The XIX International Workshop on Neutrino Telescopes

Renormalization-group equations of neutrino oscillation parameters in matter

Xin Wang

IHEP, CAS

2021.2.23

Based on:

XW and S. Zhou, JHEP 05, 035 (2019) [arXiv:1901.10882]

XW and S. Zhou, Nucl. Phys. B 950, 114867 (2020) [arXiv:1908.07304]

Neutrino Oscillation in Matter

The matter effects on neutrino oscillations in medium play an important role in understanding the neutrino oscillation data, especially in the long-baseline oscillation experiments.

L. Wolfenstein, PRD 17, 2369 (1978)
S. P. Mikheyev and A. Y. Smirnov, Sov. J. Nucl. Phys. 42, 913 (1985)
S. P. Mikheyev and A. Y. Smirnov, Prog. Part. Nucl. Phys. 23, 41 (1989)
T. K. Kuo and J. T. Pantaleone, Rev. Mod. Phys. 61, 937 (1989)

- Effective Hamiltonian

- Three flavors, ordinary matter

matter parameter $a \equiv 2\sqrt{2}G_{\rm F}N_e E$

$$H_{\rm eff} = \frac{1}{2E}\left[U\begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix}U^\dagger + \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] = \frac{1}{2E}\left[V\begin{pmatrix} \widetilde{m}_1^2 & 0 & 0 \\ 0 & \widetilde{m}_2^2 & 0 \\ 0 & 0 & \widetilde{m}_3^2 \end{pmatrix}V^\dagger\right]$$

- Renormalization-group equations (RGEs) for the effective oscillation parameters in matter

Z. Z. Xing, S. Zhou and Y. L. Zhou, JHEP 05, 015 (2018)

$$\frac{{\rm d}\widetilde{\Delta}_{21}}{{\rm d}a} = -\cos^2\widetilde{\theta}_{13}\cos 2\widetilde{\theta}_{12}$$

$$\frac{{\rm d}\widetilde{\Delta}_{31}}{{\rm d}a} = \sin^2\widetilde{\theta}_{13} - \cos^2\widetilde{\theta}_{13}\cos^2\widetilde{\theta}_{12}$$

$$\frac{{\rm d}\widetilde{\Delta}_{32}}{{\rm d}a} = \sin^2\widetilde{\theta}_{13} - \cos^2\widetilde{\theta}_{13}\sin^2\widetilde{\theta}_{12}$$

$$\widetilde{\Delta}_{ij} \equiv \widetilde{m}_i^2 - \widetilde{m}_j^2 \; (ij = 21, 31, 32)$$

$$\frac{{\rm d}\widetilde{\theta}_{12}}{{\rm d}a} = \frac{1}{2}\sin 2\widetilde{\theta}_{12}\left(\cos^2\widetilde{\theta}_{13}\widetilde{\Delta}_{21}^{-1} - \sin^2\widetilde{\theta}_{13}\widetilde{\Delta}_{21}\widetilde{\Delta}_{31}^{-1}\widetilde{\Delta}_{32}^{-1}\right)$$

$$\frac{{\rm d}\widetilde{\theta}_{13}}{{\rm d}a} = \frac{1}{2}\sin 2\widetilde{\theta}_{13}\left(\cos^2\widetilde{\theta}_{12}\widetilde{\Delta}_{31}^{-1} + \sin^2\widetilde{\theta}_{12}\widetilde{\Delta}_{32}^{-1}\right)$$

$$\frac{{\rm d}\widetilde{\theta}_{23}}{{\rm d}a} = \frac{1}{2}\sin 2\widetilde{\theta}_{12}\sin\widetilde{\theta}_{13}\cos\widetilde{\delta}\widetilde{\Delta}_{21}\widetilde{\Delta}_{31}^{-1}\widetilde{\Delta}_{32}^{-1}$$

$$\frac{{\rm d}\widetilde{\delta}}{{\rm d}a} = -\sin 2\widetilde{\theta}_{12}\sin\widetilde{\theta}_{13}\sin\widetilde{\delta}\cot 2\widetilde{\theta}_{23}\widetilde{\Delta}_{21}\widetilde{\Delta}_{31}^{-1}\widetilde{\Delta}_{32}^{-1}$$

Analytical solutions?

