Stefano Frixione

The partonic structure of the electron

Based on: 1909.03886 (SF), 1911.12040 (Bertone, Cacciari, SF, Stagnitto)
. . . and work in progress (/w Zaro, Zhao)

Genova, 11/11/2020

Assumption:

Somewhere, someone will build an e^+e^- collider

(linear or circular)

Goal: increase the accuracy in the computations of e^+e^- cross sections

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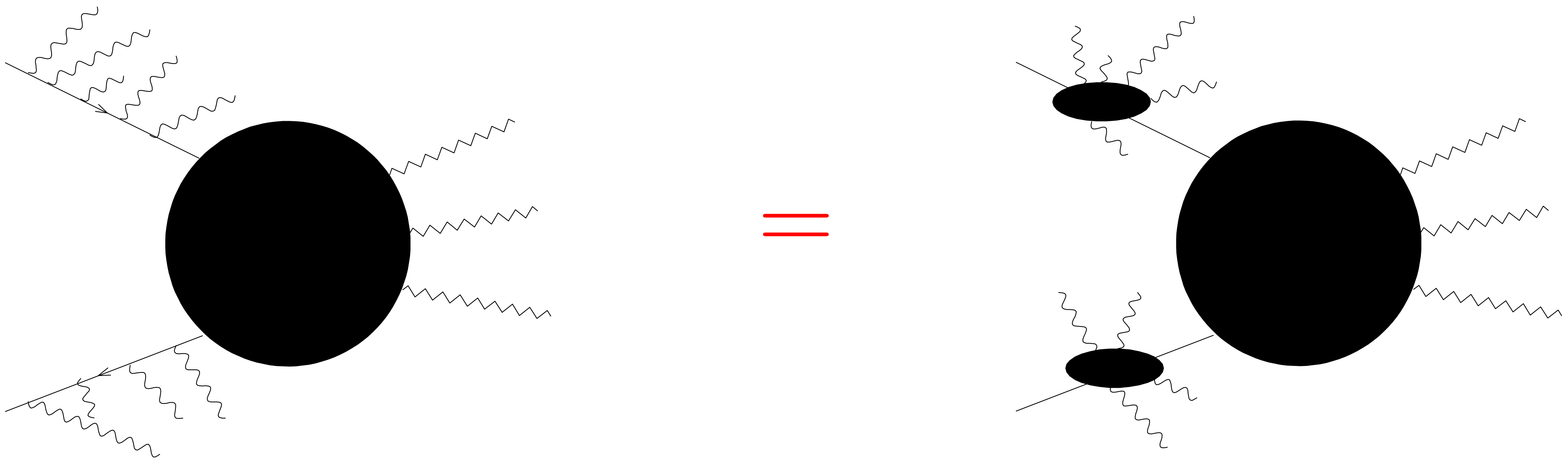
Framework: a factorisation formula

- aka structure-function approach: best to *not* use this terminology

Factorisation



$\sigma = \text{PDF}\star\text{PDF}\star\hat{\sigma}$

PDFs collect (universal) small-angle dynamics

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By means of: more accurate PDFs

- PDFs aka structure functions: best to *not* use this terminology
- improve the LL+LO accuracy, $(\alpha\log(E/m))^k$, by including NLL+NLO terms, $(\alpha\log(E/m))^k + \alpha\,(\alpha\log(E/m))^{k-1}$, in the PDFs
- the corresponding increased accuracy of short-distance cross sections is widely available, and is understood here

Current z-space LO+LL PDFs $(\alpha \log(E/m))^k$:

- $0 \leq k \leq \infty$ for $z \simeq 1$ (Gribov, Lipatov)
- $0 \leq k \leq 3$ for $z < 1$ (Skrzypek, Jadach; Cacciari, Deandrea, Montagna, Nicrosini; Skrzypek)
- matching between these two regimes

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Sought z-space NLO+NLL PDFs $(\alpha \log(E/m))^k + \alpha\,(\alpha \log(E/m))^{k-1}$:

- $0 \le k \le \infty$ for $z \simeq 1$
- $0 \le k \le \{3, 2\}$ for $z < 1 \quad \Longleftrightarrow \quad \mathcal{O}(\alpha^3)$
- matching between these two regimes
- for e^+, e^-, and γ
- both numerical and analytical

Main tool: the solution of PDFs evolution equations

(which resums collinear logs)

Why this approach?

