Crossing Statistic

Arman Shafieloo

Theoretical Physics, University of Oxford

16th September 2010

UniverseNet Meeting – Lecce, Italy

Minimization of reduced Chi square or effective Chi square is the most common approach in cosmology (and many other fields of science) to do parameter estimation and also being used in model selection.

$$\chi^2 = \sum_{i}^{N} \frac{(\mu_i^t - \mu_i^e)^2}{\sigma_i^2},$$

Consistency of a model and the data:

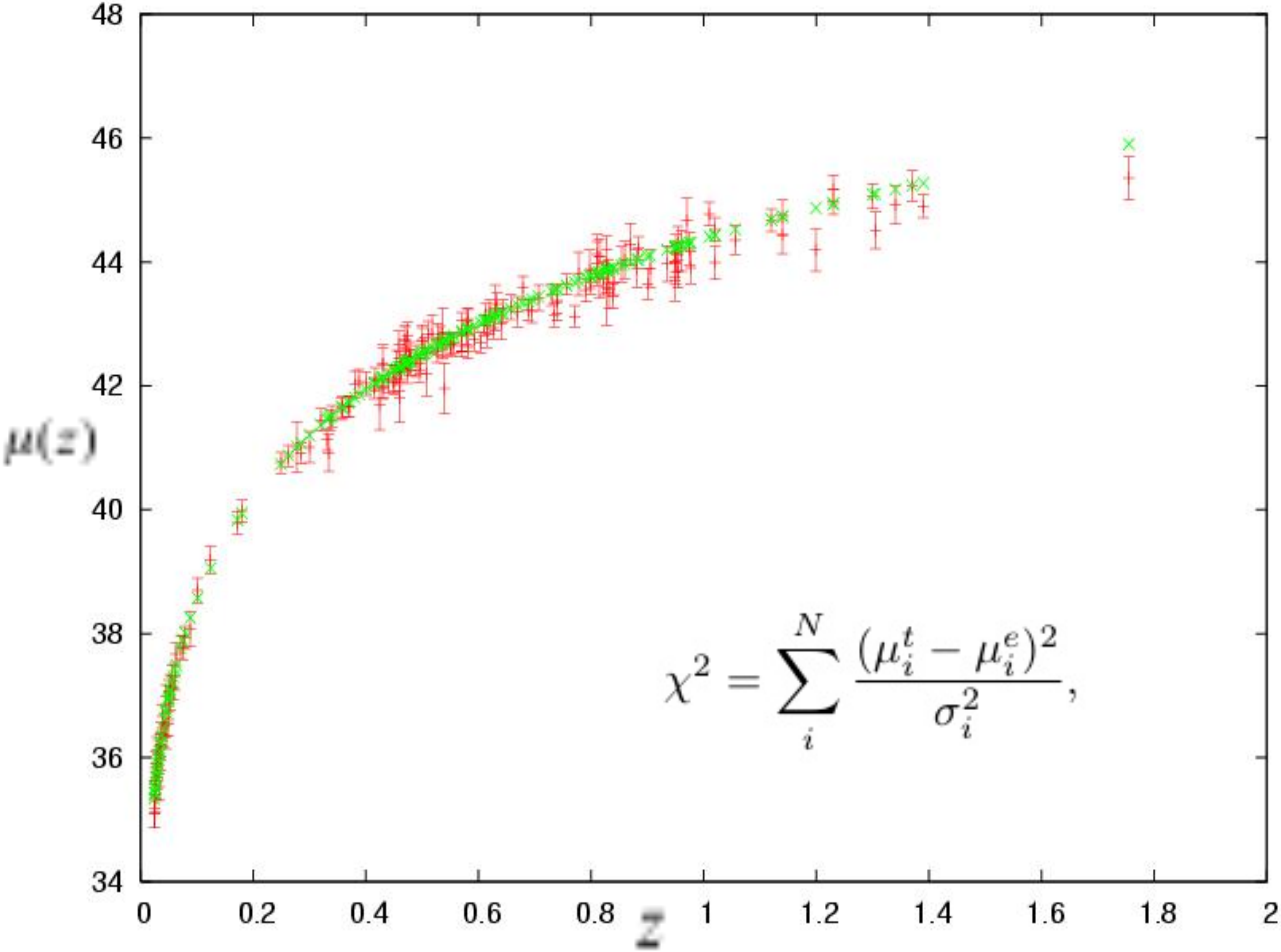
Frequentist Approach:

Assuming a proposed model, the probability of the observed data must not be insignificant.

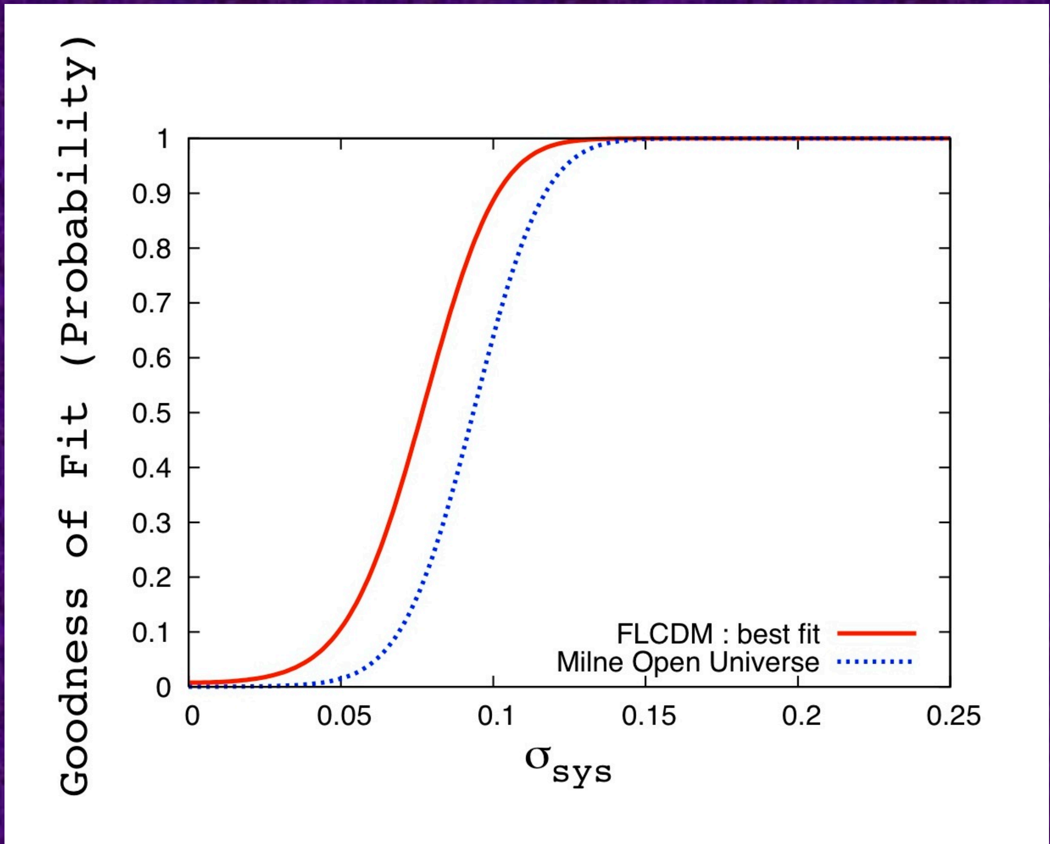
Bayesian Approach:

Priors and simplicity of the proposed model also matters (in model comparison)

Chi square analysis plays a crucial role in calculation of the likelihood in both approaches


48
46
44
42
40
38
36
34
$\mu(z)$
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
z
$\chi^2 = \sum_i^N \frac{(\mu_i^t - \mu_i^e)^2}{\sigma_i^2},$

What if the actual size of the error bars are not known?



Constitution Supernovae data (2009)

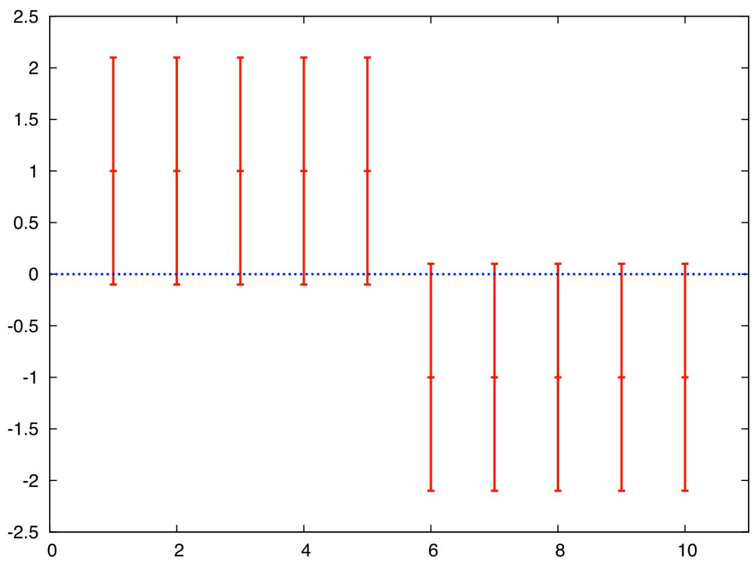
$$\chi^2 = \sum_i^N \frac{(\mu_i^t - \mu_i^e)^2}{\sigma_i^2},$$

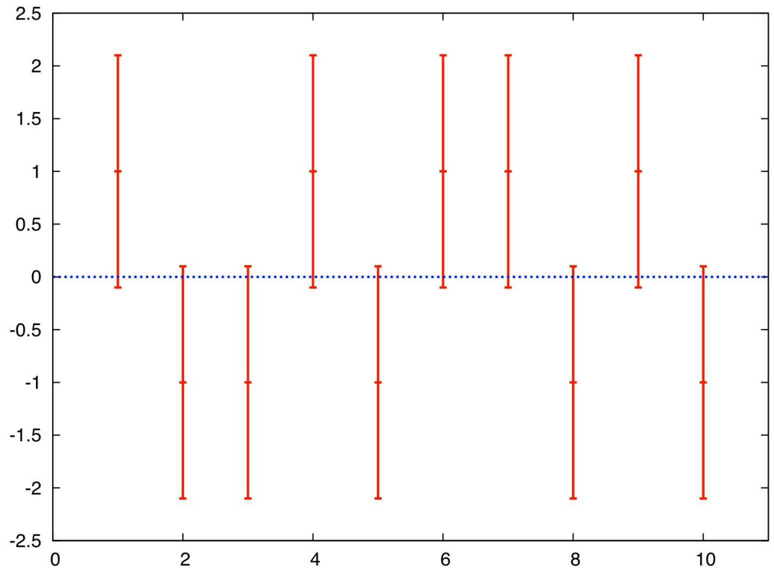
$$\sigma_i^2 = \sigma_{i(data)}^2 + \sigma_{(sys)}^2$$

$$P(\chi^2; N) = \frac{2^{-N/2}}{\Gamma(N/2)} \chi^{N-2} e^{-\chi^2/2}$$

$$Prob(\chi^2; N) = \int_{\chi^2}^{\infty} P(\chi^2; N) d\chi'^2.$$

Equal in being probable?!



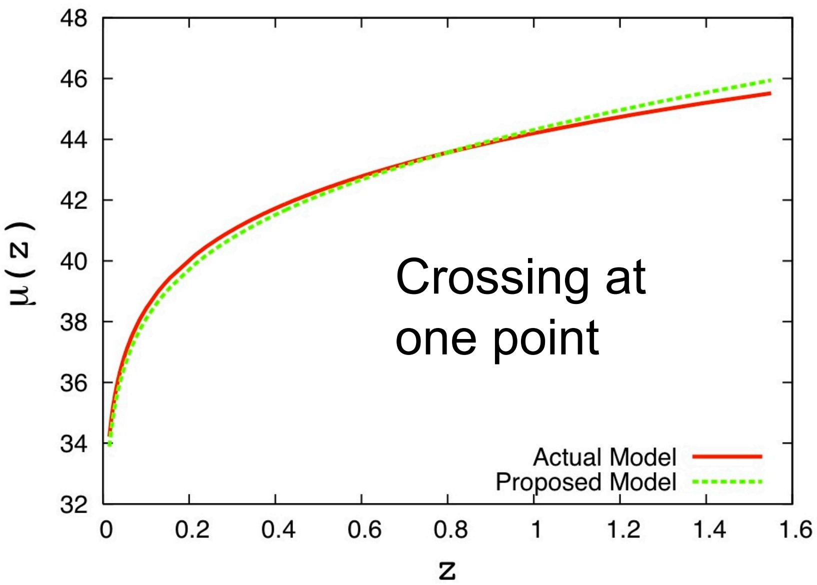


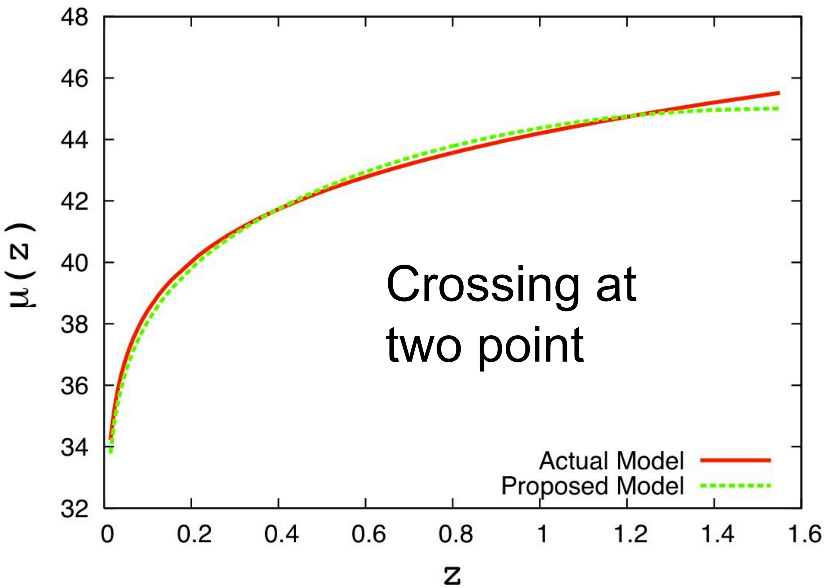
$$\chi^2 = \sum_i^N \frac{(\mu_i^t - \mu_i^e)^2}{\sigma_i^2},$$

Crossing Statistic

If a proposed model is different than the actual model, then they cross each other at one or two or three or … N points.

