Gravitational Instability of a Nonrotating Galaxy Alex Chao May 4, 2010, INFN

Is a galaxy stable under its own gravitational self-forces? Attempt to apply accelerator physics to address this question.

Beam

- Consider a collection of charged particles in a beam
- Force between particles = electromagnetic
- Beam becomes unstable when its charge is too high

Galaxy

- Consider a collection of stars in a galaxy
- Force between stars = gravitation
- Galaxy becomes unstable when its mass is too large

Replace beam by galaxy, replace particles by stars, replace electromagnetic forces by gravity.

Recall history: 19-century astrophysics has profound impact on 20/21-century accelerator physics. Two notable examples:



Nonlinear dynamics:

Henri Poincare (1854-1912) tried to prove the solar system is stable as a 3-body system. He failed, but won a King Sweden/Norway prize. He invented the Poincare section. Today, Poincare sections are what we see everyday with a beam position monitor. He discovered chaos. Today, we study "dynamic

He discovered chaos. Today, we study "dynamic aperture".



Collective instabilities:

James Clerk Maxwell (1831-1879) won the Adams Prize when he proved that the Saturn ring is unstable unless it consists of many small solid satellites instead of a single solid piece. Today, we call this mechanism "negative mass instability". Gravitational instability is well-known in classical astrophysics, but re-analyzed using modern accelerator physics tools.

If successful, more accelerator physics tools may become available for new insights.

For simplicity,

- Consider nonrelativistic Newtonian gravity
 - \rightarrow Ignore special relativity
- The galaxy is in a flat Euclidean space
 - \rightarrow Ignore general relativity
- Consider a non-rotating galaxy

Equation of motion

Consider a distribution of stars in a galaxy described by a density $\rho(\vec{x}, \vec{v}, t)$ in phase space (\vec{x}, \vec{v}) .

Consider one particular star in this galaxy. Equations of motion:



The force is obtained by integrating the R^{-2} gravitational forces from the other stars of the galaxy.

These equations do not depend on the mass of the star --- a star or a dust particle are treated the same.

Vlasov equation

Evolution of ρ is described by the Vlasov equation:

solve it for ρ .



"wake function" (electromagnetic force) in accelerator physics.

Perturbation calculation

- Let the galaxy distribution be given by an unperturbed distribution ρ_0 plus some small perturbation $\Delta \rho$.
- Let the unperturbed distribution $\rho_0 = \rho_0(\vec{v})$ be independent of \vec{x} , i.e. the galaxy has a nominal <u>uniform</u> spatial distribution in \vec{x} extending to infinity.
- The existence of instability is easy to see. Consider a uniform galaxy with all stars stationery but initially having an accidental small local concentration of stars. This extra concentration will grow indefinitely just by the extra gravitational pull.
- Let the perturbation have a structure in *t* and \vec{x} . Decompose the structure in Fourier form (no loss of generality when linearized in $\Delta \rho$):

$$\rho(\vec{x}, \vec{v}, t) = \rho_0(\vec{v}) + \Delta \rho(\vec{v}) e^{-i\omega t + i\vec{k}\cdot\vec{x}}$$

$$\omega = \text{frequency in time}$$

$$k = \text{wave number in sp}$$

space

Dispersion relation

Substitute into Vlasov equation, keeping first order in $\Delta \rho$ (linearization),

$$\begin{split} &-i(\omega-\vec{v}\cdot\vec{k})\Delta\rho(\vec{v})+G\left(\int d\vec{v}'\,\Delta\rho(\vec{v}')\right)\frac{\partial\rho_0(\vec{v})}{\partial\vec{v}}\cdot\vec{q}(\vec{k})=0\\ \text{are} \qquad &\vec{q}(\vec{k})\equiv\int d\vec{x}'\frac{e^{i\vec{k}\cdot(\vec{x}'-\vec{x})}(\vec{x}'-\vec{x})}{|\vec{x}'-\vec{x}|^3} \end{split}$$

where

 $\vec{q}(\vec{k}) = \frac{4\pi i}{|\vec{k}|^2}\vec{k}$

is the Fourier transform of the Newton kernel = "gravitational propagator",

	kernel in	propagator in
	coordinate space	momentum space
gravitational	\vec{x}	$4\pi i \vec{k}$
instability	$ \vec{x} ^3$	$ \vec{k} ^2$
accelerator	wakefields	impedances
physics	$W_{\parallel}(z), W_{\perp}(z)$	$Z_{\parallel}(\omega), Z_{\perp}(\omega)$
quantum	exchange gauge	propagators
field theory	particles	
	<u>}</u>	$\frac{1}{q^2+m^2}$

