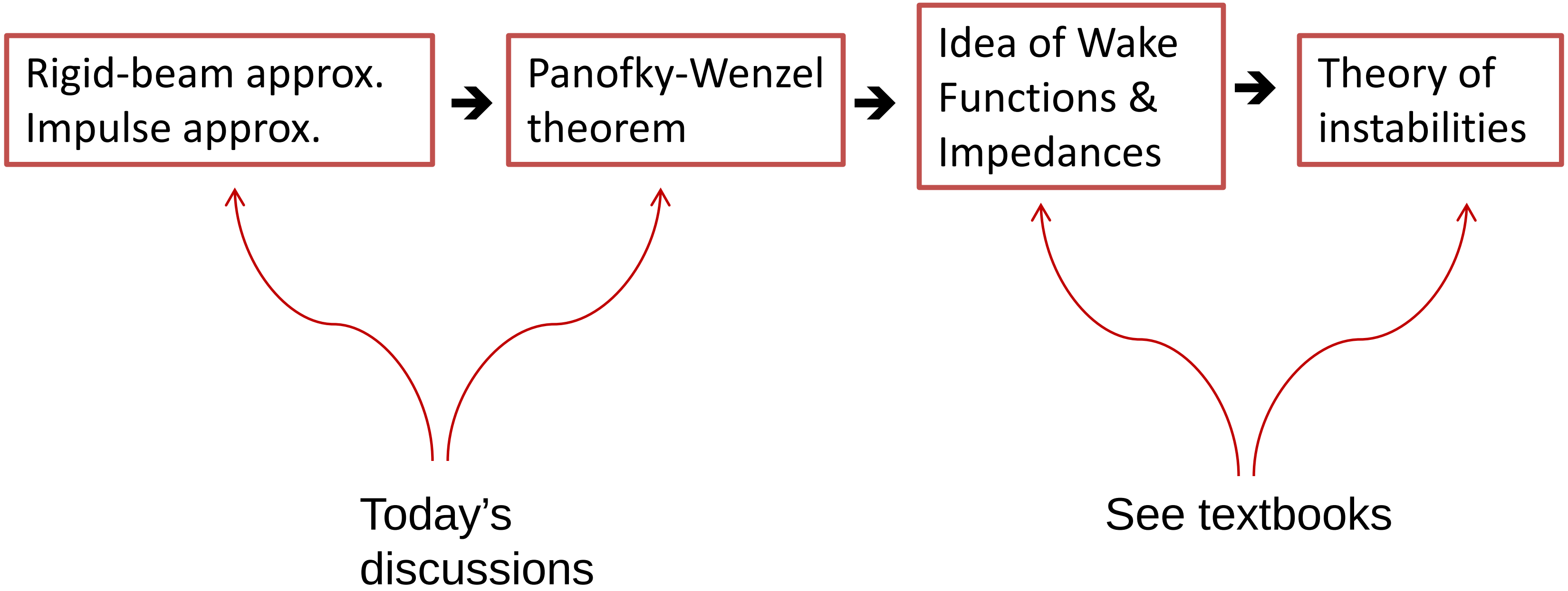
Note that although the beam is considered to be rigid during the passage, the impulse will affect the subsequent beam motion after the passage.

Note also that, by integrating over *t*, only the <u>systematic</u> part of wake force is considered. All rapid oscillating terms are integrated to zero.

Reasoning along this line turns out to be quite fruitful. In the following, we will

- derive the Panofsky-Wenzel theorem
- consider various applications

The two approximations and the Panofsky-Wenzel theorem are the basis of all beam instability analyses in high energy accelerators.


Rigid-beam approx.
Impulse approx.
Panofky-Wenzel theorem
Idea of Wake Functions & Impedances
Theory of instabilities
Today's discussions
See textbooks

Derivation of Panofsky-Wenzel theorem

$$
\begin{aligned}
\nabla \cdot \vec{E} &= 4\pi\rho \\
\nabla \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} &= 4\pi\beta\rho\hat{z} \\
\nabla \cdot \vec{B} &= 0 \\
\nabla \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} &= 0
\end{aligned}
\tag{1.6}
$$

where we have made the important rigid beam approximation $\vec{j} = \rho\vec{v}$ and $\vec{v} = \beta c\hat{z}$.

The Lorentz force, Eq.(1.1) is given by

$$
\vec{f} = e(\vec{E} + \beta\hat{z} \times \vec{B}) \tag{1.7}
$$

Rigid-beam approximation!

We want to calculate the net kick received by a test charge e with transverse position (x,y) and longitudinal position D relative to the moving beam. Both beam and test charfe move with v = βc in z-direction. Impulse approximation:

$$\Delta\vec{p}(x,y,D) = \int_{-\infty}^{\infty} dt\ \vec{f}(x,y,D+\beta ct,t)$$

Impulse Approximation!