Analytical Solutions to the RGEs

Basic Strategy

M. Freund, PRD 64, 053003 (2001)
E. K. Akhmedov *et al.*, JHEP 0404, 078 (2004)

Determine $\mathcal{F}$ and $\mathcal{G}$ via RGEs!

Series expansion

$\Delta_c \equiv \Delta_{31}\cos^2\theta_{12} + \Delta_{32}\sin^2\theta_{12}$ $\quad \hat{C}_{13} \equiv \sqrt{1 - 2A_c\cos 2\theta_{13} + A_c^2}$

$$\widetilde{\Delta}_{21} \approx \Delta_c\left[\frac{1}{2}\left(1 + A_c - \widehat{C}_{13}\right) - \alpha_c\cos 2\theta_{12}\right]$$
$$\widetilde{\Delta}_{31} \approx \Delta_c\left[\frac{1}{2}\left(1 + A_c + \widehat{C}_{13}\right) - \alpha_c\cos 2\theta_{12}\right]$$
$$\widetilde{\Delta}_{32} \approx \Delta_c\widehat{C}_{13}$$

$7.5\times 10^{-5}\,\mathrm{eV}^2$ $\quad 2.5\times 10^{-3}\,\mathrm{eV}^2$

$\boldsymbol{\alpha_c \equiv \Delta_{21}/\Delta_c \approx 0.03}$

Works well only for relatively large A_c

$\boldsymbol{A_c \equiv a/\Delta_c}$

First-order modification

$$\widetilde{\Delta}_{21} = \Delta_c\left[\frac{1}{2}(1 + A_c - \widehat{C}_{13}) + \alpha_c(\mathcal{F} - \mathcal{G})\right]$$
$$\widetilde{\Delta}_{31} = \Delta_c\left[\frac{1}{2}(1 + A_c + \widehat{C}_{13}) + \alpha_c\mathcal{F}\right]$$
$$\widetilde{\Delta}_{32} = \Delta_c\left(\widehat{C}_{13} + \alpha_c\mathcal{G}\right)$$

XW and S. Zhou, JHEP 05, 035 (2019)

- *Reproduce series expansions of $\tilde{\Delta}_{ij}$ for large A_c*
- *Regularize the effective parameters for small A_c*

Results (NO)

□ $\tilde{\theta}_{13}$

Two-flavor approximation

$$\sin^2 2\widetilde{\theta}_{13} = \frac{\sin^2 2\theta_{13}}{(A_c - \cos 2\theta_{13})^2 + \sin^2 2\theta_{13}}$$

$$\cos^2\widetilde{\theta}_{13} = \frac{1}{2}\left(1 - \frac{A_c - \cos 2\theta_{13}}{\widehat{C}_{13}}\right)$$

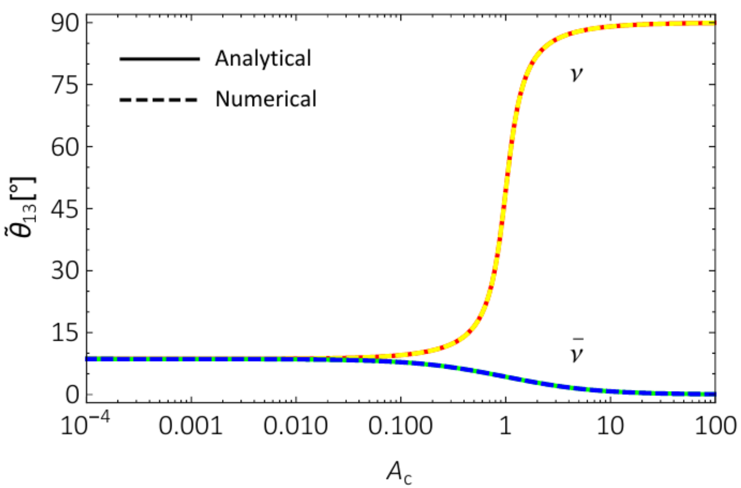
XW and S. Zhou, JHEP 05, 035 (2019)

