Moment transport equations for non-Gaussianity (with many fields)

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Motivation and background

- As observations improve (PLANCK) we are driven to consider higher order statistics
- Multi-field models have been proposed which give rise to a large non-Gaussianity

  e.g. Byrnes et al. (2008), Kim, Liddle and Seery (2010)
- A number of methods for calculating local non-Gaussianity in multi-field models currently exist, why do we need another one?

  n.b. Lyth and Rodriguez (2005), Rigopoulos et al. (2006), Vernizzi and Wands (2006), Choi et al. (2007), Battefeld and Easter (2007), Yokoyama et al. (2008), Langlois and Vernizzi (2007)

  – Favoured δN method numerically inconvenient
  – δN gives no direct evolution equation for the level of non-Gaussianity

- How do we measure non-Gaussianity?
  - ζ is the curvature perturbation on uniform density hypersurfaces
  - The non-Gaussianity of ζ is often measured by

$$f_{\rm nl} = \frac{5}{18}\frac{\langle\zeta\zeta\zeta\rangle}{\langle\zeta\zeta\rangle^2}$$

- How do we calculate $\langle\zeta\zeta\zeta\rangle$ and $\langle\zeta\zeta\rangle$?
  - δN method

$$\zeta(\mathbf{x}) \equiv \delta N(\mathbf{x}) = \sum_{n=1}^{\infty} \frac{1}{n!} \left. \frac{\partial^n N(\varphi_i^\star)}{\partial\varphi_{j_1}^\star \cdots \partial\varphi_{j_n}^\star} \right|_{\varphi_i^\star = \varphi_{i,\mathrm{fid}}^\star} \delta\varphi_{j_1}^\star(\mathbf{x}) \cdots \delta\varphi_{j_n}^\star(\mathbf{x})$$

n.b. Starobinsky (1985), Sasaki and Stewart (1996)

  - Gives

$$f_{\mathrm{nl}} \approx \frac{5}{6} \frac{N_{,i} N_{,j} N_{,ij}}{(N_{,k} N_{,k})^2}$$

Lyth and Rodriguez (2005)

- How might we do things differently?
  - The obvious alternative is to evolve the moments directly
  - But how?

Moment transport equations

– Any joint probability distribution $P(\boldsymbol{x}, \tau)$ satisfies the transport equation (when no interaction)

$$\frac{\partial P(\boldsymbol{x}, \tau)}{\partial \tau} + \frac{\partial}{\partial x_j} \left[u_j P(\boldsymbol{x}, \tau) \right] = 0$$

where

$$u_i = \frac{\mathrm{d}x_i}{\mathrm{d}\tau}$$

- The distribution can be expanded in its moments
- For zero mean, independent, nearly Gaussian variables $\boldsymbol{z}$

$$P(\boldsymbol{z}) = P_g(\boldsymbol{z}) \left(1 + \frac{\alpha^z_{ijk} H_{ijk}}{6} + ...\right)$$

where

$$H_{i_1 i_2 \cdots i_n} = (-1)^n \exp\left(\frac{z_j z_j}{2}\right) \frac{\partial^n}{\partial z_{i_1} \partial z_{i_2} \cdots \partial z_{i_n}} \exp\left(-\frac{z_k z_k}{2}\right)$$

- Can be generalised to $\mathbf{x}$, using $z_i = A^{-1}_{ij}(x_j - \Phi_j)$

– Velocity can be expanded about mean

$$u_i(\boldsymbol{x},\tau) = u_{i0} + u_{i,j}(x_j - \Phi_j) + \frac{1}{2}u_{i,jk}(x_j - \Phi_j)(x_k - \Phi_k) + \cdots$$

where

$$u_{i0} \equiv u_i\big|_{\boldsymbol{x}=\boldsymbol{\Phi}}, \quad u_{i,j} \equiv \left.\frac{\partial u_i}{\partial x_j}\right|_{\boldsymbol{x}=\boldsymbol{\Phi}}, \quad \text{and} \quad u_{i,jk} \equiv \left.\frac{\partial^2 u_i}{\partial x_j \partial x_k}\right|_{\boldsymbol{x}=\boldsymbol{\Phi}}$$

- Ordinary differential equations for the moments
  - Substitute expanded velocity and distribution into transport equation to give

$$\frac{\mathrm{d}\Phi_i}{\mathrm{d}\tau} = u_{i0} + \frac{1}{2}\Sigma_{jk}u_{i,jk}$$

$$\frac{\mathrm{d}\Sigma_{ij}}{\mathrm{d}\tau} = u_{i,k}\Sigma_{kj} + u_{j,k}\Sigma_{ki} + \frac{1}{2}\left(\alpha_{imn}u_{j,mn} + \alpha_{jmn}u_{i,mn}\right)$$

$$\begin{aligned}\frac{\mathrm{d}\alpha_{ijk}}{\mathrm{d}\tau} = u_{in}\alpha_{njk} \quad &+ \quad \Sigma_{jm}u_{i,mn}\Sigma_{nk} \\ &+ \quad \text{cyclic permutations } i \rightarrow j \rightarrow k\end{aligned}$$

  - Identify $\mathbf{x}$ with ζ, ϕ_i. $\langle\zeta\zeta\rangle = \Sigma_{\zeta\zeta}$, $\langle\zeta\zeta\zeta\rangle = \alpha_{\zeta\zeta\zeta}$ etc.

