

Ex-1

Boltzmann equation is given as -

$$\frac{dY_x}{dx} = - \frac{S(x) \langle \sigma v \rangle_x}{x H(x)} (Y_x^2 - Y_{x,eq}^2)$$

$$x = \frac{m_x}{T}, \quad S = S(m_x) x^{-3}, \quad H(T) = H(m_x) x^{-2}$$

$$\text{where } S(m_x) = \frac{2\pi^2}{45} g_s m_x^3, \quad H(m_x) = \frac{\pi}{3} \left(\frac{g_p}{10}\right)^{1/2} \frac{m_x^2}{m_p}$$

a \textcircled{a} Equilibrium number density can be expressed as -

$$n_{x,eq}(x) = \frac{g}{\pi^2} m_x^3 x^{-3/2} e^{-x}$$

\textcircled{a}

At $x = x_f$, particle density starts deviating from the thermal equilibrium.

$$\text{Therefore, } n_{x,eq}(x_f) \langle \sigma v \rangle_{x_f} = H(m_x)$$

$$\Rightarrow \frac{g}{\pi^2} m_x^3 x_f^{-3/2} e^{-x_f} \left(a + \frac{b}{x_f}\right) = H(m_x) x_f^{-2}$$

$$\Rightarrow \frac{g}{\pi^2} \frac{m_x}{H(m_x)} m_x^2 x_f^{1/2} \left(a + \frac{b}{x_f}\right) = e^{x_f}$$

$$\Rightarrow x_f = \ln \left(\frac{g}{\pi^2} \frac{m_x}{H(m_x)} \right) + \ln \left[m_x^2 x_f^{1/2} \left(a + \frac{b}{x_f}\right) \right]$$

$$\cancel{x_f = 35 - \ln(m_x)}$$

$$\begin{aligned} x_f &= \ln \left(\frac{g}{\pi^2} \cdot \frac{m_x}{\left(\frac{\pi}{3}\right) \left(\frac{g_p}{10}\right)^{1/2} \frac{m_x^2}{m_p}} \right) + \ln \left[m_x^2 x_f^{1/2} \left(a + \frac{b}{x_f} \right) \right] \\ &= \ln \left[\frac{g}{\pi^2} \cdot \frac{3}{\pi} \left(\frac{10}{g_p} \right)^{1/2} m_p \right] - \log(m_x) \\ &\quad + \ln \left[m_x^2 x_f^{1/2} \left(a + \frac{b}{x_f} \right) \right] \\ &= \ln \left(\frac{2}{\pi^2} \cdot \frac{3}{\pi} \cdot \left(\frac{10}{100} \right)^{1/2} \cdot 2.18 \times 10^{18} \right) \\ &\quad - \log \left(\frac{m_x}{100} \right) - \log 100 \\ &\quad + \ln \left[m_x^2 x_f^{1/2} \cdot \left(a + \frac{b}{x_f} \right) \right] \\ &= 35 - \ln \left(\frac{m_x}{100} \right) + \ln \left[m_x^2 \cdot x_f^{1/2} \left(a + \frac{b}{x_f} \right) \right] \end{aligned}$$

This equation can be solved graphically or by iteration.

Applying the recursion relation, we get -

$$x_f^{i+1} = 35 - \ln \left(\frac{m_x}{100 \text{ GeV}} \right) + \ln \left(m_x^2 (x_f^i)^{1/2} \left(a + \frac{b}{x_f^i} \right) \right)$$

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In the case $a = \frac{\alpha^2}{m_x^2}$, $b = 0$, we get

$$\begin{aligned} x_f &= 35 - \ln\left(\frac{m_x}{100}\right) + \ln\left(\cancel{m_x} \cdot x_f^{1/2} \cdot \frac{\alpha^2}{\cancel{m_x^2}}\right) \\ &= 35 + \ln \alpha^2 - \ln\left(\frac{m_x}{100}\right) + \frac{1}{2} \ln x_f \end{aligned}$$

For $\alpha = 10^{-2}$, we get -

$$\begin{aligned} x_f &= 35 - 4 \ln 10 - \ln\left(\frac{m_x}{100}\right) + \frac{1}{2} \ln x_f \\ &= 25.8 - \ln\left(\frac{m_x}{100}\right) + \frac{1}{2} \ln x_f \end{aligned}$$

For $m_x = 100$ GeV, the above relation iteratively converges to the following values -

$$x_f^0 = 25.8, 27.4252, 27.4557, 27.4563$$

and eventually goes to $x_f = 27.4563$ quickly.