□ $\tilde{\theta}_{12}$

$A_* \equiv a/\Delta_{21}$ $\quad \hat{C}_{12} \equiv \sqrt{1 - 2A_*\cos 2\theta_{12} + A_*^2}$

Two-flavor approximation $\quad$ θ_{13}-modification

$$\cos^2\widetilde{\theta}_{12} = \frac{1}{2}\left(1 - \frac{A_* - \cos 2\theta_{12}}{\widehat{C}_{12}}\right)\frac{2\widehat{C}_{13}\cos^2\theta_{13}}{\widehat{C}_{13} - A_c + \cos 2\theta_{13}}$$



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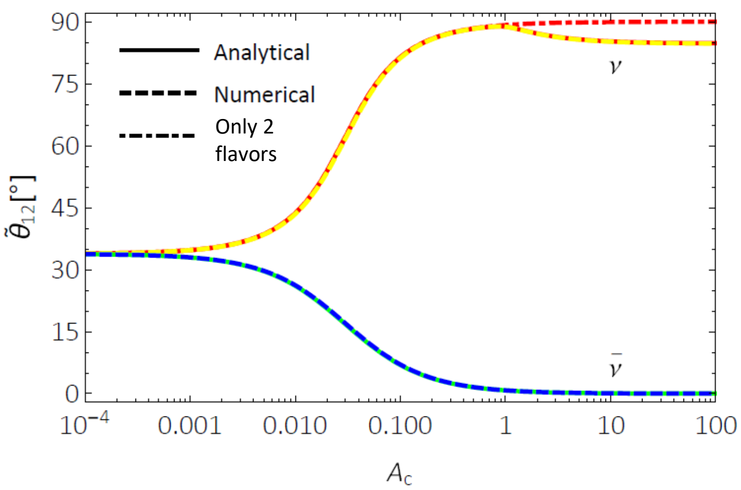
XW and S. Zhou, JHEP 05, 035 (2019)

□ $\tilde{\theta}_{12}$

$A_* \equiv a/\Delta_{21}$ $\hat{C}_{12} \equiv \sqrt{1 - 2A_*\cos 2\theta_{12} + A_*^2}$

Two-flavor approximation θ_{13}-modification

$$\cos^2\tilde{\theta}_{12} = \frac{1}{2}\left(1 - \frac{A_* - \cos 2\theta_{12}}{\widehat{C}_{12}}\right)\frac{2\widehat{C}_{13}\cos^2\theta_{13}}{\widehat{C}_{13} - A_c + \cos 2\theta_{13}}$$



Jarlskog Invariant

In the standard parametrization, the effective Jarlskog invariant in matter can be written as

$$\widetilde{\mathcal{J}} = \sin\widetilde{\theta}_{12}\cos\widetilde{\theta}_{12}\sin\widetilde{\theta}_{13}\cos^2\widetilde{\theta}_{13}\sin\widetilde{\theta}_{23}\cos\widetilde{\theta}_{23}\sin\widetilde{\delta}$$

C. Jarlskog, PRL 55, 1039 (1985)
D. D. Wu, PRD 33, 860 (1986)
P.I. Krastev and S.T. Petcov, PLB 205, 84 (1988)

- Compact and simple formula

$$\frac{\widetilde{\mathcal{J}}}{\mathcal{J}} = \frac{\sin 2\widetilde{\theta}_{12}\sin 2\widetilde{\theta}_{13}\cos\widetilde{\theta}_{13}}{\sin 2\theta_{12}\sin 2\theta_{13}\cos\theta_{13}} \approx \frac{1}{\widehat{C}_{12}\widehat{C}_{13}}$$