A. Shafieloo et al, 2010.





One point Crossing: T1

1. Assume a model

2. Construct the normalized residuals

$$q_i(z_i) = \frac{\mu_i(z_i) - \bar{\mu}(z_i)}{\sigma_i(z_i)}.$$

3. Finding the crossing point and calculating T1 by maximizing T(n1):

$$T(n_1) = Q_1(n_1)^2 + Q_2(n_1)^2,$$

$$Q_1(n_1) = \sum_{i=1}^{n_1} q_i(z_i)$$

$$Q_2(n_1) = \sum_{i=n_1+1}^{N} q_i(z_i),$$

4. Comparing the results with Monte Carlo simulations.

Two points Crossing: T2

1-2.....

3. Finding the crossing points and calculating T2 by maximizing T(n1,n2):

$$q_i(z_i) = \frac{\mu_i(z_i) - \bar{\mu}(z_i)}{\sigma_i(z_i)}.$$

$$T(n_1, n_2) = Q_1(n_1, n_2)^2 + Q_2(n_1, n_2)^2 + Q_3(n_1, n_2)^2,$$

4. Comparing the results with Monte Carlo simulations.

$$\begin{aligned} Q_1(n_1, n_2) &= \sum_{i=1}^{n_1} q_i(z_i) \\ Q_2(n_1, n_2) &= \sum_{i=n_1+1}^{n_2} q_i(z_i) \\ Q_3(n_1, n_2) &= \sum_{i=n_2+1}^{N} q_i(z_i). \end{aligned}$$

And so on we can derive T3, T4,…

Important Features:

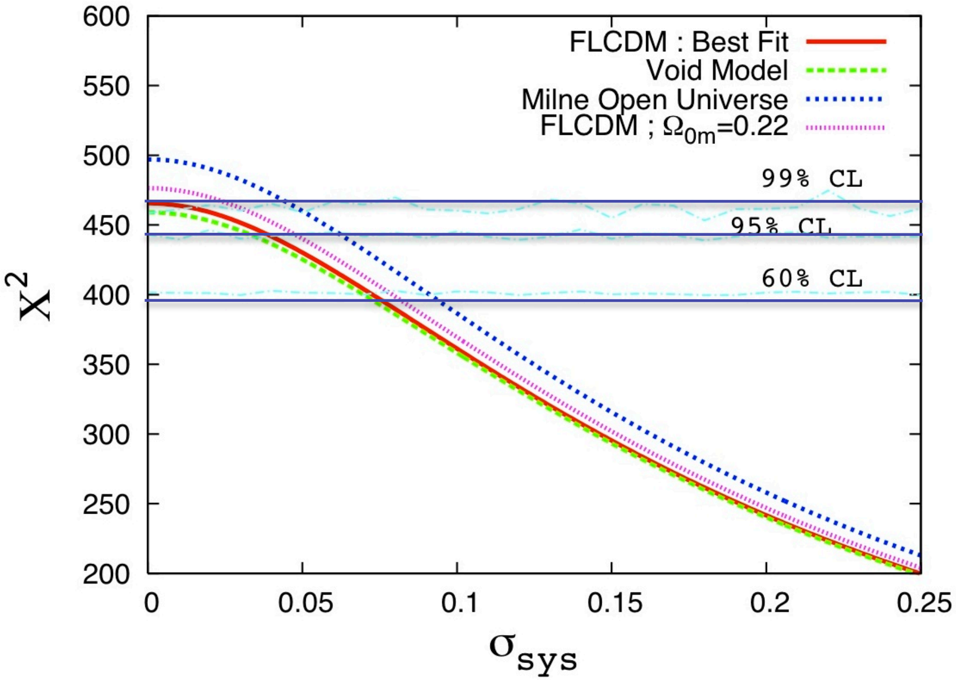
For N data points, **the last mode of Crossing Statistic** is T(N-1) which **is identical to Chi Square Statistic**

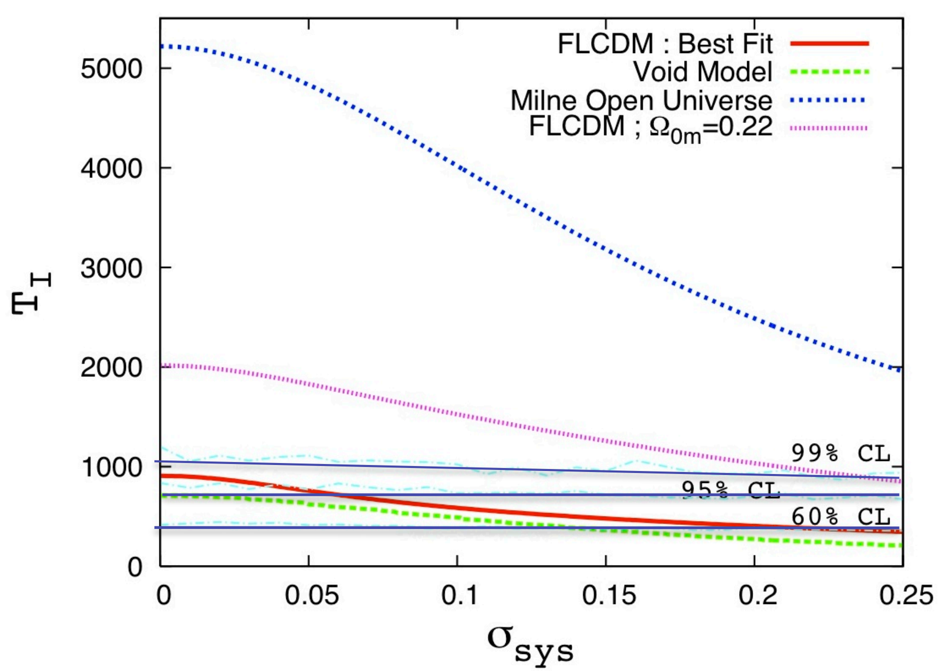
$$T_{N-1} = \sum_{i}^{N} (q_i)^2 = \chi^2$$

The zero mode of Crossing Statistic is similar to Median Statistic (Gott et al 2001)

$$T_0 = (\sum_{i}^{N} q_i)^2$$

Some Applications:



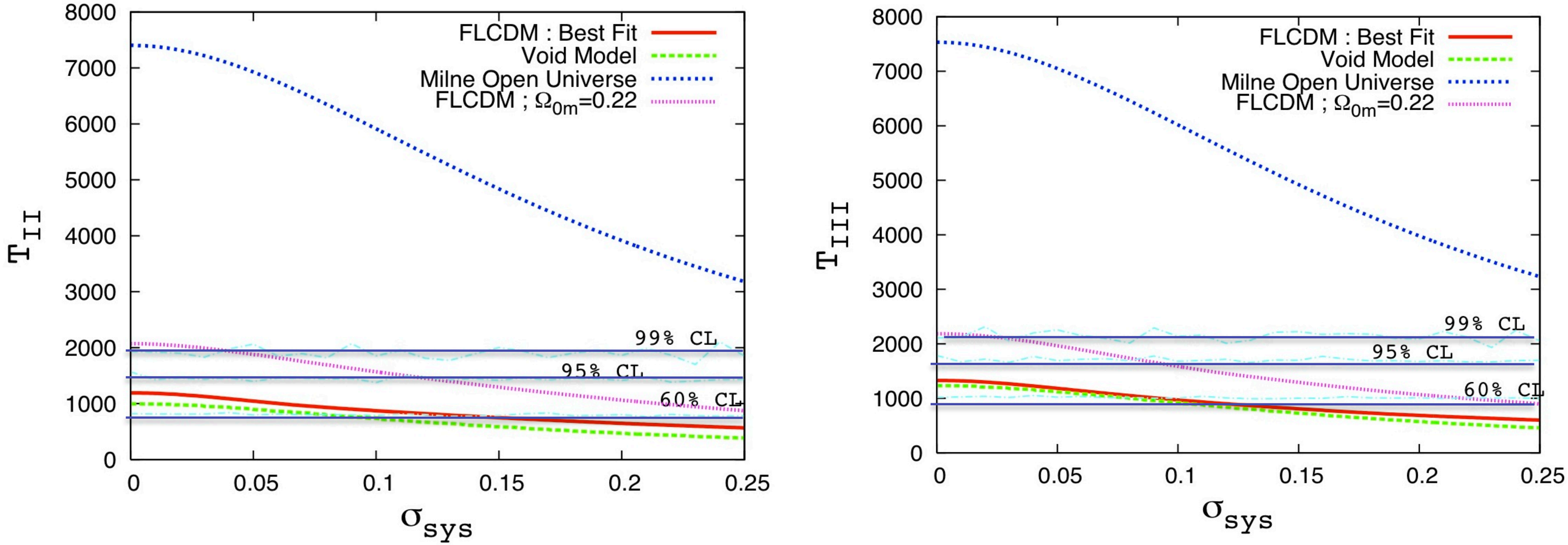


Constitution
Supernovae data

$$\sigma_i^2 = \sigma_{i(data)}^2 + \sigma_{(sys)}^2$$

A. Shafieloo et al, 2010.

Some Applications:



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Supernovae data

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A. Shafieloo et al, 2010.

Summary:

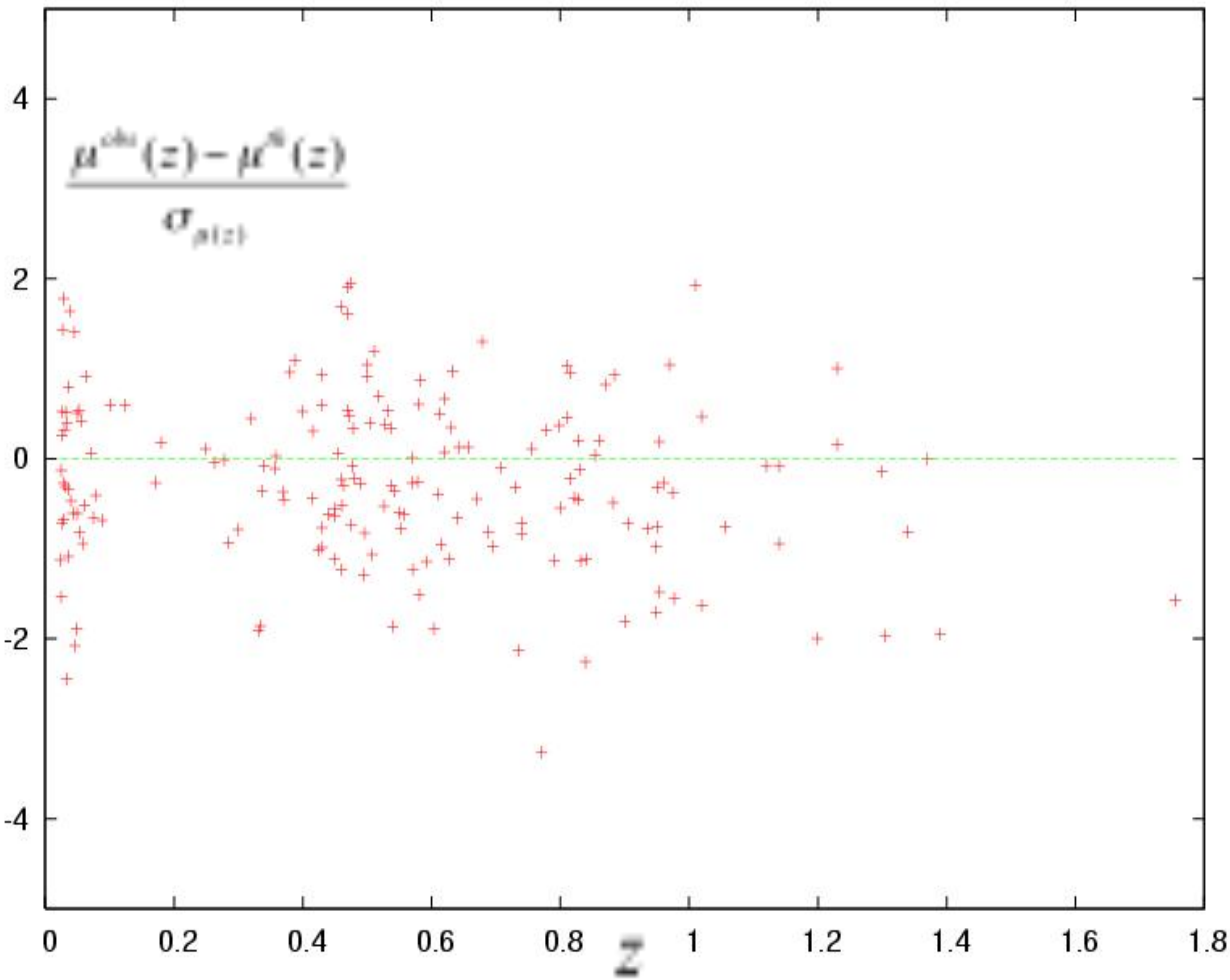
- Designed to be used in situations where the intrinsic dispersion of a data set is not well known (e.g supernovae data). ***Crossing statistic is in general less sensitive than χ2 to the intrinsic dispersion of the data***.
- ***Crossing Statistic can easily distinguish between different models in cases where the χ2 statistic fails.***
- The last mode of Crossing Statistic is identical to χ2, so that one can consider it as a ***generalization of χ2***. In fact we are extracting more information from the data by using Crossing Statistic.

Dealing with observational uncertainties in matter density

- Small uncertainties in the value of matter density may affect the reconstruction exercise quiet dramatically.
- Hubble parameter is not affected to a very high degree by the value of matter density.
- Any uncertainties in matter density is bound to affect the reconstructed w(z).

$$H(z) = \left[\frac{d}{dz}\left(\frac{d_L(z)}{1+z}\right)\right]^{-1}$$

$$\omega_{DE} = \frac{\left(\frac{2(1+z)}{3}\frac{H'}{H}\right) - 1}{1 - \left(\frac{H_0}{H}\right)^2 \Omega_{0M}(1+z)^3}$$


4
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1.6
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z

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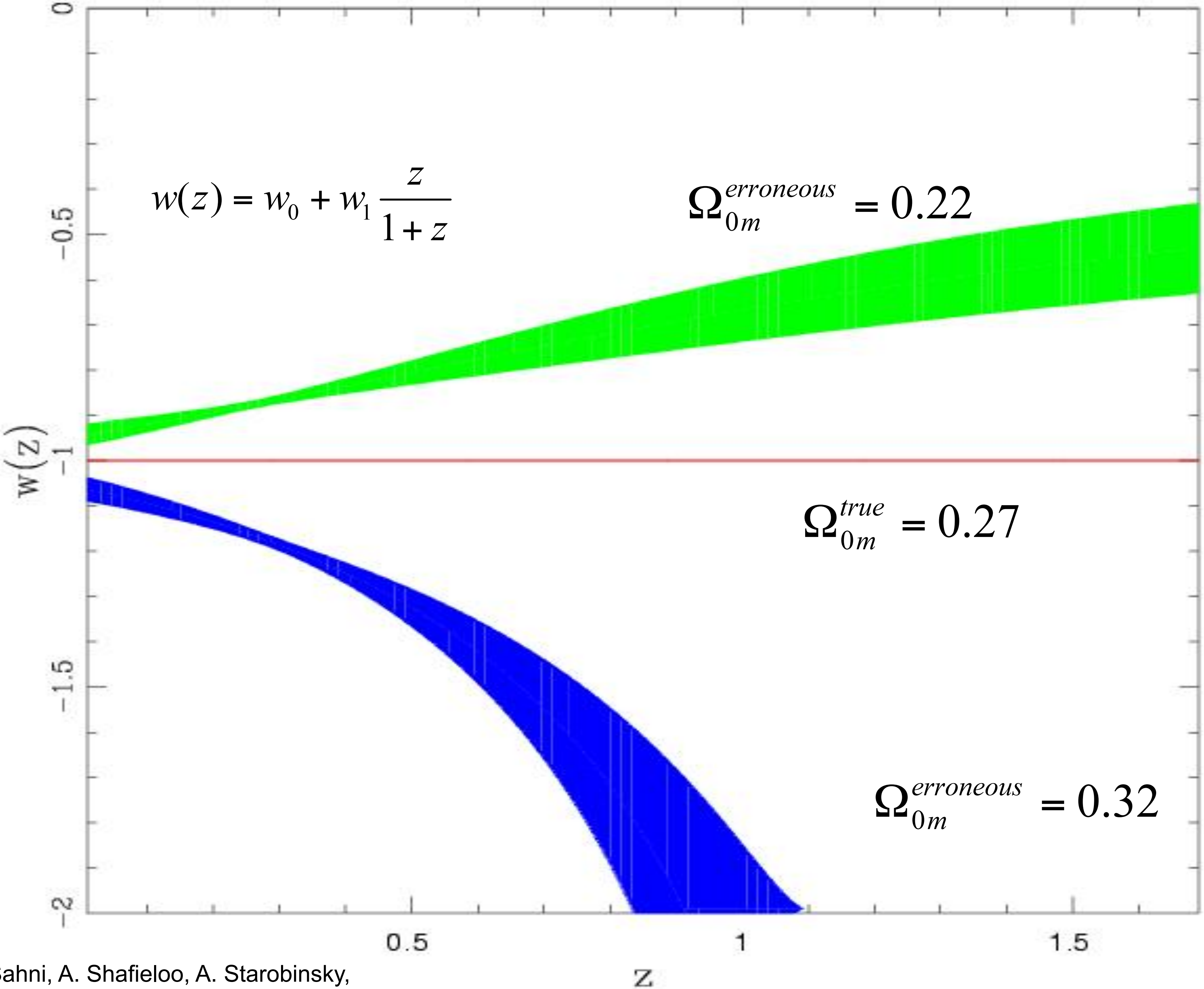
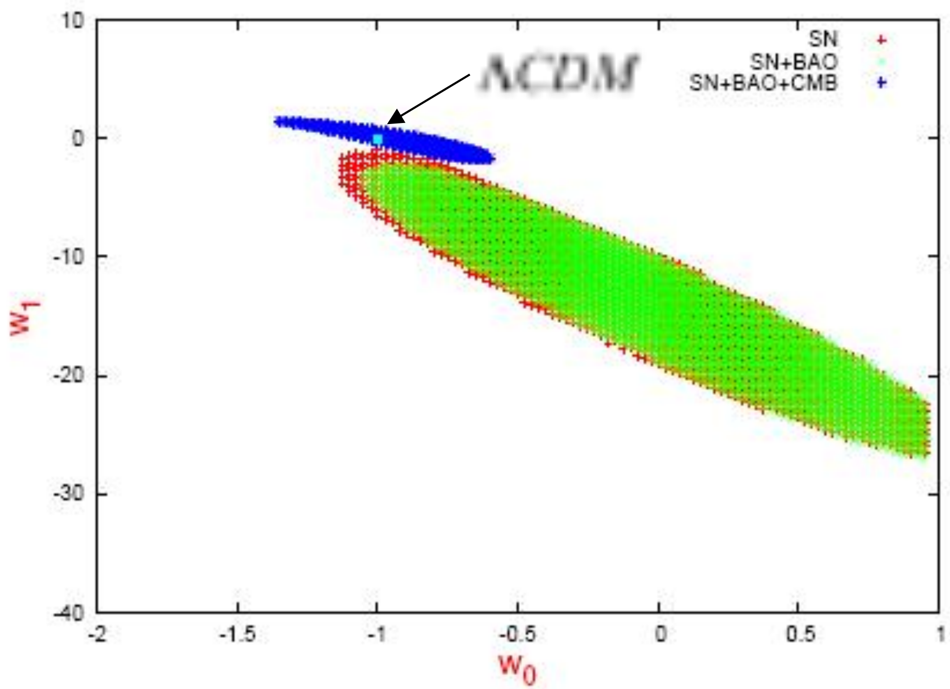
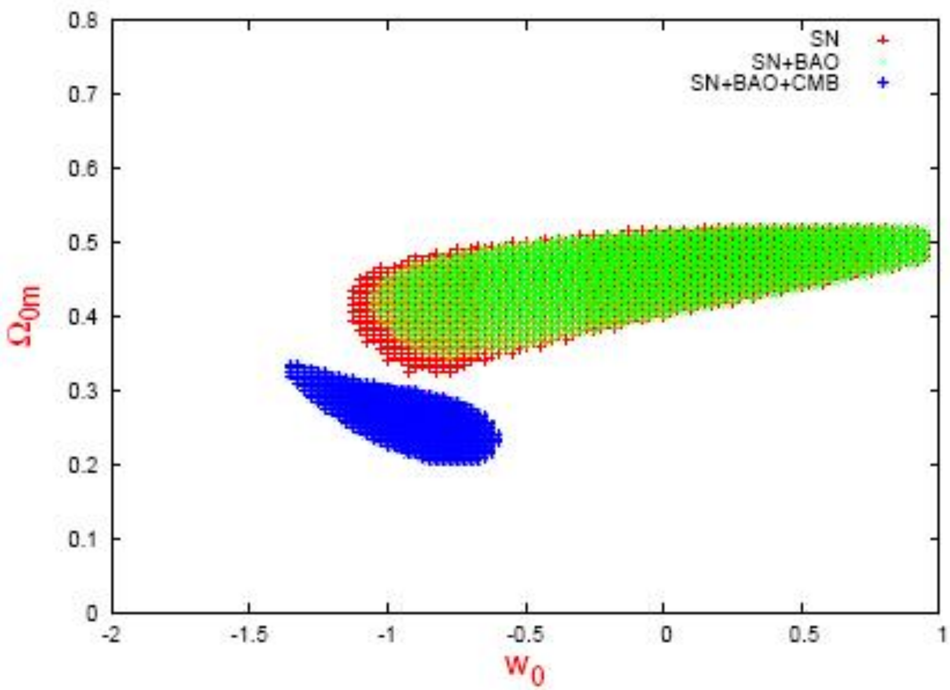


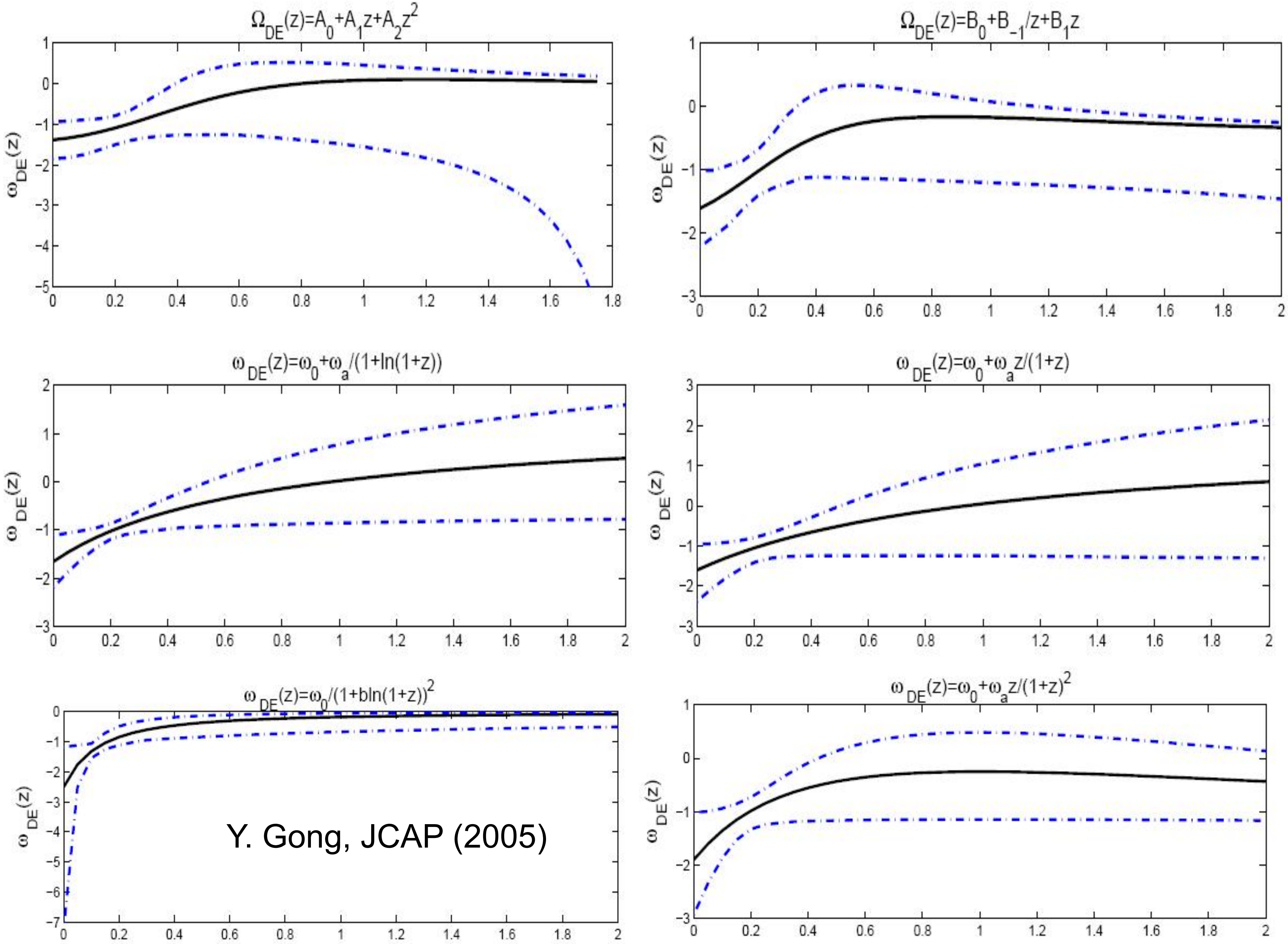
V. Sahni, A. Shafieloo, A. Starobinsky,
Phys. Rev. D (2008)