Because it allows one to exploit a significant amount of the technical knowledge we have acquired in two decades of LHC physics

Consider the production of a system X at an e^+e^- collider:

$$e^+(P_{e^+}) + e^-(P_{e^-}) \longrightarrow X$$

Its cross section is written as follows:

$$d\Sigma_{e^+e^-}(P_{e^+}, P_{e^-}) = \sum_{kl} \int dy_+ dy_- \, \mathcal{B}_{kl}(y_+, y_-) \, d\bar{\sigma}_{kl}(y_+ P_{e^+}, y_- P_{e^-})$$

To be definite, let's stipulate that:

$$k \in \{e^+, \gamma\}, \qquad l \in \{e^-, \gamma\}$$

which is immediate to generalise, if need be. Then:

- $d\Sigma_{e^+e^-}$: the collider-level cross section
- $d\bar{\sigma}_{kl}$: the particle-level cross section
- $\mathcal{B}_{kl}(y_+, y_-)$: describes beam dynamics
- e^+, e^- on the lhs: the beams
- e^+, e^-, γ on the rhs: the particles

I'll only talk about particles and particle-level cross sections

The parametrisation of beam dynamics is supposed to be given

I sum over polarisations

Write any particle cross section by means of a factorisation formula, quite similar to its QCD counterpart $\longrightarrow$

$$d\bar{\sigma}_{kl}(p_k, p_l) = \sum_{ij=e^+,e^-,\gamma} \int dz_+ dz_- \, \Gamma_{i/k}(z_+, \mu^2, m^2)\, \Gamma_{j/l}(z_-, \mu^2, m^2) \times d\hat{\sigma}_{ij}(z_+ p_k, z_- p_l, \mu^2)$$

where one calculates Γ and $d\hat{\sigma}$ to predict $d\bar{\sigma}$

- $k\,, l = e^+\,, e^-\,, \gamma$ on the lhs: the particles that emerge from beamstrahlung
- $i\,, j = e^+\,, e^-\,, \gamma$ on the rhs: the partons
- $d\bar{\sigma}_{kl}$: the particle-level (ie observable) cross section
- $d\hat{\sigma}_{ij}$: the subtracted parton-level cross section.

  Generally with $m = 0 \Longrightarrow$ power-suppressed terms in $d\bar{\sigma}$ discarded
- $\Gamma_{i/k}$: the PDF of parton i inside particle k
- μ: the hard scale, $m^2 \ll \mu^2 \sim s$

Differences of QED wrt QCD:

- PDFs and power-suppressed terms can be computed perturbatively
- An object (e.g. e^-) may play the role of both particle and parton

As in QCD, a particle is a physical object, a parton is not

An aside

The fact that, unlike in QCD, PDFs and PSTs can be computed perturbatively in QED does *not* mean that cross sections computed at a given order in α are physically sensible

ISR effects spoil "convergence", and must be resummed
$\longrightarrow$ PDFs (or YFS)

This fact is conveniently forgotten: while there has been a significant activity in cross section computations, we've been living with KK MCs and LL PDFs since the early 90s

$$d\bar{\sigma}_{kl}(p_k, p_l) = \sum_{ij=e^+,e^-,\gamma} \int dz_+ dz_- \, \Gamma_{i/k}(z_+, \mu^2, m^2) \, \Gamma_{j/l}(z_-, \mu^2, m^2) \times d\hat{\sigma}_{ij}(z_+ p_k, z_- p_l, \mu^2)$$

This formula can actually be used in two ways:

A: standard: for the computation of the particle cross section, given the parton cross section and the (evolved) PDFs

B: to solve for the PDFs (at given scales and $\mathcal{O}(\alpha^k)$), given the particle and parton cross sections for some (simple) definite process

Henceforth, I consider the dominant production mechanism at an e^+e^- collider, namely that associated with partons inside an electron*

[This is equivalent to neglecting hard contributions initiated by beamstrahlung-generated photons and leptons of opposite charge wrt that of the beam. Such contributions can be treated with the same techniques as those presented here]

Simplified notation:

$$\Gamma_i(z, \mu^2) \equiv \Gamma_{i/e^-}(z, \mu^2)$$

*The case of the positron is identical, at least in QED, and will be understood

NLO initial conditions (1909.03886)