Examples

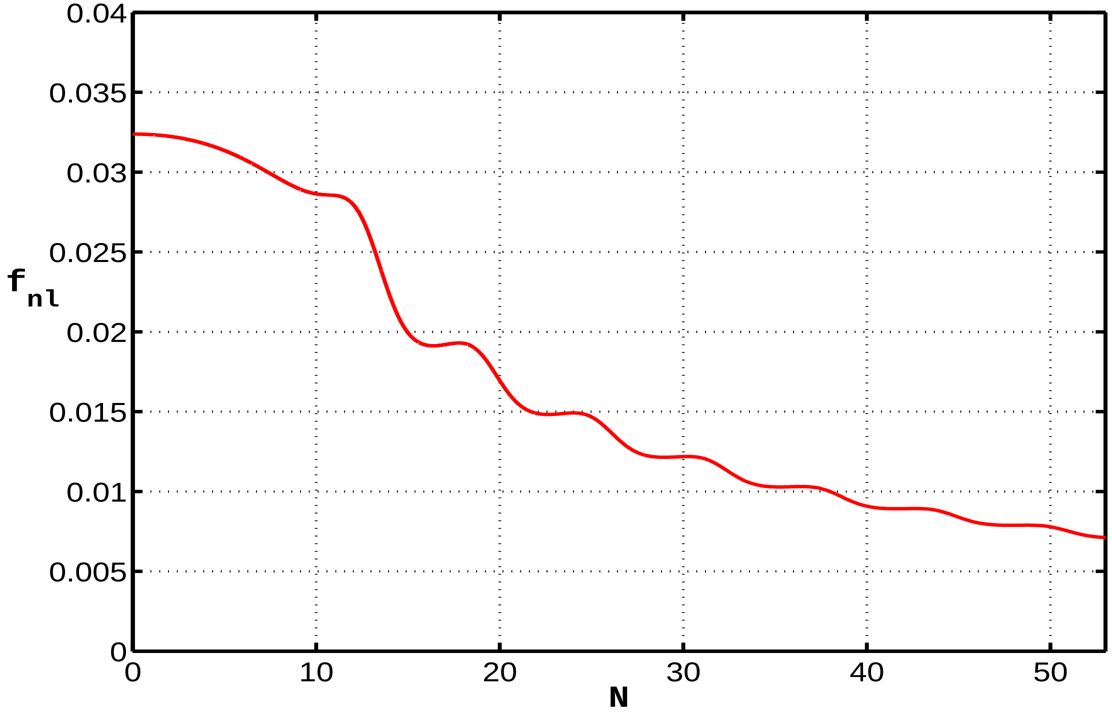
- Method in practice
  - Potential 1

$$V = \sum_i \frac{1}{2} m_i^2 \phi_i^2$$

  - Potential 2

$$V = \sum_i \Lambda_i^4 \left(1 - \cos\left(\frac{2\pi\phi_i}{f_i}\right)\right)$$

Dimopoulos et al. (2005), Kim et al. (20010)



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0.035
0.03
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10
20
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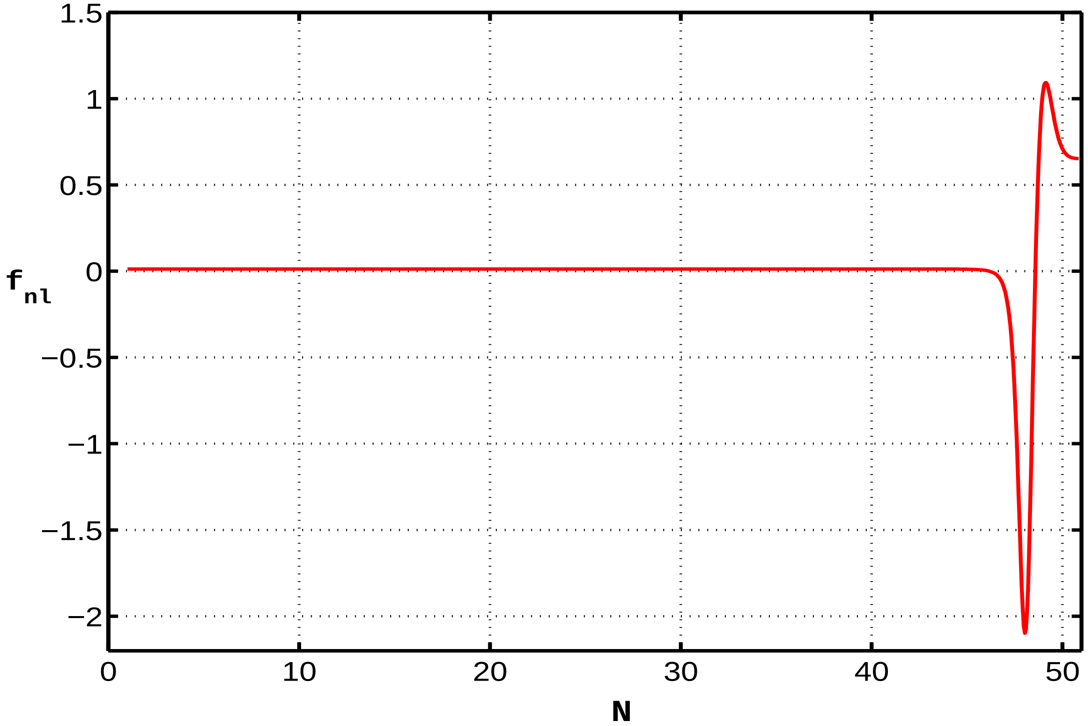

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0.5
0
−0.5
−1
−1.5
−2
f nl
0
10
20
30
40
50
N

Conclusions

- I have sketched a new method for calculating the non-Gaussianity produced in multiple field models
- It has some conceptual advantages over existing methods
- And a considerable computational advantage
- Provides an evolution equation for measures of non-Gaussianity such as f_{nl}