For $\alpha^2 = 16\pi^2$, we get the following relation,

$$x_f = 40.06 - \ln\left(\frac{m_x}{100}\right) + \frac{1}{2} \ln x_f$$

For $m_x = 100$ GeV, gives ~~the~~ iteratively the series:

$$x_f^0 = 40.06, 41.90723, 43.774959, 45.6646899, 47.573150, \dots$$

converging not so quickly to $x_f \sim 50$

So, we see that the larger the coupling, the later the particle decouples from thermal equilibrium and the smaller is the particle density at x_f :

$$n_x^{eq}(x_f) = \frac{g}{\pi^2} m_x^3 x_f^{-3/2} e^{-x_f}$$

$$= \frac{g}{\pi^2} m_x^3 \cdot \begin{cases} 8.3 \times 10^{-5} & \text{for } x_f = 27.45 \\ \sim 0 & \text{for } x_f = 50. \end{cases}$$

(c)

If we take, $Y_{x,eq} = 0$, the differential equation is solvable via separation of variables:

$$\frac{dY_x}{Y_x^2} = - \frac{S(x) \langle \sigma v \rangle_x}{x H(x)} dx$$

After integration:

$$\int^{Y(x')} \frac{dY_x}{Y_x^2} = - \frac{1}{Y(x')} + K = - \frac{S(m_x)}{H(m_x)} \int^{x'} \frac{\langle \sigma v \rangle_x}{x^2}$$

K is the constant of integration.

So, finally, we get -

$$Y_x(x') = \frac{K}{1 + K \cdot \frac{S(m_x)}{H(m_x)} \int_{x_f}^{x'} \frac{\langle \sigma v \rangle_x}{x^2} dx}$$

(d)

In order to match at x_f we have to impose $Y_x(x_f) = Y_{x,eq}(x_f)$ on the initial condition and this just fixes $K = Y_{x,eq}(x_f)$ and the integration ~~interval~~ interval to be (x_f, x') :

$$Y_x(x') = \frac{Y_{x,eq}(x_f)}{1 + Y_{x,eq}(x_f) \frac{S(m_x)}{H(m_x)} \int_{x_f}^{x'} \frac{\langle \sigma v \rangle_x}{x^2} dx}$$

$$\int \left(a + \frac{b}{x} \right) \frac{1}{x^2} dx.$$

$$= \int_{x_f}^{x'} \left(\frac{a}{x^2} + \frac{b}{x^3} \right) dx.$$

$$= a \left(\frac{1}{x} \right)_{x_f}^{x'} + b \frac{x^{-3+1}}{-3+1} \Big|_{x_f}^{x'}$$

$$= a \left(\frac{1}{x_f} - \frac{1}{x'} \right) + \frac{b}{2} \left(\frac{1}{x_f^2} - \frac{1}{x'^2} \right)$$

$$\approx \frac{a}{x_f} + \frac{b}{2x_f^2} = \frac{1}{x_f} \cdot \left(a + \frac{b}{2x_f} \right)$$



$$Y_x(x') = \frac{Y_{x,eq}(x_f)}{1 + Y_{x,eq}(x_f) \cdot \frac{S(m_x)}{H(m_x)} \int \frac{1}{x_f} \left(a + \frac{b}{2x_f}\right)}$$

↓
Assuming this is greater than 1.

$$Y_x(x') \approx \frac{H(m_x)}{S(m_x)} \cdot \frac{x_f}{\left(a + \frac{b}{2x_f}\right)}$$

So, the approximately solution is indeed a constant, depending on a power of x_f , which we estimated previously. For the first case in (6) we get.

$$Y_x(x') \approx \frac{15}{2g\pi} \cdot \sqrt{g_p} \cdot \frac{m_x}{m_p} \cdot x_f$$

$$\approx 3.5 \times 10^{-11} \left(\frac{m_x}{100 \text{ GeV}}\right)$$

This is quite a good estimate for the full result, that can be obtained by numerically integrating the Boltzmann equation.