CPL parameterization strains to fit the data simultaneously at low and high redshifts.





A. Shafieloo, V. Sahni, A. Starobinsky, Phys. Rev. D Rapid Communication (2009)


$\Omega_{DE}(z)=A_0+A_1z+A_2z^2$
$\Omega_{DE}(z)=B_0+B_{-1}/z+B_1z$
$\omega_{DE}(z)=\omega_0+\omega_a/(1+\ln(1+z))$
$\omega_{DE}(z)=\omega_0+\omega_a z/(1+z)$
$\omega_{DE}(z)=\omega_0/(1+b\ln(1+z))^2$
$\omega_{DE}(z)=\omega_0+\omega_a z/(1+z)^2$
$\omega_{DE}(z)$
Y. Gong, JCAP (2005)

Agnostic Approach
Binned Normalized Difference Statistics (BND)

- Directly applicable on the distance moduli data.
- "yes-no" statistic for each model. No comparison with alternative models or parameterizations.
- Focuses on specific features of the data with respect with best fit model.
- Insensitive to the uncertainties of the matter or curvature densities.

Method

1. Assume a model. Obtain the best fit parameters of the model to the data and the corresponding distance moduli.

   $$\boxed{\bar{\mu}(z)}$$

2. Construct the “error normalized difference”

   $$\boxed{q_i(z_i) = \frac{\mu_i(z_i) - \bar{\mu}(z_i)}{\sigma_i(z_i)}}$$

3. Construct “binned normalized difference”

   $$\boxed{Q(N) = C_N \sum_{i=1}^{N} q_i(z_i)}$$

4. Increase the bin size “N” until “Q(N)” changes sign for the first time.

   $$\boxed{N = N_C ; z = z_C}$$

5. Generate many realization of the data, assuming the model with its best fit parameters as the fiducial model.

6. Repeat the analysis for each realization of the data.

7. Find out the fraction of realizations leading to redshift of crossing less than or equal to the redshift of crossing in the actual case.

   $$\boxed{z_{sim} \le z_c}$$

Results

- Models:

Phantom Divide Line (PDL) $w_0 = -1.4 \quad w_1 = 2$

Lambda Cold Dark Matter (LCDM) $w_0 = -1.0 \quad w_1 = 0$

- Data:

Gold 2006 (182 SnIa)

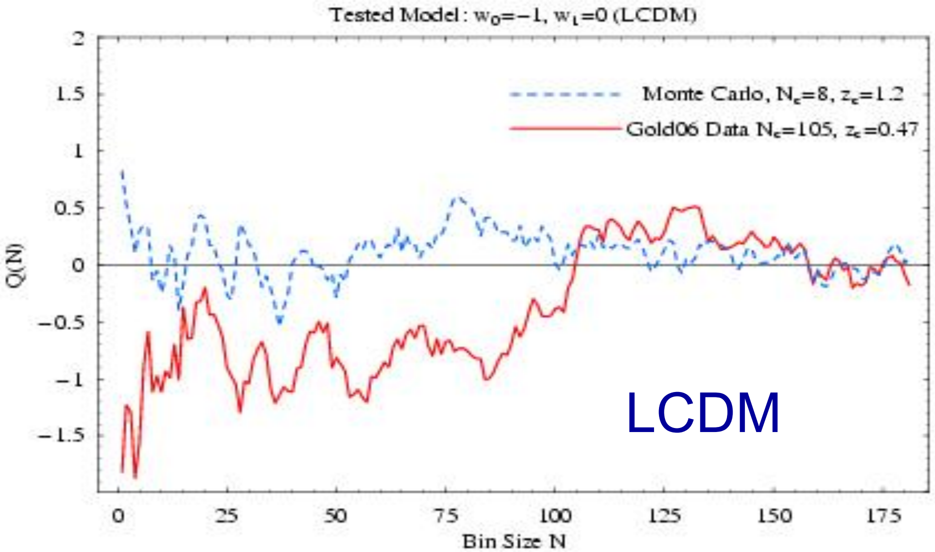
Union 2008 (307 SnIa)

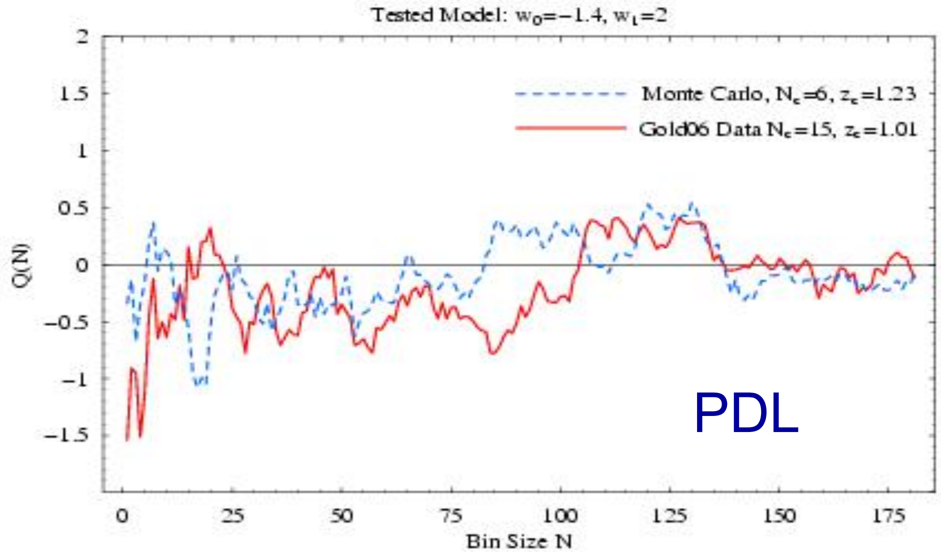
Constitution (397 SnIa)

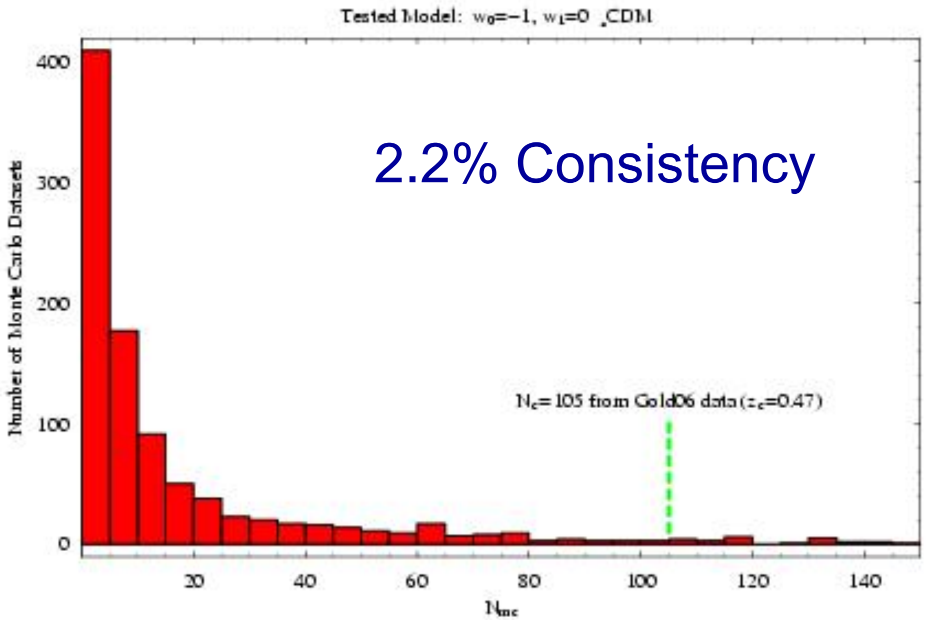
$$w(z) = w_0 + w_1 \frac{z}{1+z}$$

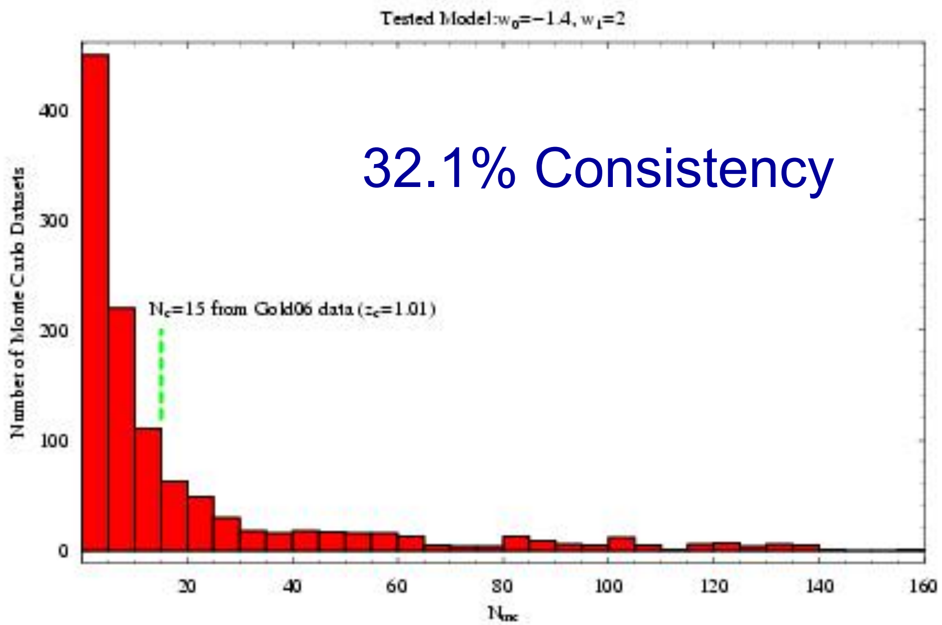
Gold data (2006)