Rewrite:

$$\Delta \rho(\vec{v}) = -iG\left(\int d\vec{v}' \,\Delta \rho(\vec{v}')\right) \,\frac{\frac{\partial \rho_0(\vec{v})}{\partial \vec{v}} \cdot \vec{q}(\vec{k})}{\omega - \vec{v} \cdot \vec{k}}$$

Integrating both sides over \vec{v} and canceling out $\int d\vec{v}' \Delta \rho(\vec{v}')$ gives a <u>dispersion relation</u>

$$1 = -iG \int d\vec{v} \frac{\frac{\partial \rho_0(\vec{v})}{\partial \vec{v}} \cdot \vec{q}(\vec{k})}{\omega - \vec{v} \cdot \vec{k}}$$

Given $\rho_0(\vec{v})$, the dispersion relation can be solved to give ω as function of \vec{k} .
$$\Rightarrow \quad \omega(\vec{k})$$

Singularity, beware!

Instability

The galaxy is unstable if ω contains a positive imaginary part α . Perturbation ~ $e^{-i\omega t}$ ~ $e^{\alpha t}$ grows with time. An infinitesimal perturbation grows exponentially.

Using the dispersion relation, we obtain $\omega(\vec{k})$ and answer the question:

For what values of \vec{k} is the galaxy unstable?

As will be shown later, the galaxy becomes unstable when $|\vec{k}| <$ threshold value k_{th} . This means galaxies larger than ~ $2\pi/k_{\text{th}}$ are unstable. <u>All stable galaxies must be smaller than $2\pi/k_{\text{th}}$.</u>

Uniform isotropic galaxy

• Consider a galaxy with *isotropic* unperturbed distribution:

$$\rho_0 = \rho_0(|\vec{v}|^2)$$

• Normalization:

$$\int_0^\infty 4\pi v^2 dv \,\rho_0(v^2) = \rho_m \quad \text{in units of g/cm}^3$$

• Dispersion relation becomes

$$1 = \frac{16\pi^2 G}{k} \int_0^\infty v^3 dv \, \rho_0'(v^2) \int_{-1}^1 du \, \frac{u}{\omega - kvu}$$

Singularity is undefined as is. Must apply <u>Landau damping</u> analysis, i.e. $\omega \rightarrow \omega + i\varepsilon$, as in accelerator/plasma physics.

Landau Damping

Analysis amounts to $\omega \rightarrow \omega + i\epsilon$,



where P.V. means taking principal value of the integral => singularity is then well-defined.

The seemingly real integral now acquires an explicit imaginary part!

We now specialize the problem further to <u>uniform-sphere isotropic</u> galaxy, waterbag model" in

$$\rho_0(v^2) = \begin{cases} \frac{3\rho_m}{4\pi v_0^3} & \text{if } v^2 < v_0^2 \\ 0 & \text{otherwise} \end{cases} \xrightarrow{\text{accelerator physics}} v_0^2 \text{ is related to the internal "temperature" of the galaxy.}$$

 \Rightarrow Dispersion relation becomes

$$\lambda = \frac{1}{2 + x \ln \left| \frac{x - 1}{x + 1} \right| + i\pi x H(1 - |x|)} \tag{1}$$

where

$$\lambda = \frac{6\pi G\rho_m}{k^2 v_0^2} \quad \text{and} \quad x = \frac{\omega}{k v_0}$$

Problem is solved in principle:

x as function of $\lambda \implies \omega$ as function of k. But this requires inverting $\lambda(x)$ to obtain $x(\lambda)$, not easy to do.

Trick is to examine the stability diagram.

Another standard technique in accelerator physics

Stability diagram

In general, solution $x(\lambda)$ is complex. Instability occurs when x has a positive imaginary part.

At the <u>edge</u> of instability, however, x is (barely) real.

