Theoretical Aspects of Astroparticle Physics,
Cosmology and Gravitation
The Galileo Galilei Institute for Theoretical Physics
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Course: Particle dark matter: WIMPs, axions and axion like particles

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Exercise Sheet 3: Axions and FIMPs

Exercise 1

The QCD Lagrangian includes a non-trivial topological term given as

$$\mathcal{L} = \mathcal{L}_{QCD} + \theta \frac{g_s^2}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} G^a_{\mu\nu} \tilde{G^a}_{\mu\nu} \tag{1}$$

where $\tilde{G^a}_{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} G^a_{\rho\sigma}$, $\epsilon^{\mu\nu\rho\sigma}$ is the completely antisymmetric tensor and

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^{abc} G^b_\mu G^c_\nu \,. \tag{2}$$

This last term breaks the CP symmetry in the QCD sector. Adding also the CP violation in the EW sector, the effective $\bar{\theta}$ parameter becomes

$$\begin{aligned} \mathcal{L}_\theta &= (\theta + arg(det[M])) \frac{g_s^2}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} G^a_{\mu\nu} G^a_{\rho\sigma} \\ &= \bar{\theta} \frac{g_s^2}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} G^a_{\mu\nu} G^a_{\rho\sigma} \end{aligned} \tag{3}$$

where M_{ij} is the quark mass matrix.

(a) Show that one can write,

$$\frac{g_s^2}{32\pi^2} G^a_{\mu\nu} \tilde{G^a_{\mu\nu}} = \partial^\mu K_\mu \tag{4}$$

where

$$K_\mu = \frac{g_s^2}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} \left(A^a_\nu G^a_{\rho\sigma} - \frac{g}{3} f_{abc} A^a_\nu A^b_\rho A^c_\sigma \right) .$$

The integral of this term will give a non-trivial boundary contribution even for vanishing field $G^a_{\mu\nu}$ at infinity. Show that such boundary term is proportional to an integer number, labelling inequivalent topological field configurations.

(b) Introduce a broken chiral $U(1)_{PQ}$ symmetry with a Nambu-Goldstone boson field, ϕ. The Lagrangian has a shift symmetry $\phi \to \phi + 2\pi\beta f_{PQ}$ ($\beta \in \mathbb{Z}$) and can be written as,

$$\mathcal{L} = \mathcal{L}_{QCD} + \bar{\theta}\frac{g_s^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}G^a_{\mu\nu}\tilde{G^a}_{\mu\nu} + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\phi}{f_{PQ}}\frac{g_s^2}{32\pi^2}G^a_{\mu\nu}\tilde{G^a}_{\mu\nu}\,. \quad (5)$$

ϕ is a dynamical field and can absorb the $\bar{\theta}$ term. After redefining the field as $\phi \to \phi + f_{PQ}\bar{\theta}$, instanton effects breaking the chiral symmetry generate a potential

$$V(\phi) = \Lambda^4\left(1 - \cos\left(\frac{\phi}{f_{PQ}}\right)\right) \quad (6)$$

with the minimum at $\phi = 0$.

(c) Determine the mass of ϕ (m_ϕ) from the above potential in terms of Λ and f_{PQ}.

(d) Finally the Lagrangian in curved space-time is as follows,

$$\mathcal{L} = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\,. \quad (7)$$

where $g^{\mu\nu}$ is the FRW metric. Determine the corresponding energy momentum tensor, using the following relation:

$$T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\partial_\nu\phi - g_{\mu\nu}\mathcal{L} \quad (8)$$

Comparing with the case of a perfect fluid, estimate the density and the pressure given by the scalar field.

(e) Write down the equation of motion of the scalar field for a FRW metric and show that for an homogeneous field it is given by:

$$\ddot{\phi}(t) + 3H(t)\dot{\phi}(t) + V'(\phi) = 0 \quad (9)$$

(f) Consider only the quadratic term in the potential and determine the solutions of the field equations for two regimes $H \gg m_\phi$ and $H \ll m_\phi$.

Exercise 2

Consider a model where we have following kind of process

$$A \to BC \quad (10)$$

where A, B are thermal bath particles and C is the DM candidate. The above process is out of equilibrium when

$$\frac{\langle\Gamma_{A\to BC}\rangle_x}{H} \ll 1 \tag{11}$$

where the thermal averaged decay rate is

$$\langle\Gamma_{A\to BC}\rangle_x = \Gamma_{A\to BC}\frac{K_1(x)}{K_2(x)} ,$$

$x = \frac{m_A}{T}$ and $H = 1,66\sqrt{g_\rho}\,\frac{T}{M_{Pl}}$ ($M_{Pl} = 1.22 \times 10^{19}$ GeV). Here $K_i(x)$ denote the modified Bessel functions of the second kind.

(a) Write down the Boltzmann equation for the number density of the Dark Matter assuming that the mother particle A is in equilibrium and that the decay is the only source of DM density.

(b) Rewrite the equation for $Y_C = n_C/s$ as a function of the parameter x, and show that as long as one can neglect the back-reaction, the solution can be written as a simple integral. Recall that $s = s(m_A)x^{-3}$ is the entropy density and $H(T) = H(m_A)x^{-2}$ the Hubble parameter during radiation domination, i.e. $s(m_A) = \frac{2\pi^2}{45} g_S\, m_A^3$, $H(m_A) = \frac{\pi}{3}\left(\frac{g_\rho}{10}\right)^{1/2}\frac{m_A^2}{M_P}$ with $g_S \sim g_\rho$ counting the relativistic degrees of freedom.

(c) Obtain the present density of the DM from the previous result expressed as $\Omega_C h^2$.

Use: $\int x^3 K_1(x) = \frac{3\pi}{2}$.

(d) Consider a theory with the Yukawa coupling $g\phi_A\bar\psi_B\psi_C$ and determine the decay width for the process $\phi_A \to \bar\psi_B\psi_C$ when $m_{\phi_A} > m_{\psi_B} + m_{\psi_C}$.

(e) Consider $m_{\phi_A} = 300$ GeV, $m_{\psi_B} = m_{\psi_C} = 100$ GeV and determine the maximum coupling g which is needed in order to keep the process $\phi_A \to \bar\psi_B\psi_C$ out of equilibrium at temperature $T \sim m_{\phi_A}$.

Use: $\left(\frac{K_1(x)}{K_2(x)}\right)_{T=m_{\phi_A}} = 0.3$

(f) For the above mentioned masses determine the range of the coupling g which gives the relic density of ψ_C in the range $0.1172 \le \Omega_C h^2 \le 0.1226$.