L. Perivolaropoulos & A. Shafieloo Phys. Rev. D (2009)



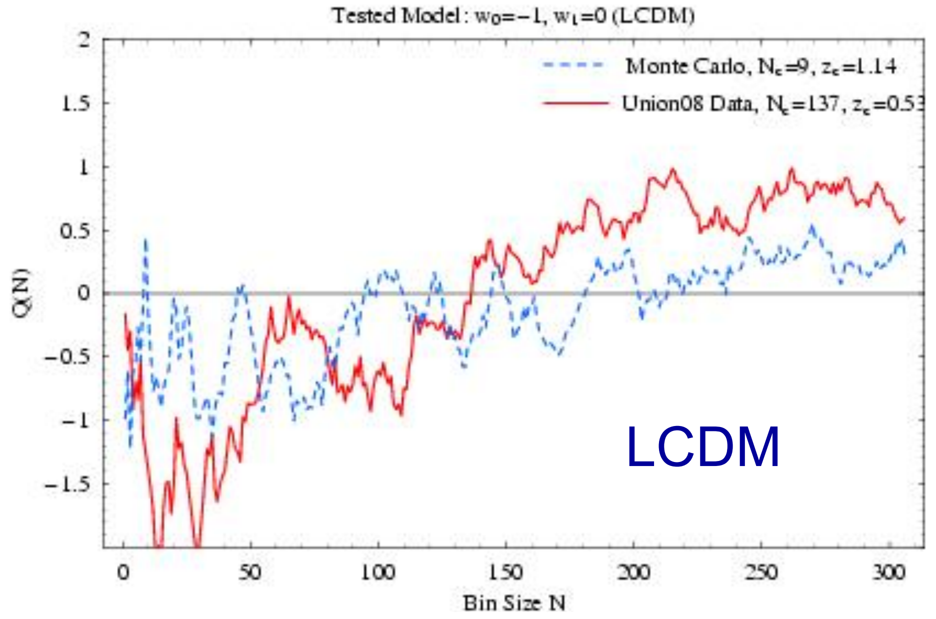


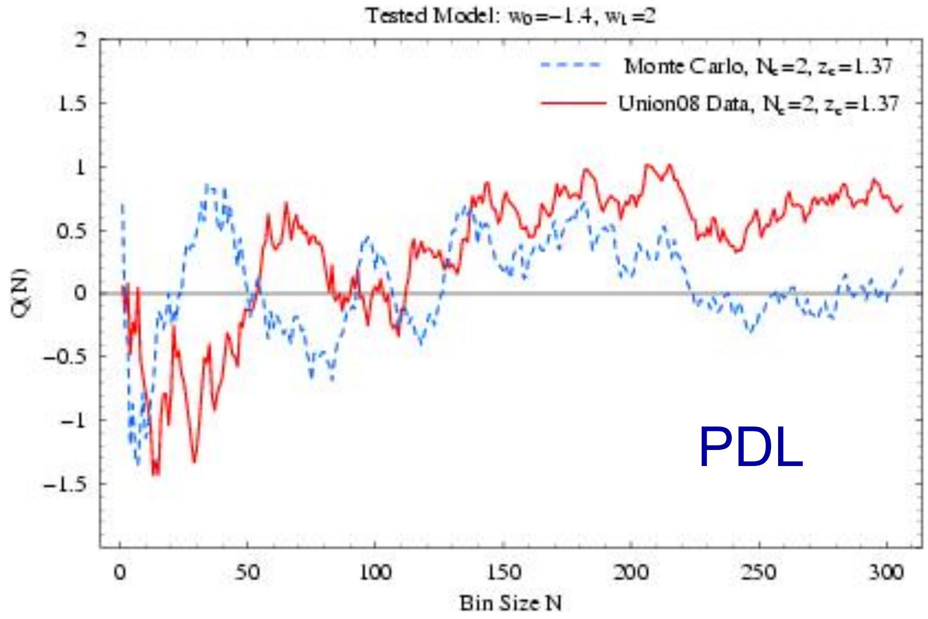


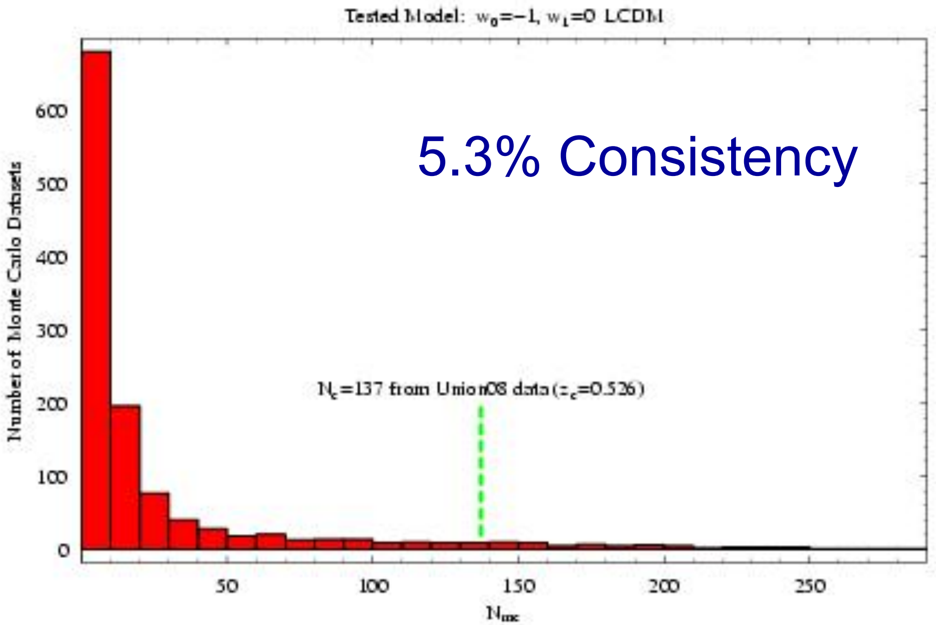


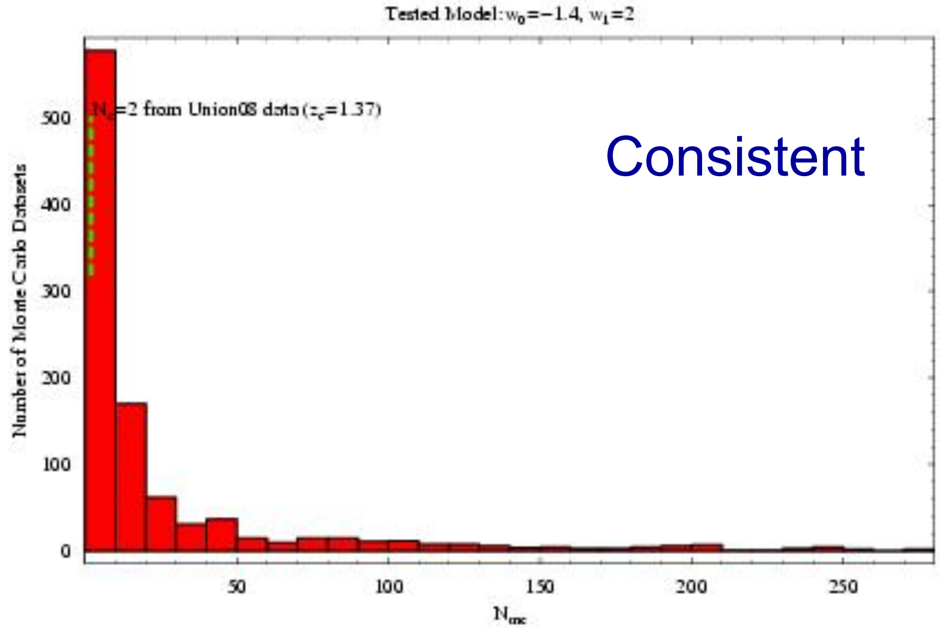
Union data (2008)

L. Perivolaropoulos & A. Shafieloo Phys. Rev. D (2009)

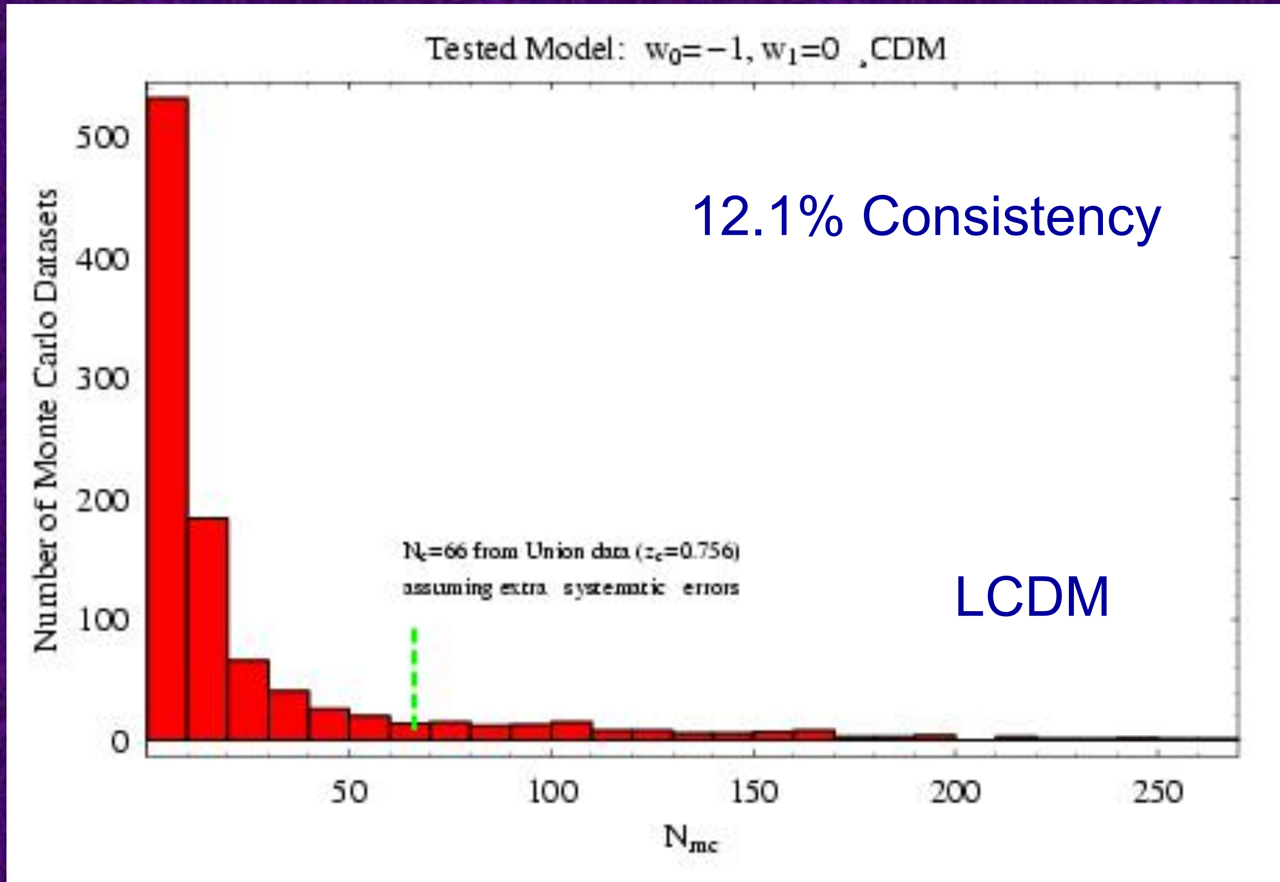








Effect of unknown systematics

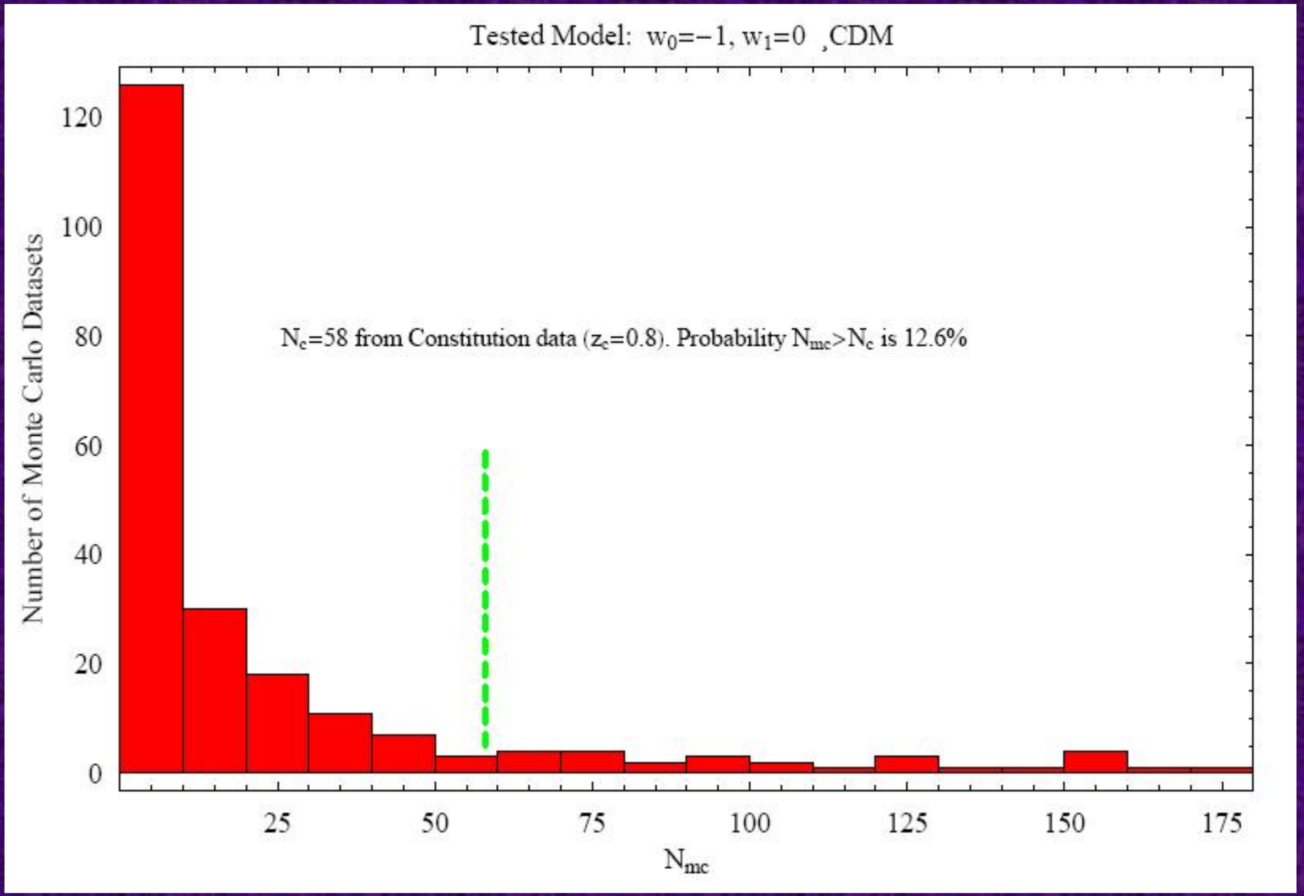


L. Perivolaropoulos & A. Shafieloo Phys. Rev. D (2009)

Union08 Data

Assuming extra systematic errors: $\sigma_{sys} = 0.20$

Constitution data



12.6 % consistency with LCDM

Summary:

- The nature of dark energy is unknown and one of the biggest puzzles of cosmology.
- To study the behavior of dark energy we need to reconstruct the expansion history of the universe.
- Parametric and Non-Parametric approaches are both useful and each has some advantages and some disadvantages over the other one.
- First target can be testing the standard ‘Vanilla’ model. If it is not ‘*Lambda*’ then we can look further.
- It is important to avoid degeneracies and higher derivatives of the data to have a more clear understanding of dark energy.

