EXERCICES SOLUTIONS: COSMIC RAYS AND MULTIMESSENGER ASTRONOMY

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Cosmic rays: lectures 1 and 2

EXERCISE 1 - FERMI ACCELERATION

(i) We have at $t = 0$, N_0 particles with energy E_0.

$$N_0 = N_{in} + N_{out} \ , \tag{1}$$

so that,

$$\frac{\mathrm{d}N_{out}}{\mathrm{d}E} = \frac{\mathrm{d}(N_0 - N_{in})}{\mathrm{d}E} = -\frac{\mathrm{d}N_{in}}{\mathrm{d}t}\Big/\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\tau_{\rm acc}}{\tau_{\rm esc}}\frac{N_{in}}{E} = \frac{\tau_{\rm acc}}{\tau_{\rm esc}}\frac{N_0 - N_{out}}{E} \tag{2}$$

Integrating this differential equation gives:

$$N_{out}(E) = N_0\left(1 - \left(\frac{E}{E_0}\right)^{-\frac{\tau_{\rm acc}}{\tau_{\rm esc}}}\right) \tag{3}$$

Therefore $dN_{out}/dE \propto E^{-\alpha}$ with $\alpha = 1 + \tau_{\rm acc}/\tau_{\rm esc}$.

(ii) In the strong shock limit with $\gamma = 5/3$:

$$\lim_{\mathcal{M}\to\infty}\frac{v_1}{v_2} = \frac{\gamma+1}{\gamma-1} = 4 \tag{4}$$

The spectral index is

$$\alpha = 1 + \frac{\tau_{\rm acc}}{\tau_{\rm esc}} = 1 + \frac{3}{v_1/v_2 - 1} = 1 + \frac{3}{3} = 2 \tag{5}$$

EXERCISE 2 - EXTENDED AIR SHOWER

(i) From $p = k_BTn$ we get:

$$n(0) = \frac{p}{k_BT} \simeq \frac{10^5\mathrm{N/m^2}}{273\mathrm{K}1.4\times10^{-23}\mathrm{J/K}} \simeq 2.6\times10^{25}\mathrm{m^{-3}}\,. \tag{6}$$

(ii) Equal chemical potentials across atmospheric layers imply:

$$k_BT\ln(n(0)/n_Q) = k_BT\ln(n(h)/n_Q) + Mgh\,. \tag{7}$$

This can be solved for $n(h) = n(0)e^{-h/\ell}$ with $\ell = k_BT/Mg$. Note that the quantum density drops out in the difference of logarithms. Note, that this is only an approximation. For instance, the rotational degrees of freedom of dinitrogen contribute to the total chemical potential as well.

(iii) For nitrogen we have

$$\ell = \frac{k_BT}{Mg} \simeq \frac{273\mathrm{K}1.4\times10^{-23}\mathrm{J/K}}{28u9.81\mathrm{m/s^2}} \simeq 8.2\mathrm{km} \tag{8}$$

The integral gives $1 = \sigma_{\rm CR}\ell n(0)\exp(-h_{\rm CR}/\ell)$ which can be solved as

$$h_{\rm CR} = \ell\ln(\sigma_{\rm CR}\ell n(0)) \simeq 19.4\mathrm{km}\,. \tag{9}$$

This is the right order of magnitude. A more sophisticated atmospheric model gives somewhat larger values.

EXERCISE 3 - SOLAR POTENTIAL

(i) Using spherical coordinates and a spherical wind we have

$$\nabla \cdot (\mathbf{K} \cdot \nabla n) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 K \frac{\partial n}{\partial r}\right) \tag{10}$$

$$\nabla \cdot (\mathbf{V} n) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 V n\right) \tag{11}$$

$$\frac{\partial}{\partial p}\left(\frac{p}{3}(\nabla \cdot \mathbf{V})n\right) = \frac{\partial}{\partial p}\left(\frac{p}{3}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V)n\right) \tag{12}$$

Using $n = 4\pi p^2 f$, we can rewrite the last term using the expression

$$\frac{\partial}{\partial p}\left(\frac{p^3}{3}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V)f\right) = p^2 \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V f) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{p^3}{3} V \frac{\partial f}{\partial p}\right) - \frac{V}{3}\frac{\partial}{\partial p \partial r}\left(p^3 f\right) \tag{13}$$

Combining these terms in the steady-state equation gives the desired expression.

(ii) Inserting the expressions for $K_\odot$ and $V_\odot$ gives:

$$e\mathcal{V}_\odot = \frac{1\mathrm{GeV}}{3} \times 4 \times 10^7 \frac{\mathrm{cm}}{\mathrm{s}} \times 10^{-22}\frac{\mathrm{s}}{\mathrm{cm}^2} \times \int_{1\mathrm{AU}}^{\infty} \mathrm{d}r' e^{-\frac{r'}{1\mathrm{AU}}} \simeq 7\mathrm{MeV}\,, \tag{14}$$

(iii) With this modified ansatz we get:

$$\frac{\partial p}{\partial r} = \frac{V_\odot(r)}{3}\frac{p^{2/3}p_0^{1/3}}{K_\odot(r, p_0)}\,, \tag{15}$$

This can be solve via:

$$\int_{p(r_\oplus)}^{p(\infty)} \mathrm{d}p' p'^{-2/3} = \frac{p_0^{1/3}}{3}\int_{r_\oplus}^{\infty} \mathrm{d}r' \frac{V_\odot(r')}{K_\odot(r', p_0)} \tag{16}$$

We obtain:

$$(cp(\infty))^{1/3} - (cp(r_\oplus))^{1/3} = \frac{(cp_0)^{1/3}}{9}\int_{r_\oplus}^{\infty} \mathrm{d}r' \frac{V_\odot(r')}{K_\odot(r', p_0)} \tag{17}$$

EXERCISE 4 - DEFLECTION OF UHE CRS

(i) Larmor radius of 10^{20} eV protons ($Z = 1$) in magnetic field $B = 10^{-9}$ G is:

$$R_L = \frac{cp}{ZeB} \simeq 3.34 \times 10^{24}\mathrm{m} \simeq 108\ \mathrm{Mpc} \tag{18}$$

The maximal deflection angle $\Delta\psi$ off the true position of a source at $d = 10$ Mpc follows from a magnetic field perpendicular to the line-of-sight:

$$\sin \Delta\psi = \frac{d}{2R_L} \tag{19}$$

For small angles one can Taylor-expand $\sin \Delta\psi \simeq \Delta\psi + \mathcal{O}((\Delta\psi)^3)$ and arrive at

$$\Delta\psi \simeq \frac{d}{2R_L} \simeq 2.7^\circ \tag{20}$$

(i) The arc length is $2\Delta\psi R_L$ and therefore the CR delay compared to light emitted at the same time is

$$\Delta t = \frac{2\Delta\psi R_L - d}{c} \tag{21}$$

Now, here we have to be a bit careful since we are subtracting two large numbers to find a small difference. Using our first order Taylor-term for $\Delta\psi$ would give us $\Delta t = 0$, so we have to go to the next order:

$$\sin \Delta\psi \simeq \Delta\psi - \frac{1}{6}(\Delta\psi)^3 + \mathcal{O}((\Delta\psi)^5)\,, \tag{22}$$

giving

$$\Delta\psi \simeq \frac{d}{2R_L} + \frac{1}{6}(\Delta\psi)^3\,, \tag{23}$$

To leading order we get therefore

$$\Delta t \simeq \frac{2\Delta\psi R_L - d}{c} \simeq \frac{1}{3}\frac{R_L}{c}(\Delta\psi)^3 \simeq 11600\text{yr} \tag{24}$$