With f given in (1.7) and fields satisfying (1.6), one then proceeds to calculate the quantity $\nabla \times \Delta\vec{p}$. It is then found that

$$\boxed{\nabla \times \Delta\vec{p} = \vec{0}}$$

∇ refers to taking derivative with respect to coordinates (x, y, D)

Panofsky-Wenzel theorem!

One thing amazing: we have not yet assumed anything of the beam, or the vacuum chamber boundary. They are arbitrary. We only made two approximations, and the Panofsky-Wenzel theorem followed.

Note also: the original P-W paper (1956) reads completely different/obscure.

Discussion 1

Decompose the P-W theorem into longitudinal and transverse components =>

$$\nabla \cdot (\hat{z} \times \Delta\vec{p}) = 0 \tag{1.15}$$

$$\frac{\partial}{\partial D}\Delta\vec{p}_\perp = \nabla_\perp \Delta p_z \tag{1.16}$$

Eq.(1.15) says something about the transverse components of $\Delta\vec{p}$. Eq.(1.16) says that the transverse gradient of the longitudinal wake potential is equal to the longitudinal gradient of the transverse wake potential.

Discussion 2

Specialize to ultrarelativistic case β=1 ➔ $\nabla_\perp \cdot \Delta\vec{p}_\perp = 0$ (1.17)

Application 1: Cartesian coordinates

Apply to Cartesian coordinates, (1.15) reads

$$\frac{\partial \Delta p_x}{\partial y} = \frac{\partial \Delta p_y}{\partial x}$$

And when β=1, (1.17) gives $$\frac{\partial \Delta p_x}{\partial x} + \frac{\partial \Delta p_y}{\partial y} = 0$$

Combining two equations then gives

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Delta p_{x,y} = 0 \text{ if } \beta = 1$$

It is clear that P-W theorem imposes very strong conditions on the wake potential.

Application 2: Cylindrical coordinates

A more common application is for a cylindrically symmetric pipe.

(1.15) ➔

$$\frac{\partial}{\partial r}(r\Delta p_\theta) = \frac{\partial}{\partial \theta}\Delta p_r$$

(1.16) ➔

$$\begin{cases} \frac{\partial}{\partial D}\Delta p_r = \frac{\partial}{\partial r}\Delta p_z \\ \frac{\partial}{\partial D}\Delta p_\theta = \frac{1}{r}\frac{\partial}{\partial \theta}\Delta p_z \end{cases}$$

(1.17) ➔

$$\frac{\partial}{\partial r}(r\Delta p_r) = -\frac{\partial}{\partial \theta}\Delta p_\theta \qquad (\beta = 1)$$

These P-W equations are surprisingly simple. They do not contain any beam source terms. The beam can have any shape or distribution. Neither do they depend on the boundary conditions. The boundary can be perfectly conducting or resistive metal, or dielectric, or even a gradually fading plasma surface. The boundary also does not have to consist of a single piece.

The only inputs needed for the Panofsky-Wenzel theorem are the Maxwell equations in free space and the two approximations.

Solving the P-W equations in cylindrical coordinates for the case driven by an ultrarelativistic cos mθ ring beam ➔

$$
\begin{aligned}
c\Delta\vec{p}_\perp &= -eI_m W_m(D) m r^{m-1}(\hat{r}\cos m\theta - \hat{\theta}\sin m\theta) \\
c\Delta p_z &= -eI_m W'_m(D) r^m \cos m\theta
\end{aligned}
$$

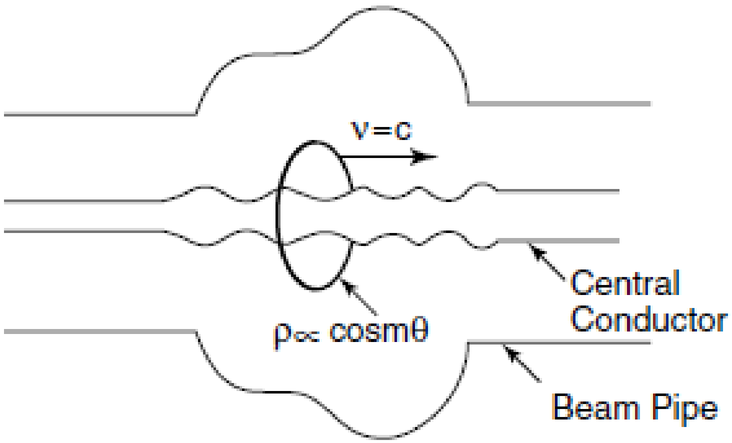
Here $W_m(D)$ is the called wake function. Fourier transform of $W_m(D)$ is the transverse impedance. Fourier transform of $W'_m(D)$ is the longitudinal impedance.

The 3-D dependence of wake potential Δp on (r, θ, D) no longer requires a 3-D PIC calculation. Its r- and θ-dependences are explicitly solved, thanks to Panofsky-Wenzel theorem.