Conventions for the perturbative coefficients:

$$\Gamma_i = \Gamma_i^{[0]} + \frac{\alpha}{2\pi}\Gamma_i^{[1]} + \mathcal{O}(\alpha^2)$$

Results:

$$\begin{aligned}
\Gamma_i^{[0]}(z,\mu_0^2) &= \delta_{ie^-}\delta(1-z) \\
\Gamma_{e^-}^{[1]}(z,\mu_0^2) &= \left[\frac{1+z^2}{1-z}\left(\log\frac{\mu_0^2}{m^2} - 2\log(1-z) - 1\right)\right]_+ + K_{ee}(z) \\
\Gamma_{\gamma}^{[1]}(z,\mu_0^2) &= \frac{1+(1-z)^2}{z}\left(\log\frac{\mu_0^2}{m^2} - 2\log z - 1\right) + K_{\gamma e}(z) \\
\Gamma_{e^+}^{[1]}(z,\mu_0^2) &= 0
\end{aligned}$$

Note:

- Meaningful only if $\mu_0 \sim m$
- In $\overline{\rm MS}$, $K_{ij}(z) = 0$; in general, these functions *define* an IR scheme

NLL evolution (1911.12040)

General idea: solve the evolution equations starting from the initial conditions computed previously

$$\frac{\partial \Gamma_i(z,\mu^2)}{\partial \log \mu^2} = \frac{\alpha(\mu)}{2\pi} \left[P_{ij} \otimes \Gamma_j \right] (z,\mu^2) \quad \Longleftrightarrow \quad \frac{\partial \Gamma(z,\mu^2)}{\partial \log \mu^2} = \frac{\alpha(\mu)}{2\pi} \Big[\mathbb{P} \otimes \Gamma \Big] (z,\mu^2) \,,$$

Done conveniently in terms of non-singlet, singlet, and photon

Two ways:

- Mellin space: suited to both numerical solution and all-order, large-z analytical solution (called *asymptotic solution*)
- Directly in z space in an integrated form: suited to fixed-order, all-z analytical solution (called *recursive solution*)

A technicality: owing to the running of α, it is best to evolve in t rather than in μ, with: (~ Furmanski, Petronzio)

$$
\begin{aligned}
t &= \frac{1}{2\pi b_0} \log \frac{\alpha(\mu)}{\alpha(\mu_0)} \\
&= \frac{\alpha(\mu)}{2\pi} L - \frac{\alpha^2(\mu)}{4\pi} \left(b_0 L^2 - \frac{2b_1}{b_0} L \right) + \mathcal{O}(\alpha^3) \,, \qquad L = \log \frac{\mu^2}{\mu_0^2} \,.
\end{aligned}
$$

Note:

- $t \longleftrightarrow \mu$; notation-wise, the dependence on t is equivalent to the dependence on μ
- $t = 0 \iff \mu = \mu_0$
- L is my "large log"
- Tricky: fixed-α expressions are obtained with $t = \alpha L/(2\pi)$ (and not $t = 0$)

Mellin space

Introduce the evolution operator $\mathbb{E}_N$

$$\Gamma_N(\mu^2) = \mathbb{E}_N(t)\,\Gamma_{0,N}\,, \qquad \mathbb{E}_N(0) = I\,, \qquad \Gamma_{0,N} \equiv \Gamma_N(\mu_0^2)$$

The PDFs evolution equations are then re-expressed by means of an evolution equation for the evolution operator:

$$\begin{aligned}\frac{\partial \mathbb{E}_N(t)}{\partial t} &= \frac{b_0\alpha^2(\mu)}{\beta(\alpha(\mu))}\sum_{k=0}^{\infty}\left(\frac{\alpha(\mu)}{2\pi}\right)^k \mathbb{P}_N^{[k]}\,\mathbb{E}_N(t) \\ &= \left[\mathbb{P}_N^{[0]} + \frac{\alpha(\mu)}{2\pi}\left(\mathbb{P}_N^{[1]} - \frac{2\pi b_1}{b_0}\,\mathbb{P}_N^{[0]}\right)\right]\mathbb{E}_N(t) + \mathcal{O}(\alpha^2)\end{aligned}$$

- Can be solved numerically
- Can be solved analytically in a closed form under simplifying assumptions. Chiefly: large-z is equivalent to large-N
- I'll show results for the non-singlet $\equiv$ singlet. The photon is feasible as well (see 1911.12040), but technically very involved