Thank you

Summary:

- Om can be derived directly from the Hubble parameter and not its derivatives.
- Errors in the reconstruction of Om are smaller than those appearing in the EOS.
- Om is not sensitive to the value of matter density.
- Om can be determined using parametric as well as non-parametric reconstruction methods.
- Om and can be applied on non-uniform data.

- Low redshift supernovae are very important in cosmological parameter estimation.

- CPL parameterization strains to fit the data at low and high redshifts simultaneously.

- If w is assumed to be constant, then |1+w| < 0.06 (1 \sigma) from independent tests. If the latter (not justified) assumption is omitted, many unexpected and interesting things may occur at recent z < 0.3. previously, we thought that this may be temporal phantom DE behaviour, but new data, if interpreted cosmologically, points to just the opposite direction.

- significant increase of "*Om*", "*q*" and "*w*" as compared to the LCDM, possible decay of DE into something else.

Summary:

- According to the BND statistic, Gold06 and Union08 datasets have probability 2.2% and 5.3% to have emerged in the context of LCDM cosmology.

- Inconsistency between the data and LCDM model exist at the high redshifts. At low redshifts BND statistic does not show any inconsistency. The tension must be due to data points at high redshifts that seems to be systematically brighter than LCDM predictions.

- The inconsistency can be interpreted either as:

1. More deceleration at high z than expected in the context of LCDM.
2. Statistical fluctuation.
3. Systematic effect perhaps due to a mild SnIa evolution at high z.

- Our results indicates a potential challenge for LCDM cosmology and provides a motivation for obtaining additional SnIa data at high redshifts $z > 1$.

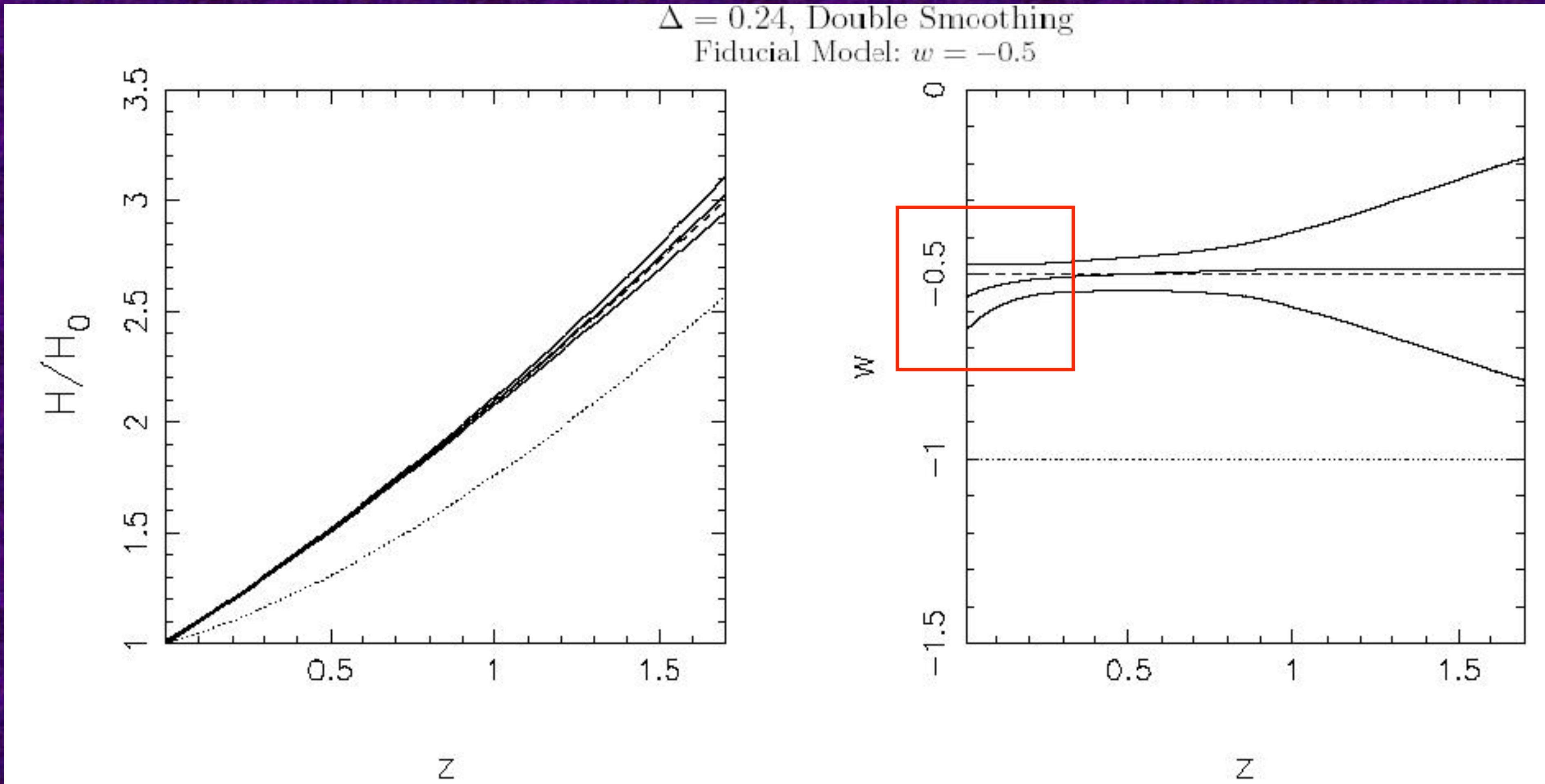
TABLE I: Mean values and marginalized 68% confidence levels for the cosmological parameters. The set of $w(z)_i$ represent the measured values of the dark energy equation of state in uncorrelated redshift bins.

Parameter	WMAP+UNION+BAO	WMAP+Constitution+BAO	all dataset	future datasets
$\Omega_b h^2$	0.02281 ± 0.00057	0.02278 ± 0.00058	0.02304 ± 0.00056	0.02270 ± 0.00015
$\Omega_c h^2$	0.1128 ± 0.0059	0.1144 ± 0.0060	0.1127 ± 0.0018	0.1100 ± 0.0012
Ω_Λ	0.728 ± 0.018	0.715 ± 0.017	0.728 ± 0.016	0.751 ± 0.008
n_s	0.964 ± 0.014	0.963 ± 0.014	0.971 ± 0.014	0.962 ± 0.004
τ	0.085 ± 0.017	0.084 ± 0.016	0.088 ± 0.017	0.084 ± 0.05
Δ^2_R	$(2.40 \pm 0.10) \cdot 10^{-9}$	$(2.40 \pm 0.10) \cdot 10^{-9}$	$(2.40 \pm 0.10) \cdot 10^{-9}$	$(2.40 \pm 0.10) \cdot 10^{-9}$
$w(z = 1.7)$	$--$	$--$	$--$	$-1.55^{+0.46}_{-0.44}$
$w(z = 1)$	$-1.72^{+0.73}_{-0.81}$	$-1.68^{+0.73}_{-0.85}$	$-1.07^{+0.21}_{-0.20}$	-1.03 ± 0.10
$w(z = 0.75)$	$-0.71^{+0.44}_{-0.47}$	$-0.47^{+0.34}_{-0.33}$	$-0.86^{+0.025}_{-0.26}$	-0.98 ± 0.08
$w(z = 0.5)$	$-0.65^{+0.29}_{-0.30}$	$-1.06^{+0.41}_{-0.40}$	-0.86 ± 0.14	-1.00 ± 0.05
$w(z = 0.25)$	-1.05 ± 0.10	-1.04 ± 0.07	-1.00 ± 0.07	-1.00 ± 0.02
$w(z = 0)$	-0.97 ± 0.22	-0.86 ± 0.13	$-1.02^{+0.17}_{-0.18}$	-0.99 ± 0.05
σ_8	0.814 ± 0.055	0.815 ± 0.057	0.810 ± 0.024	0.811 ± 0.012
Ω_m	0.272 ± 0.018	0.285 ± 0.017	0.272 ± 0.016	0.249 ± 0.008
H_0	70.7 ± 2.0	69.4 ± 1.7	70.8 ± 2.0	73.1 ± 1.0
z_{reion}	10.8 ± 1.4	10.8 ± 1.4	11.0 ± 1.5	10.7 ± 0.4
t_0	13.65 ± 0.14	13.67 ± 0.15	13.67 ± 0.13	13.60 ± 0.06

P. Serra et al, arXiv:0908.3186

“No Evidence for Dark Energy Dynamics from a Global Analysis of Cosmological Data”

Refinement of the method:



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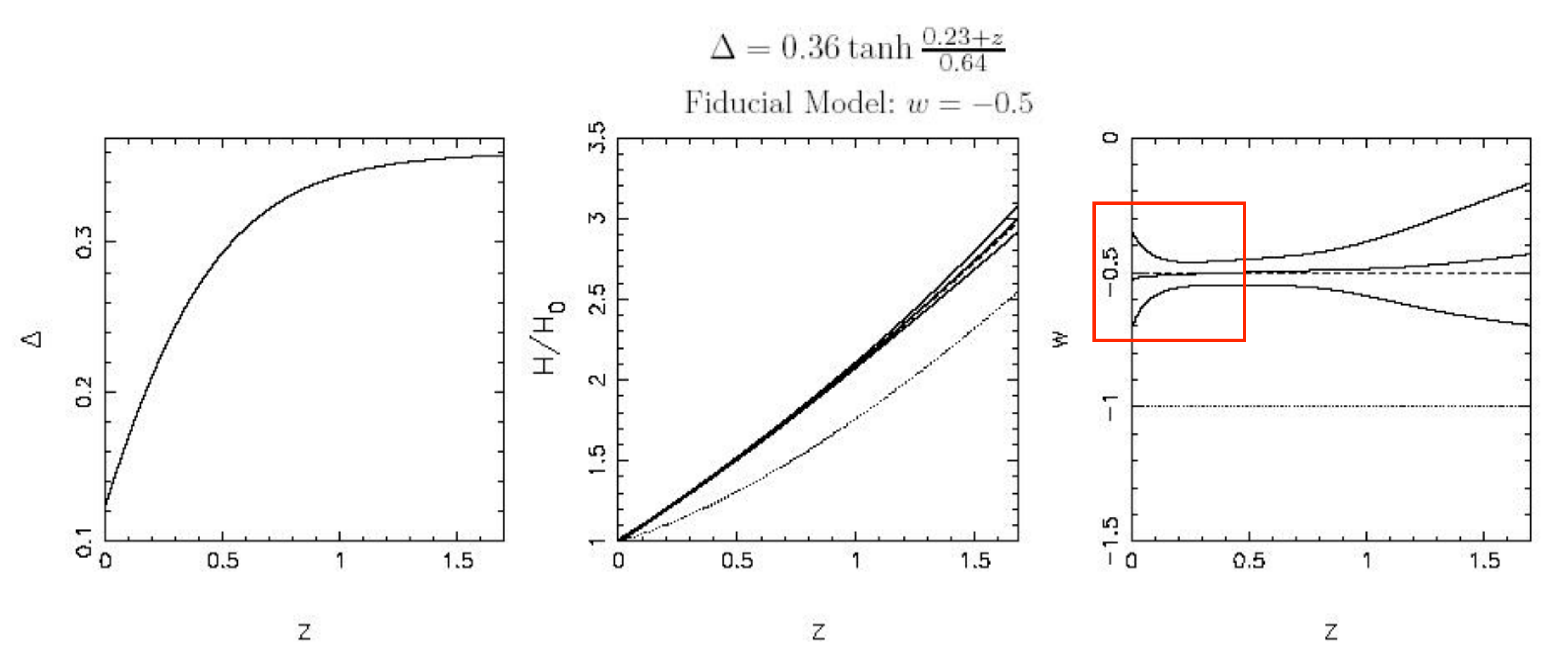


Variable Delta

Accuracy of the results and the relation with the quality and quantity of the data

$$\frac{\Delta h}{h} \propto \frac{\sigma}{N^{1/2}\Delta^{3/2}}$$

$$\frac{\Delta w}{w} \propto \frac{\sigma}{N^{1/2}\Delta^{5/2}}$$

M. Tegmark, PRD 2002

Tegmark et al, astro-ph/9805117

In this method, a large value of delta produces a smooth result, but the accuracy of reconstruction worsens, while a small delta gives a more accurate, but noisy result.