This drastic simplification is applicable only to Δp, and is not application to E, B, or f.

For discussions beyond this point, see textbooks.

Application 3: Cylindrical pipe with a central conductor



Presence of a central conductor profoundly changes the wakefields. In particular, another set of wake functions now emerges which was forbidden because they diverge at *r=0*.

Special case when central conductor is a smooth wire of radius *a* ➔

$$\Delta p_r = eI_0 \frac{W_0(D)}{r \ln(a/b)}$$
$$\Delta p_\theta = 0$$
$$\Delta p_z = -eI_0 W_0'(D) \left[1 - \frac{\ln(r/b)}{\ln(a/b)}\right]$$

One must be careful when measuring impedances using the "wire method" – a questionable technique to measure impedances!

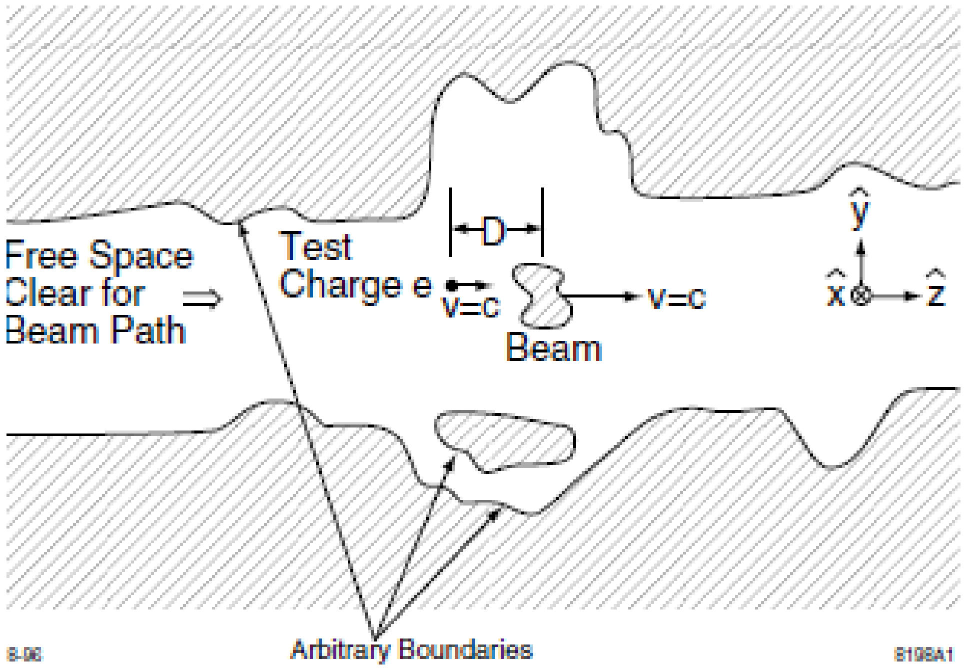
Application 4:   Planar wake theorem

For ultimate linear collider :

➔Higher RF frequency for higher gradient
➔Smaller accelerating structures
➔Stronger wakefields
➔Need a way to desensitize the wakefields
➔Rectangular structures instead of cylindrical ones (Rosenzweig)

➔Also, small structures are to be made by laser cutting.
➔Rectangular structures allows easer fabrication.

Consider a 2-D arrangement with planar boundaries and rod beams.



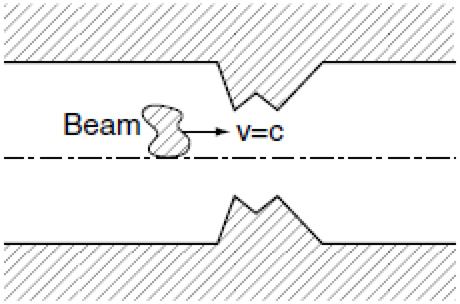
Following similar derivation as P-W theorem, one obtains an interesting result:

$$\frac{\partial \Delta p_y}{\partial y} = \frac{\partial \Delta p_y}{\partial D} = \frac{\partial \Delta p_z}{\partial y} = 0$$

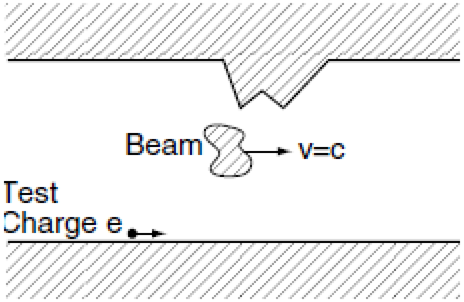
Planar wake theorem

It follows from this Planar Wake theorem that

(1) If the 2-D boundaries have an up-down symmetry, then there will be no transverse wake!



(2) If one side of the 2-D boundaries is a perfect plane, there will be no longitudinal wake!



(3) Unfortunately there is no 2-D design that eliminates both transverse and longitudinal wakes.