Show first that this formalism allows one to quickly re-obtain the known LL result:

$$\Gamma_{0,N}^{[0]} = 1 \quad \Longrightarrow \quad \Gamma_{\rm LL}(z,\mu^2) = M^{-1}\big[\exp\big(\log E_N\big)\big]$$

From the explicit expression of the AP ff kernel:

$$\log E_N = \frac{\alpha}{2\pi}\,P_N^{[0]}\,L \;\stackrel{N\to\infty}{\longrightarrow}\; -\eta_0\left(\log\bar{N} - \lambda_0\right)$$

$$\eta_0 = \frac{\alpha}{\pi}L\,, \qquad \bar{N} = N\,e^{\gamma_{\rm E}}\,, \qquad \lambda_0 = \frac{3}{4}$$

The computation of the inverse Mellin transform is trivial:

$$\Gamma_{\rm LL}(z,\mu^2) = \frac{e^{-\gamma_{\rm E}\eta_0}\,e^{\lambda_0\eta_0}}{\Gamma(1+\eta_0)}\,\eta_0(1-z)^{-1+\eta_0}$$

The usual form, bar for the "-1" of soft origin (we're resumming collinear logs here)

The NLL case is only slightly more complicated; we use:

$$\Gamma_{\rm NLL}(z,\mu^2) = M^{-1}\big[\exp\big(\log E_N\big)\big] \otimes \Gamma_{\rm NLO}(z,\mu_0^2)$$

which is convenient because the form of the evolution operator is functionally the same as at the LL:

$$\log E_N \stackrel{N\to\infty}{\longrightarrow} -\xi_1 \log \bar{N} + \hat{\xi}_1$$

with:

$$\begin{aligned}
\xi_1 &= 2t - \frac{\alpha(\mu)}{4\pi^2 b_0}\left(1 - e^{-2\pi b_0 t}\right)\left(\frac{20}{9}n_F + \frac{4\pi b_1}{b_0}\right) \\
&= 2t + \mathcal{O}(\alpha t) = \eta_0 + \ldots \\
\hat{\xi}_1 &= \frac{3}{2}\,t + \frac{\alpha(\mu)}{4\pi^2 b_0}\left(1 - e^{-2\pi b_0 t}\right)\left(\lambda_1 - \frac{3\pi b_1}{b_0}\right) \\
&= \frac{3}{2}\,t + \mathcal{O}(\alpha t) = \lambda_0 \eta_0 + \ldots \\
\lambda_1 &= \frac{3}{8} - \frac{\pi^2}{2} + 6\zeta_3 - \frac{n_F}{18}(3 + 4\pi^2)
\end{aligned}$$

Thence:

$$
\begin{aligned}
\Gamma_{\rm NLL}(z,\mu^2) = {} & \frac{e^{-\gamma_{\rm E}\xi_1} e^{\hat{\xi}_1}}{\Gamma(1+\xi_1)}\, \xi_1 (1-z)^{-1+\xi_1} \\
& \times \left\{ 1 + \frac{\alpha(\mu_0)}{\pi} \left[\left(\log\frac{\mu_0^2}{m^2} - 1 \right) \left(A(\xi_1) + \frac{3}{4} \right) - 2B(\xi_1) + \frac{7}{4} \right.\right. \\
& \qquad \left.\left. + \left(\log\frac{\mu_0^2}{m^2} - 1 - 2A(\xi_1) \right) \log(1-z) - \log^2(1-z) \right] \right\}
\end{aligned}
$$

where:

$$
\begin{aligned}
A(\kappa) &= -\gamma_{\rm E} - \psi_0(\kappa) \\
B(\kappa) &= \frac{1}{2}\,\gamma_{\rm E}^2 + \frac{\pi^2}{12} + \gamma_{\rm E}\,\psi_0(\kappa) + \frac{1}{2}\,\psi_0(\kappa)^2 - \frac{1}{2}\,\psi_1(\kappa)
\end{aligned}
$$

z space

Use integrated PDFs (so as to simplify the treatment of endpoints)

$$\mathcal{F}(z,t) = \int_0^1 dy\, \Theta(y-z)\, \Gamma(y,\mu^2) \quad \Longrightarrow \quad \Gamma(z,\mu^2) = -\frac{\partial}{\partial z}\mathcal{F}(z,t)$$

in terms of which the formal solution of the evolution equation is:

$$\mathcal{F}(z,t) = \mathcal{F}(z,0) + \int_0^t du\, \frac{b_0 \alpha^2(u)}{\beta(\alpha(u))} \left[\mathbb{P} \,\overline{\otimes}\, \mathcal{F}\right](z,u)$$