Bias: $\beta \propto \Delta^2$

Shafieloo et al, MNRAS 2006

The Method of Smoothing

error-sensitive

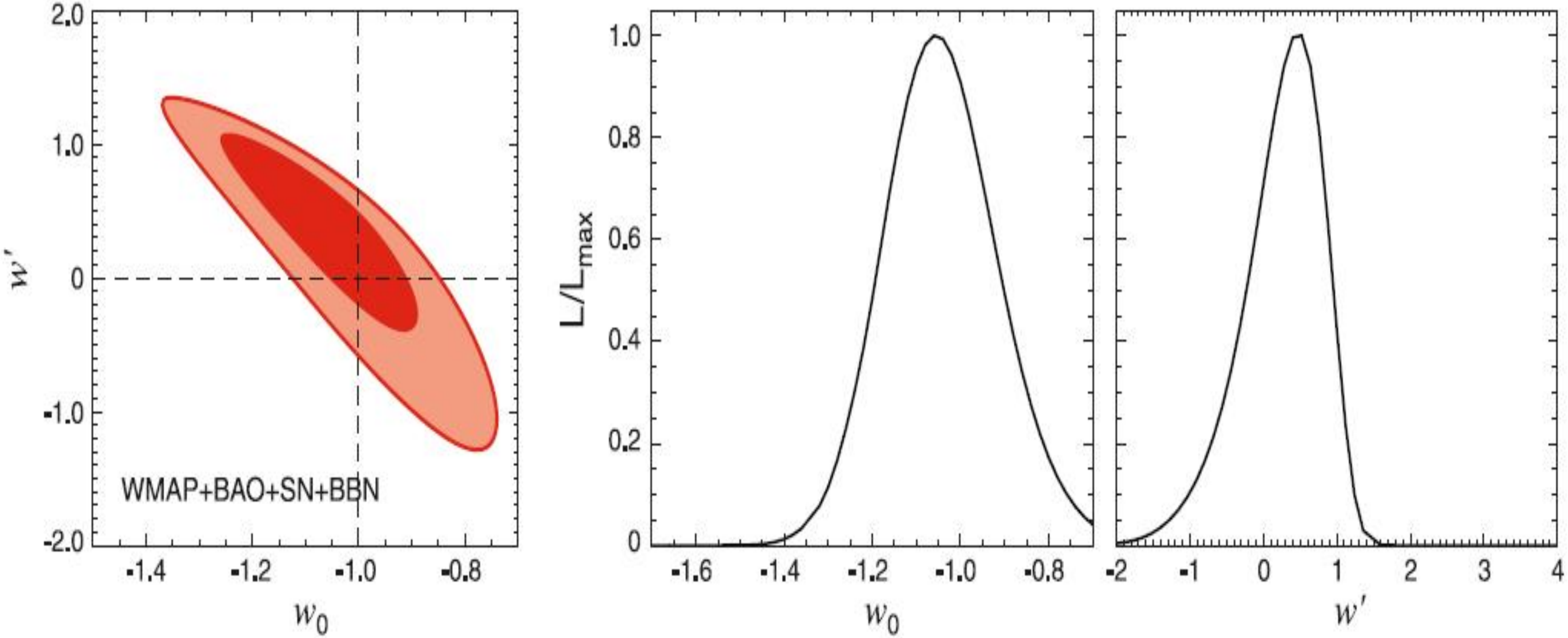
$$\ln d_L(z,\Delta) = \ln d_L(z)^g + N(z)\sum_i [\ln d_L(z_i) - \ln d_L(z_i)^g]\exp\left[-\frac{\ln^2\left(\frac{1+z_i}{1+z}\right)}{2\Delta^2}\right]\left(\frac{1}{\sigma_{\ln d_L(z_i)}}\right)^2$$

$$N(z)^{-1} = \sum_i \exp\left[-\frac{\ln^2\left(\frac{1+z_i}{1+z}\right)}{2\Delta^2}\right]\left(\frac{1}{\sigma_{\ln d_L(z_i)}}\right)^2$$

Smoothing function
(error-sensitive)

$$H(z) = \left[\frac{d}{dz}\left(\frac{d_L(z)}{1+z}\right)\right]^{-1}$$

$$\omega_{DE} = \frac{\left(\frac{2(1+z)}{3}\frac{H'}{H}\right) - 1}{1 - \left(\frac{H_0}{H}\right)^2 \Omega_{0M}(1+z)^3}$$



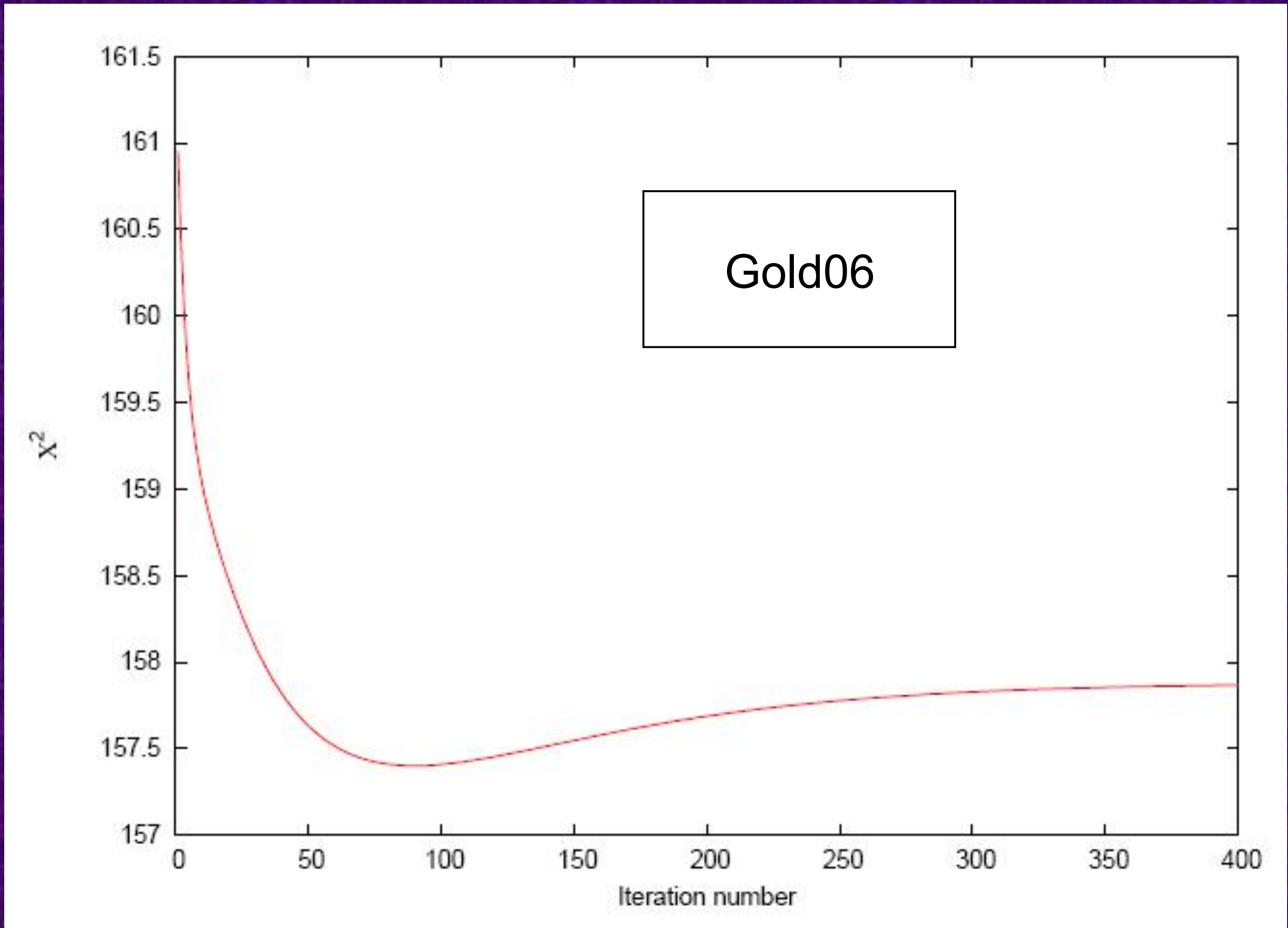
E. Komatsu et al, APJS 2008

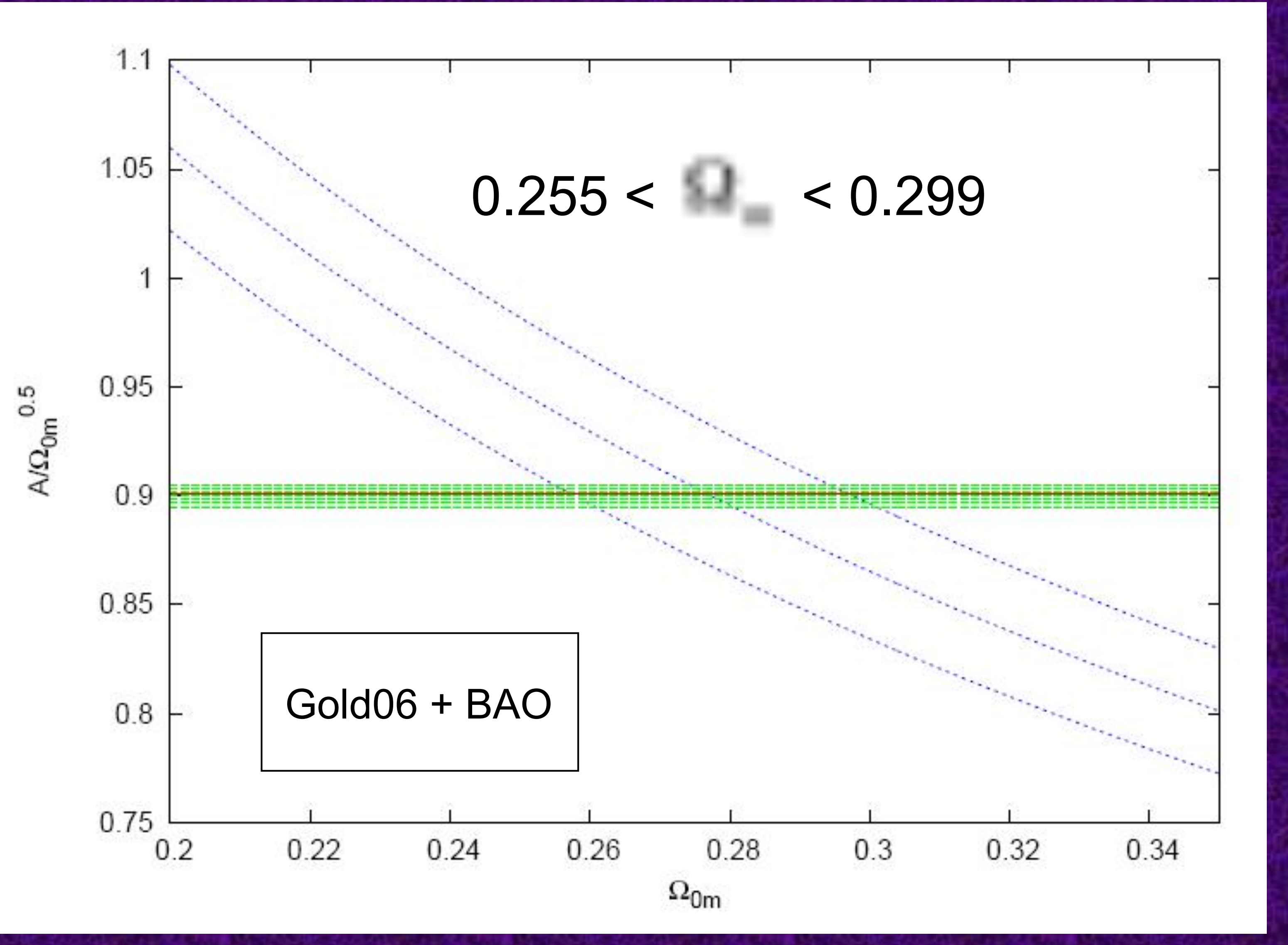
WMAP 5

Gold data, Riess 06

Gold 06

182 SN Ia
0.024 < z < 1.75





A. Shafieloo, MNRAS 2007

Beyond the Standard Model of Cosmology

- The universe may be more complicated than its current standard model (Vanilla Model) which is a flat LCDM with power law form of the primordial perturbation spectrum.
- There might be some extensions to the standard model in defining the cosmological quantities.
- This needs proper investigation, using advanced statistical methods, high performance computational facilities and high quality observational data.