If we scan x = real from - ∞ to ∞ , the RHS of (1) traces out a curve on a complex plane:



- If λ lies exactly on the trace, the galaxy is at edge of instability.
- If λ lies inside the trace, galaxy is stable.
- If λ lies outside the trace, galaxy is unstable.

In accelerator physics, the LHS of (1) replaces λ by the <u>impedance</u>, which is a complex quantity. The impedance will have to stay inside the stability region if the beam is to be stable. Stability mechanism is Landau damping.

Here, λ is real and positive. The stability condition reduces to

$$\lambda < \frac{1}{2},$$
 i.e. $\frac{6\pi G\rho_m}{k^2 v_0^2} < \frac{1}{2}$

For stability, a galaxy wants to be <u>dilute</u> (small ρ_m) and <u>hot</u> (large v_0). Given the density of a galaxy, its size \propto (temperature)^{1/2}

Galaxy is stable only if perturbations have

$$k > k_{\rm th},$$
 where $k_{\rm th} = \frac{\sqrt{12\pi G\rho_m}}{v_0}$

Stable galaxy must have a size $\lesssim \frac{2\pi}{k_{\rm th}}$.

Instability growth rate

How fast does the instability grow when $\lambda > 1/2$? Growth rate $\tau^{-1} = Im(\omega)$

$$\frac{6\pi G\rho_m}{k^2 v_0^2} = \frac{1}{2 - \frac{2\tau^{-1}}{k v_0} \tan^{-1}\left(\frac{k v_0}{\tau^{-1}}\right)}$$
$$\frac{\tau^{-1}}{\sqrt{6\pi G}}$$

- Growth is faster for small *k*, i.e. long wavelength.
- Fastest growth when *k* = 0, with

$$(\tau^{-1})_{\max} = \sqrt{4\pi G \rho_m}$$



Numerical example

For the Milky Way, take

$$\label{eq:rhom} \begin{split} \rho_m &= 2 \ x \ 10^{\text{-}23} \ \text{g/cm}^3 \\ v_0 &= 200 \ \text{km/s} \end{split}$$

=>

maximum instability growth time $\tau_{max} = 7 \times 10^6$ yrs stable size = 14,000 light yrs

Reasonable agreement with observed size of the Milky Way.

If Milky Way started with a size > 14,000 light yrs, it would have shaken off the excess in 7 million years.

Extensions

- Extend application from (stars in a galaxy) to (galaxies in a galaxy-cluster): Take $\rho_m = 10^{-28}$ g/cm³, $v_0 = 1000$ km/s, we obtain $\tau_{max} = 3 \times 10^9$ yrs and maximum cluster size = 3×10^7 light-yrs.
- Extension to <u>finite</u> galaxies by a 1-D nonrotating model: Each star is an infinite plane and moves only in the *z*-dimension. Unperturbed distribution has a finite temperature to balance out the gravitational pull. The needed temperature provides stability by Landau damping. This model is always stable. Simulation result:



• Extension to a 2-D <u>rotating</u> model: Each star is a line mass infinitely long in *z*-dimension and free to move in the *x*- and *y*-dimensions. Unperturbed distribution is when the rotating centrifugal force exactly balances the gravitational pull. The rotation temperature stabilizes the galaxy. This model is always stable. Simulation result:



- Extension to two colliding galaxies, analogy to the "two stream instability" in accelerator physics.
- Growth for very large wavelengths will be reduced when special relativity is included and action-at-a-distance is avoided.
- Replacing Euclidean metrics by Einstein metrics, might study gravitational instability near black hole.

Summary

- Galaxy instability problem reanalyzed using standard accelerator physics techniques --- Vlasov equation, linearization, dispersion relation, Landau damping, stability diagrams, and a rich arsenal of various instability mechanisms to draw analogies from.
- Dilute, hot galaxies are more stable.
- Isotropic nonrotating stable galaxies must have sizes $<\frac{2\pi v_0}{\sqrt{12\pi G\rho_m}}$
- When unstable, the maximum growth rate occurs at long wavelengths, and

$$(\tau^{-1})_{\max} = \sqrt{4\pi G \rho_m}$$

• Model is simplistic. Extensions are needed/possible.