By inserting the representation:

$$\mathcal{F}(z,t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(\mathcal{J}_k^{\mathrm{LL}}(z) + \frac{\alpha(t)}{2\pi}\, \mathcal{J}_k^{\mathrm{NLL}}(z) \right)$$

on both sides of the solution, one obtains recursive equations, whereby a $\mathcal{J}_k$ is determined by all $\mathcal{J}_p$ with $p < k$. The recursion starts from $\mathcal{J}_0$, which are the integrated initial conditions

For the record, the recursive equations are:

$$
\begin{aligned}
\mathcal{J}_k^{\mathrm{LL}} &= \mathbb{P}^{[0]} \,\overline{\otimes}\, \mathcal{J}_{k-1}^{\mathrm{LL}} \\
\mathcal{J}_k^{\mathrm{NLL}} &= (-)^k (2\pi b_0)^k \mathcal{F}^{[1]}(\mu_0^2) \\
&\quad + \sum_{p=0}^{k-1} (-)^p (2\pi b_0)^p \Bigg(\mathbb{P}^{[0]} \,\overline{\otimes}\, \mathcal{J}_{k-1-p}^{\mathrm{NLL}} + \mathbb{P}^{[1]} \,\overline{\otimes}\, \mathcal{J}_{k-1-p}^{\mathrm{LL}} \\
&\qquad\qquad - \frac{2\pi b_1}{b_0} \mathbb{P}^{[0]} \,\overline{\otimes}\, \mathcal{J}_{k-1-p}^{\mathrm{LL}} \Bigg)
\end{aligned}
$$

We have computed these for $k \le 3$ ($\mathcal{J}^{\mathrm{LL}}$) and $k \le 2$ ($\mathcal{J}^{\mathrm{NLL}}$), ie to $\mathcal{O}(\alpha^3)$

Results in 1911.12040 and its ancillary files

Large-z singlet and photon

As for the non-singlet, start from the asymptotic AP kernel expressions:

$$\mathbb{P}_{\mathrm{S},N} \quad \xrightarrow{N\to\infty} \quad \begin{pmatrix} -2\log\bar{N} + 2\lambda_0 & 0 \\ 0 & -\frac{2}{3}\,n_F \end{pmatrix} + \frac{\alpha}{2\pi}\begin{pmatrix} \frac{20}{9}n_F\log\bar{N} + \lambda_1 & 0 \\ 0 & -n_F \end{pmatrix} + \mathcal{O}(1/N) + \mathcal{O}(\alpha^2)$$

This implies

$$\begin{aligned} (\mathbb{E}_N)_{SS} &= E_N \\ M^{-1}\left[\,(\mathbb{E}_N)_{\gamma\gamma}\,\right] &= \frac{\alpha(\mu_0)}{\alpha(\mu)}\,\delta(1-z) \end{aligned}$$

$\Rightarrow$ Singlet $\equiv$ non-singlet

Photon $\equiv$ initial condition $+\ \alpha(0)$ scheme

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$$\Gamma_\gamma(z,\mu^2)=\frac{1}{2\pi}\,\frac{\alpha(\mu_0)^2}{\alpha(\mu)}\,\frac{1+(1-z)^2}{z}\left(\log\frac{\mu_0^2}{m^2}-2\log z-1\right)\,.$$

Or: $\sim$ Weizsaecker-Williams function, plus the natural emergence of a small scale in the argument of α

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But: vastly different from the numerical (exact) solution

$\longrightarrow$ $1/N$ suppression of off-diagonal terms in the evolution operator is over-compensated by the δ-like peak of the electron initial-condition

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By solving the 2×2 system ($\xi_{1,0} = 2 + \mathcal{O}(\alpha)$):

$$\Gamma_\gamma(z,\mu^2) = \frac{\alpha(\mu_0)^2}{\alpha(\mu)}\,\frac{3}{2\pi\xi_{1,0}}\,\log(1-z) - \frac{\alpha(\mu_0)^3}{\alpha(\mu)}\,\frac{1}{2\pi^2\xi_{1,0}}\,\log^3(1-z)$$