XW and S. Zhou, JHEP 05, 035 (2019)

$$\hat{C}_{12} \equiv \sqrt{1 - 2A_* \cos 2\theta_{12} + A_*^2}$$
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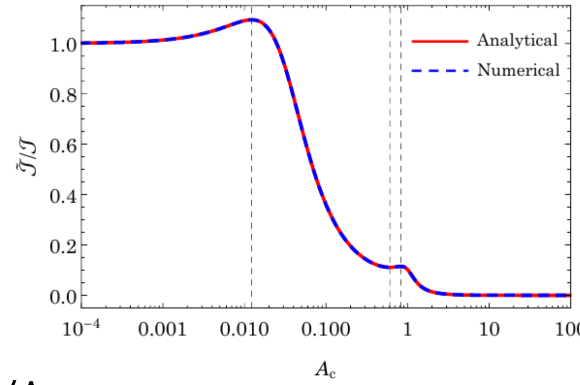

$$\frac{\widetilde{\mathcal{J}}}{\mathcal{J}} = \frac{\Delta_{21}\Delta_{31}\Delta_{32}}{\widetilde{\Delta}_{21}\widetilde{\Delta}_{31}\widetilde{\Delta}_{32}}$$

V. A. Naumov, Int. J. Mod. Phys. D 1, 379 (1992)
P. F. Harrison and W. G. Scott, PLB 476, 349 (2000)
Z. Z. Xing, PLB 487, 327 (2000)

$$A_* = a/\Delta_{21}$$

$\Rightarrow$ $A_* = a\,\mathbf{\cos^2\theta_{13}}/\Delta_{21}$

Significantly improves the accuracy!

$\Delta\tilde{\mathcal{J}}/\tilde{\mathcal{J}}$: 2% → 0.04%

$$\widetilde{\Delta}_{21} = \Delta_c\left[\frac{1}{2}(1 + A_c - \widehat{C}_{13}) + (\widehat{C}_{12} - A_*)\alpha_c\right]$$
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P. B. Denton and S. J. Parke, PRD 100, 053004 (2019)
XW and S. Zhou, NPB 950, 114867 (2020)

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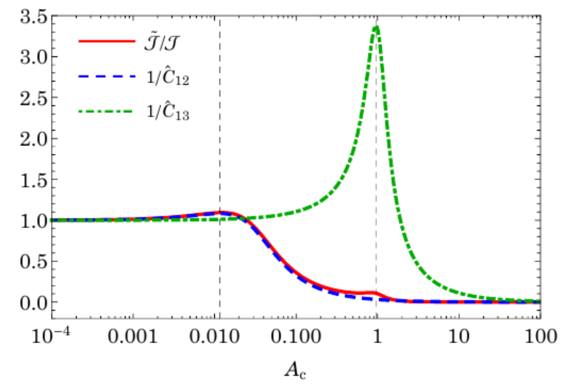

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⇨ $A_* = a\,\mathbf{\cos^2\theta_{13}}/\Delta_{21}$

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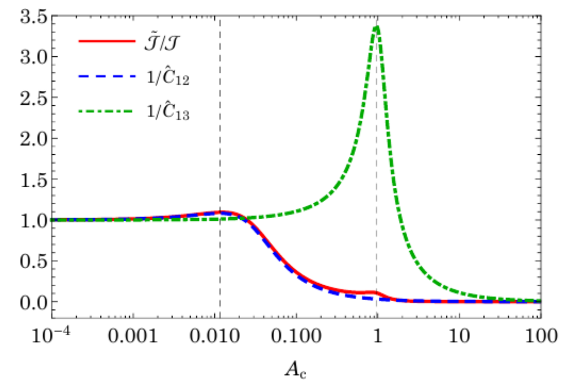
$$\hat{C}_{12} \equiv \sqrt{1 - 2A_* \cos 2\theta_{12} + A_*^2}$$
$$\hat{C}_{13} \equiv \sqrt{1 - 2A_{\rm c} \cos 2\theta_{13} + A_{\rm c}^2}$$