Some of the extensions to the standard model:

- Properties of dark energy.
- Non power-law form of the primordial spectrum.
- More complicated reionization process.
- Topology of the universe.
- Properties of dark matter.
- Varying universal constants.
- …

Probes of Dark Energy

- ***Standard candles***:
  measure luminosity distance.

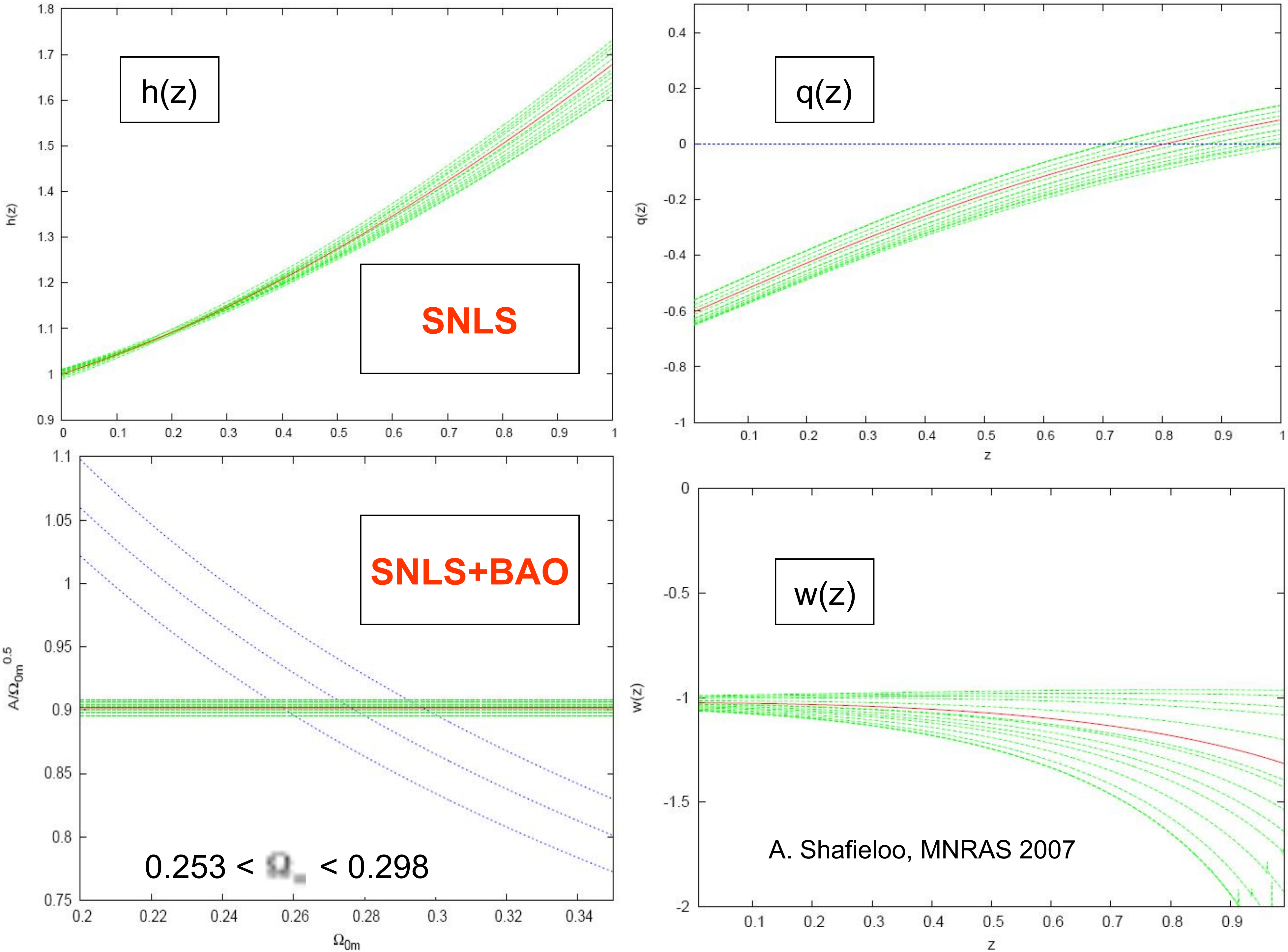
$$d_L(z) = (1+z)\int_0^z \frac{dz'}{H(z')}$$

- ***Standard rulers***:
  measure angular diameter distance.

$$d_A(z) = \frac{d_L(z)}{(1+z)^2}$$

$$\frac{H(z)}{H(0)} = \left[\frac{(1+z)^2 \delta'^2(0)}{\delta'^2(z)} - 3\Omega_m \frac{(1+z)^2}{\delta'^2(z)} \int_0^z \frac{\delta\,|\delta'|}{1+z}\,dz\right]^{\frac{1}{2}}$$

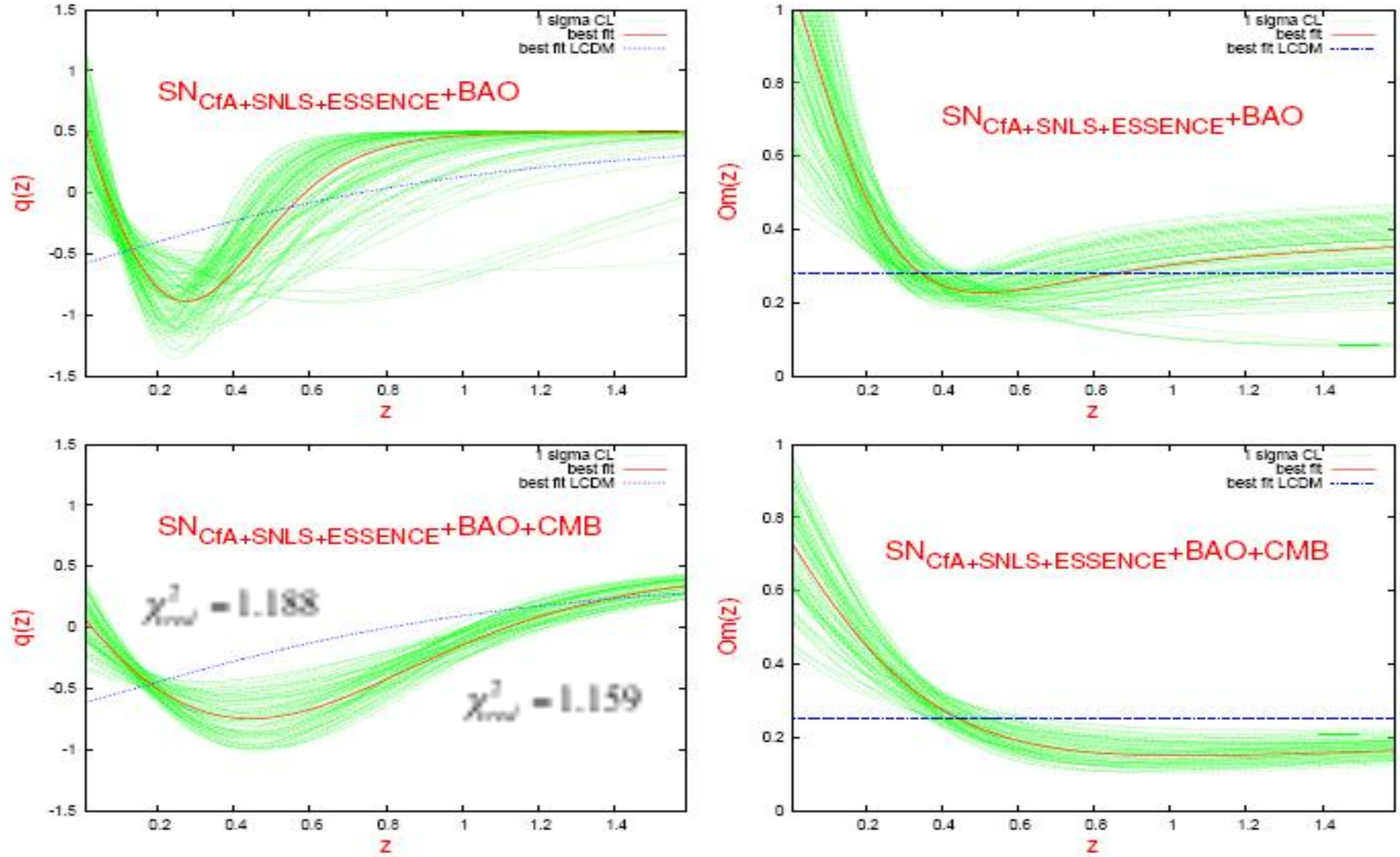
- ***Growth of fluctuations***:
  testing modified gravity or to distinguish between physical and geometrical models of Dark Energy.



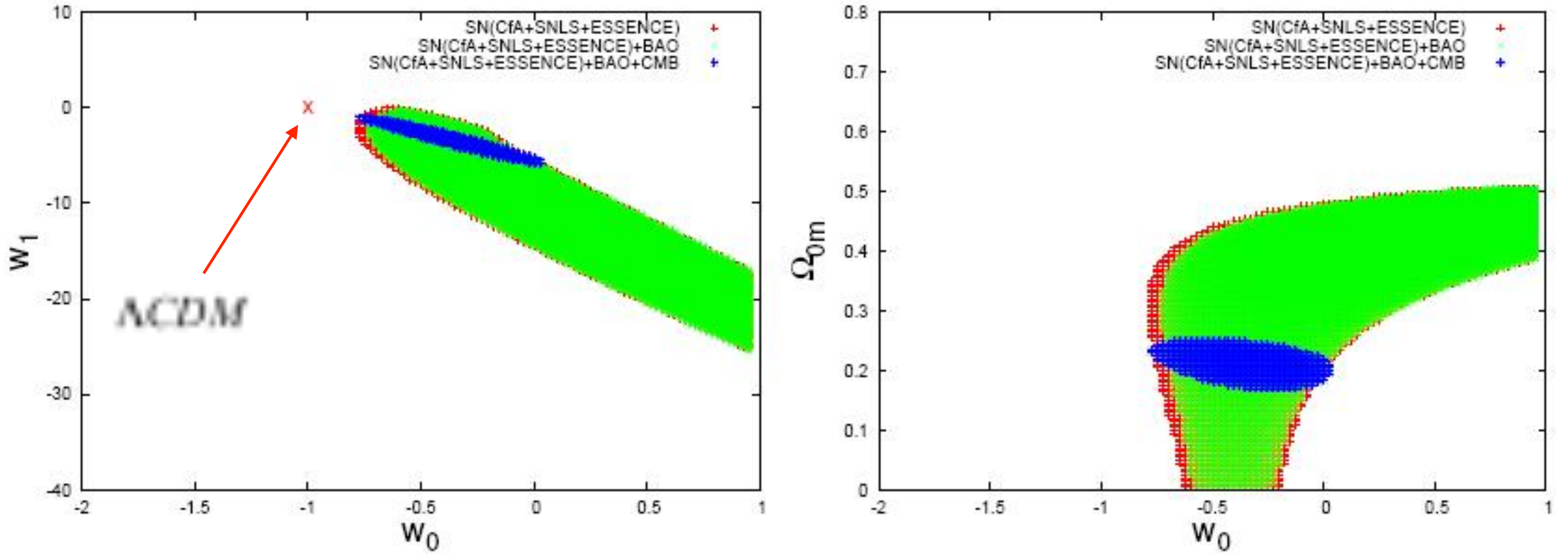
h(z)
SNLS
q(z)
z
SNLS+BAO
$A/\Omega_{0m}^{0.5}$
Ω_{0m}
0.253 < Ω < 0.298
w(z)
A. Shafieloo, MNRAS 2007
z

Consistency Check:

SNLS+ESSENCE+CfA SN Ia + BAO + CMB



Consistency Check



A. Shafieloo, V. Sahni, A. Starobinsky, Phys. Rev. D Rapid Communication 2009