A remarkable fact

Our asymptotic solutions, expanded in α, feature *all* of the terms:

$$\frac{\log^q(1-z)}{1-z} \qquad \mathrm{singlet,\ non-singlet}$$

$$\log^q(1-z) \qquad \mathrm{photon}$$

of our recursive solutions

Non-trivial; stems from keeping subleading terms (at $z \to 1$) in the AP kernels

However:

It turns out that, at the NLL, all of the $\log^k(1-z)$ terms not proportional to $\log \mu_0^2/m^2$ are artifacts of the $\overline{\mathrm{MS}}$ scheme

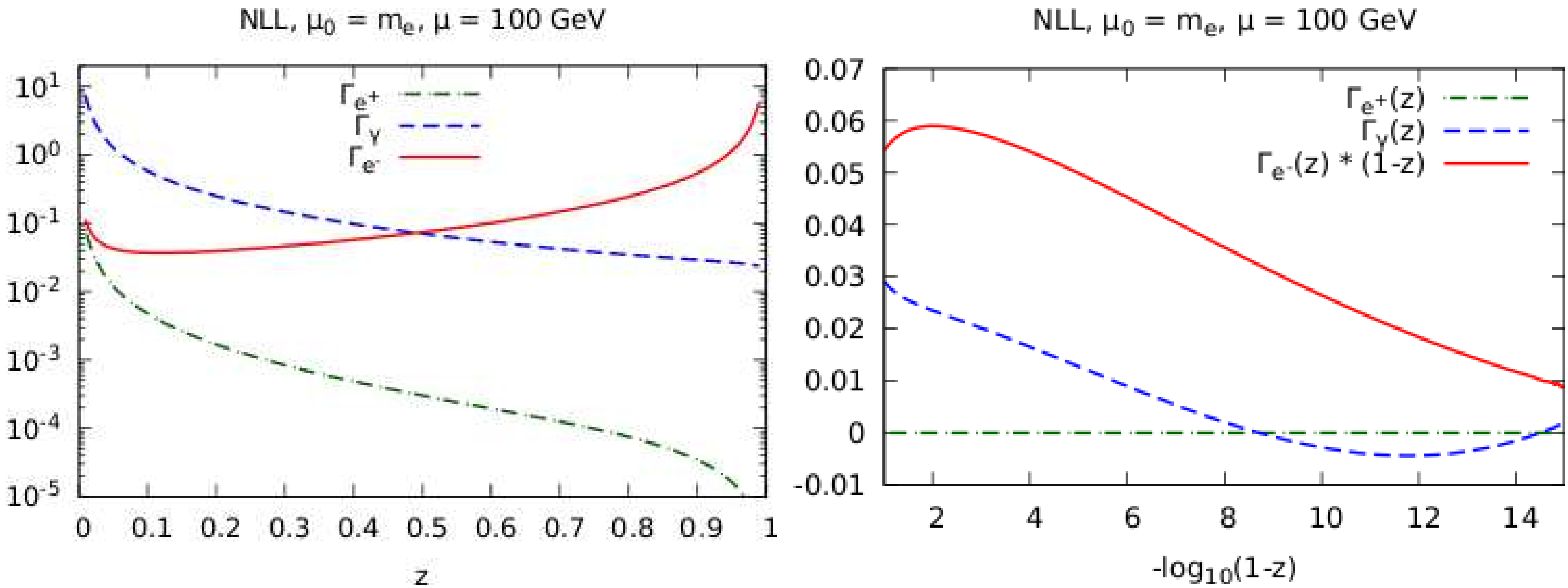
They are artifacts because they are non-physical: at the NLO, they cancel against similar terms in the cross sections

They do contribute at higher (ie out of control) orders. It might be convenient to get rid of them from the beginning

