Theoretical Aspects of Astroparticle Physics, Cosmology and Gravitation 2021
Tutorial 2
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1 PBHs from ultra slow-roll inflation

For a given model of ultra slow-roll inflation, the power spectrum amplitude at a scale corresponding to 1 solar-mass PBHs is $\mathcal{P}_\mathcal{R} = 0.0052$.

(a) Assuming a monochromatic population of PBHs, what is the value of the mass fraction β at this scale? You may also assume $\sigma^2 \sim \mathcal{P}_\mathcal{R}$, note that this is only really valid for a monochromatic power spectrum, which is not actually realistic!

Solution: Exact answer will depend on expression used, I used:

$$\beta = \mathrm{erfc}\left(\frac{\delta_c}{\sqrt{2}\sigma}\right) = 4.365 \times 10^{-10}. \tag{1}$$

(b) What is the corresponding value of f_{PBH}?

Solution:

$$f_{\mathrm{PBH}} = \left(\frac{M_{PBH}}{M_{eq}}\right)^{-\frac{1}{2}} \frac{\beta}{\Omega_{\mathrm{DM}}} \approx 1 \tag{2}$$

(c) There is a constraint on the power spectrum at this scale, meaning that the amplitude must satisfy $\mathcal{P}_\mathcal{R} < 0.004$. How much smaller are the values of β and f_{PBH} under this constraint?

Solution: I find $\beta = 1.1 \times 10^{-12}$ and $f_{PBH} = 0.0025$. Your answer for f_{PBH} should be around 3 orders of magnitude smaller than part (a).

(d) If the amplitude of the power spectrum depends on the number of e-folds of the ultra-slow-roll phase N_{USR} as

$$\mathcal{P}_\mathcal{R} \propto \exp(N_{\mathrm{USR}})^6, \tag{3}$$

how many fewer e-folds of ultra-slow-roll are required so as to adhere to this constraint?

Solution:

$$\frac{\mathcal{P}_1}{\mathcal{P}_2} = \exp(6(N_1 - N_2)), \tag{4}$$

and plugging the numbers in from above we find $N_1 - N_2 = 0.044$.

Use $\delta_c = 0.45$ for this question.

2 Additional problems

(a) The power spectrum is not actually directly proportional to $\exp(N_{USR})^6$ as in equation (3). Other than the amplitude of the peak, how else will the spectrum change if the number of e-folds of ultra-slow-roll inflation changes?

Solution: If the start of the USR phase is kept fixed, changing the number of e-folds of USR will also change the position of the peak. This means that you can't just tune the number of PBHs of a given mass that a specific inflationary model and resulting power spectrum produces by altering the number of USR e-folds alone - you'd then also need to change the start point of USR, which corresponds to the field value of the feature in the potential.

(b) If there is an early-matter dominated phase, then the relationship between primordial black hole abundance and the power spectrum is

$$\beta_0 \simeq 0.056\sigma^5 \tag{5}$$

where $\beta_0 = (t_1/t_i)^{\frac{2}{3}}\beta$, t_i is the time of formation, and t_1 is the time that the early-matter dominated phase ends.

If PBHs of 10^{20} g make up all of the dark matter (still possible!) then what is the required amplitude of the power spectrum, if there was an early matter dominated phase that ended at $t = 10^{-14}$ s?

Solution: First, calculate the time at which PBHs with mass 10^{20}g are formed:

$$t_i = \frac{4GM_{PBH}}{3c^3} = 3.3 \times 10^{-19}\text{s}. \tag{6}$$

Then for $f_{PBH} = 1$, the standard PBH mass fraction, is $\beta = 9.8 \times 10^{-17}$. This means that β_0, which is the 'matter-domination mass fraction' which accounts for the fact that Ω_{PBH} remains constant during a matter dominated phase, is

$$\beta_0 = \left(\frac{10^{-14}}{3.3 \times 10^{-19}}\right)^{\frac{2}{3}} \times 9.8 \times 10^{-17} = 9.55 \times 10^{-14}. \tag{7}$$

Putting this into equation (5) and again using $\mathcal{P} \sim \sigma^2$, we find

$$\mathcal{P}_{\mathcal{R}} = 1.96 \times 10^{-5}, \tag{8}$$

which is much smaller than any amplitude which would produce even a negligible fraction of the dark matter if they formed during radiation domination.