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$$\boldsymbol{A}_{\mathbf{c}} \ll \mathbf{cos\,2}\boldsymbol{\theta}_{\mathbf{13}}\text{:}\quad \widetilde{\Delta}_{21}/\Delta_{21} \approx \widehat{C}_{12}$$
$$(\widetilde{\Delta}_{31}\widetilde{\Delta}_{32})/(\Delta_{31}\Delta_{32}) \approx \widehat{C}_{13}$$

$$\boldsymbol{A}_{\mathbf{c}} \to \boldsymbol{\infty}\text{:}\quad \Delta_{21}/\widetilde{\Delta}_{21} \to \alpha_{\rm c}$$
$$(\Delta_{31}\Delta_{32})/(\widetilde{\Delta}_{31}\widetilde{\Delta}_{32}) \to 1/A_{\rm c}^2$$

Same asymptotic behavior as $1/(\hat{C}_{12}\hat{C}_{13})$ does

$$A_* = a/\Delta_{21}$$

$$\Rightarrow\quad A_* = a\,\mathbf{cos^2}\,\boldsymbol{\theta}_{\mathbf{13}}/\Delta_{21}$$

Significantly improves the accuracy!

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Summary

- Understanding the matter effects on neutrino oscillations in an analytical way has attracted a lot of attention.

H. W. Zaglauer and K. H. Schwarzer, Z. Phys. C 40, 273 (1988)
Z. Z. Xing, PRD 64, 073014 (2001)
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H. Yokomakura *et al.*, PLB, 496, 175 (2000)
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E. K. Akhmedov et al., JHEP 0404, 078 (2004)
S. K. Agarwalla *et al.*, JHEP 04, 047 (2014)
X. J. Xu, JHEP 1510, 090 (2015)
H. Minakata and S. J. Parke, JHEP 1601, 180 (2016)
P. B. Denton *et al.*, JHEP 1606, 051 (2016)
Z. Z. Xing and J. Y. Zhu, JHEP 1607, 011 (2016)
Y. F. Li *et al.*, JHEP 1612, 109 (2016)
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Z. Z. Xing and J. Y. Zhu, NPB 949, 114803 (2019)
S. Luo, PRD 101, no.3, 033005 (2020)
J. y. Zhu, JHEP 05, 097 (2020)
P. B. Denton *et al.*, PRD 101, no.9, 093001 (2020)
…

- We establish a set of compact and simple expressions of the effective oscillation parameters in matter by solving their RGEs.

$$\cos^2\widetilde{\theta}_{13} = \frac{1}{2}\left(1 - \frac{A_c - \cos 2\theta_{13}}{\widehat{C}_{13}}\right) \qquad \widetilde{\Delta}_{21} = \Delta_c\left[\frac{1}{2}(1 + A_c - \widehat{C}_{13}) + (\widehat{C}_{12} - A_*)\alpha_c\right]$$

$$\cos^2\widetilde{\theta}_{12} = \frac{1}{2}\left(1 - \frac{A_* - \cos 2\theta_{12}}{\widehat{C}_{12}}\right)\frac{2\widehat{C}_{13}\cos^2\theta_{13}}{\widehat{C}_{13} - A_c + \cos 2\theta_{13}} \qquad \widetilde{\Delta}_{31} = \Delta_c\left[\frac{1}{2}(1 + A_c + \widehat{C}_{13}) + \frac{1}{2}(\widehat{C}_{12} - A_* - \cos 2\theta_{12})\alpha_c\right]$$

$$\widetilde{\theta}_{23} \approx \theta_{23} \quad \widetilde{\delta} \approx \delta \qquad \widetilde{\Delta}_{32} = \Delta_c\left[\widehat{C}_{13} + \frac{1}{2}(A_* - \widehat{C}_{12} - \cos 2\theta_{12})\alpha_c\right]$$

$$\frac{\widetilde{\mathcal{J}}}{\mathcal{J}} = \frac{1}{\widehat{C}_{12}\widehat{C}_{13}}$$

Thank you!