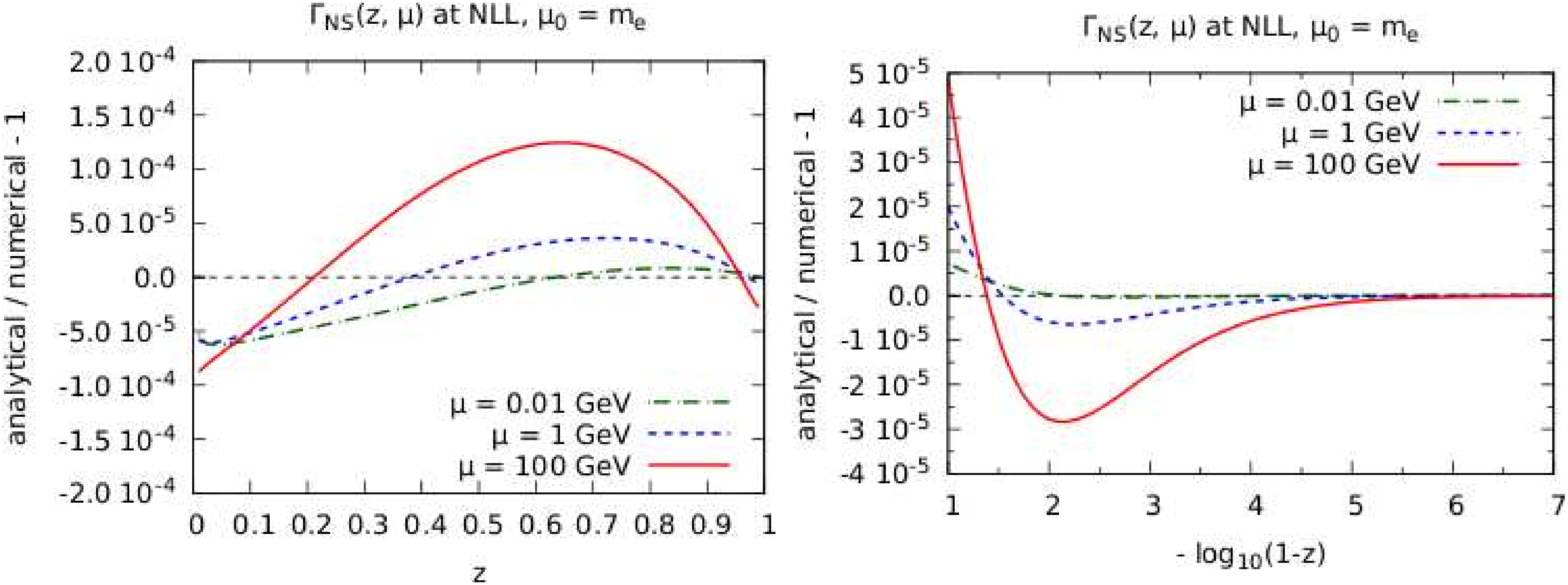
$\longrightarrow$ Different IR scheme: work in progress

Illustrative results for PDFs

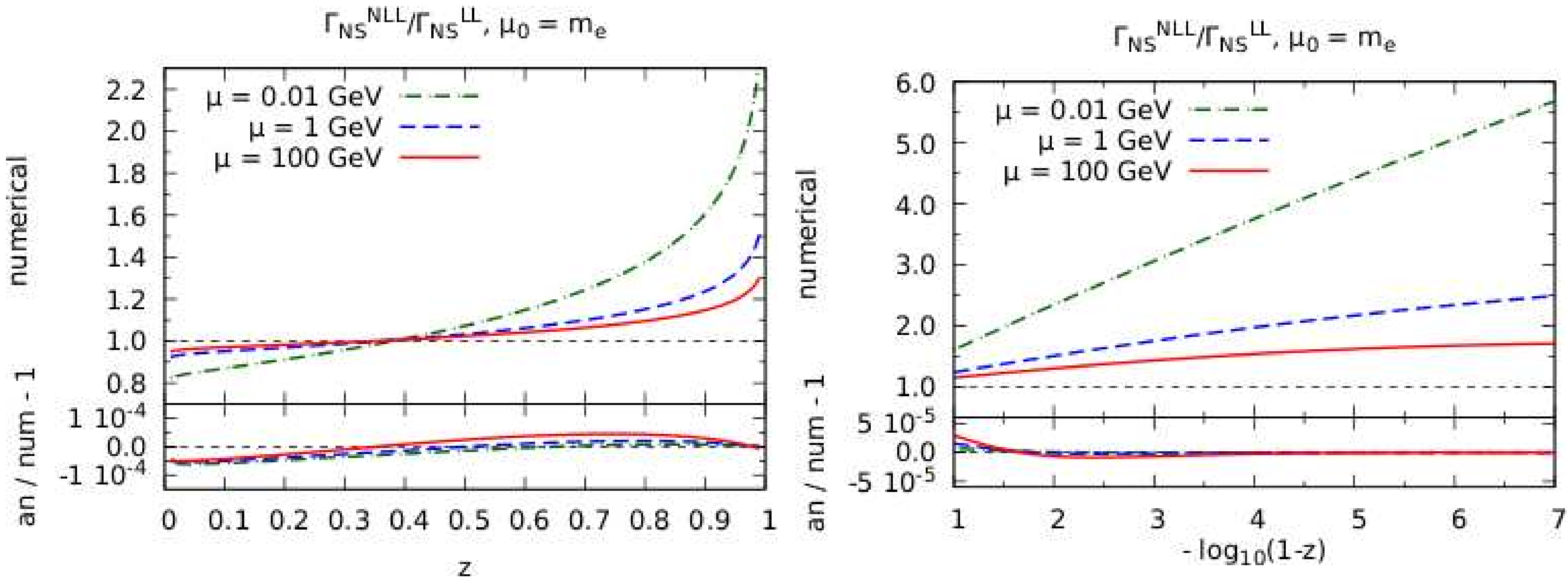
- Analytical results obtained by means of an additive matching between the recursive and the asymptotic solutions
- All are in $\overline{\mathrm{MS}}$
- Bear in mind that PDFs are unphysical quantities



e^- vs γ vs e^+. Note that e^- in the right-hand panel is strongly damped



Numerical vs analytical, non-singlet



NLL vs LL, non-singlet. The insets show the double ratio, ie numerical vs analytical

In order to understand the large-z bit of the previous plots:

$$\Gamma_{\rm LL}(z,\mu^2) = \frac{e^{-\gamma_{\rm E}\eta_0} e^{\lambda_0 \eta_0}}{\Gamma(1+\eta_0)}\, \eta_0 (1-z)^{-1+\eta_0}$$

$$\Gamma_{\rm NLL}(z,\mu^2) = \frac{e^{-\gamma_{\rm E}\xi_1} e^{\hat{\xi}_1}}{\Gamma(1+\xi_1)}\, \xi_1 (1-z)^{-1+\xi_1}$$
$$\times \left\{ 1 + \frac{\alpha(\mu_0)}{\pi} \left[\left(\log\frac{\mu_0^2}{m^2} - 1 \right) \left(A(\xi_1) + \frac{3}{4} \right) - 2B(\xi_1) + \frac{7}{4} \right.\right.$$
$$\left.\left. + \left(\log\frac{\mu_0^2}{m^2} - 1 - 2A(\xi_1) \right) \log(1-z) - \log^2(1-z) \right] \right\}$$

with:

$$\xi_1 \simeq \eta_0 \,, \qquad \hat{\xi}_1 \simeq \lambda_0 \eta_0$$

$$A(\kappa) \;=\; \frac{1}{\kappa} + \mathcal{O}(\kappa) \;\Longrightarrow\; \log(1-z) \text{ dominates}$$
$$B(\kappa) \;=\; -\frac{\pi^2}{6} + 2\zeta_3 \kappa + \mathcal{O}(\kappa^2)$$

Cross sections

These results are not yet public; we are double-checking them.
Some *preliminary* findings are the following:

- The inclusion of NLL contributions into the electron PDF has an impact between 0.1% and 0.5%. We expect this to be somewhat observable dependent
- This estimate does not include the effects of the photon PDF
- The usage of the $\overline{\mathrm{MS}}$ scheme is suboptimal in this context. We have introduced a different IR scheme which is physically more appealing

Conclusions

- We have computed all NLO initial conditions for PDFs and FFs (1909.03886), unpolarised

- We have NLL-evolved those relevant to the electron PDFs (1911.12040), both analytically and numerically

- These can be obtained at:

  `https://github.com/gstagnit/ePDF`

Many results are based on establishing a "dictionary" QCD $\longrightarrow$ QED, which works at any order in α_S and α

Being done/to be done

- Assess the impact of PDFs NLL effects on physical cross sections

- The inclusion of these results in MadGraph5_aMC@NLO v3.X is the only missing ingredient in the latter for the computation of NLO QED corrections in e^+e^- collisions

  NLO QCD+EW in hh collisions and NLO QCD in e^+e^- collisions already OK

- γ PDFs; soft effects; alternative IR schemes; FFs

- Polarisations?