Improvements on perturbative oscillation formulas including NSI

Based on:

M E Chaves *et al.* J. Phys. G: 48 015001 (2021)

See also arXiv:1810.04979v2

M. E. Chaves *mchaves@ifi.unicamp.br*

D. R. Gratieri * *drgratieri@id.uff.br*

O. L. G.Peres *orlando@ifi.unicamp.br*

The ν_α time evolution within the NSI hypothesis is:

$$i\frac{d}{dt}|\nu_\alpha\rangle \;=\; H|\nu_\alpha\rangle = \left(\mathcal{H} + H_{\rm NSI}\right)|\nu_\alpha\rangle \;; \qquad \alpha = e, \mu, \tau$$

Defining: $\Delta_{31} = \Delta m^2_{31}/2\,E_\nu$, $r_\Delta = \dfrac{\Delta m^2_{21}}{\Delta m^2_{31}}$, $r_A = \dfrac{A}{\Delta m^2_{31}}$

We can write:

$$\mathcal{H} = \Delta_{31}\left[U\begin{pmatrix}0 & 0 & 0\\ 0 & r_\Delta & 0\\ 0 & 0 & 1\end{pmatrix}U^\dagger + r_A\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right]$$

Where, $A = 2E_\nu V_{\rm CC}$, $V_{\rm CC} = \sqrt{2}G_{\rm F}\,n_e$ and $n_e = N_A\rho\langle Z/A\rangle$.

The ν_α time evolution within the NSI hypothesis is:

$$i\frac{d}{dt}|\nu_\alpha\rangle \;=\; H|\nu_\alpha\rangle = \left(\mathcal{H} + H_{\rm NSI}\right)|\nu_\alpha\rangle \;; \qquad \alpha = e, \mu, \tau$$

Defining: $\Delta_{31} = \Delta m^2_{31}/2\, E_\nu$, $r_\Delta = \dfrac{\Delta m^2_{21}}{\Delta m^2_{31}}$, $r_A = \dfrac{A}{\Delta m^2_{31}}$

We can write:

$$H_{\rm NSI} \equiv \Delta_{31} r_A \begin{pmatrix} \epsilon_{\rm ee} & \epsilon_{e\mu} & \epsilon_{e\tau} \\ \epsilon_{\mu e} & \epsilon_{\mu\mu} & \epsilon_{\mu\tau} \\ \epsilon_{\tau e} & \epsilon_{\tau\mu} & \epsilon_{\tau\tau} \end{pmatrix}$$

Where, $A = 2E_\nu V_{\rm CC}$, $V_{\rm CC} = \sqrt{2}G_{\rm F}\, n_e$ and $n_e = N_A \rho\langle Z/A\rangle$.

Analytic oscillation formulas $\longrightarrow$ better understanding

Perturbation Theory Through Dyson Series:

$$\sin^2\theta_{13} \approx r_\Delta \approx 0.03 \to \kappa = 0.03 \longrightarrow \textit{ small number}$$

In the *propagation basis*:

$$|\widetilde{\nu}_\alpha\rangle = [R(\theta_{23})]^\dagger |\nu_\alpha\rangle \quad \longrightarrow \quad \widetilde{H} = [\mathrm{R}(\theta_{23})]^\dagger H \mathrm{R}(\theta_{23})$$

And the NSI contribution reads

$$\widetilde{H}_{NSI} \equiv \Delta_{31} r_A \widetilde{\epsilon}\ ; \quad \text{and} \quad \widetilde{\epsilon}_{\alpha\beta} = |\widetilde{\epsilon}_{\alpha\beta}| e^{i\phi_{\alpha\beta}}$$

is a short for a function of the NSI parameters and θ_{23}

Splitting $\widetilde{\mathcal{H}}$ order by order in powers of $\mathcal{O}(\kappa)$

$$\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}^{(0)} + \widetilde{\mathcal{H}}^{(a)} + \widetilde{\mathcal{H}}^{(b)} + \widetilde{\mathcal{H}}^{(c)} + \widetilde{\mathcal{H}}^{(d)} \longrightarrow \textit{block-diagonal form !!}$$

$$\widetilde{\mathcal{H}}^{(0)} = \Delta_{31} \begin{pmatrix} r_A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \mathcal{O}(\kappa^0)$$

$$\widetilde{\mathcal{H}}^{(a)} = \Delta_{31} \begin{pmatrix} 0 & 0 & s_{13} e^{-i\delta_{\mathrm{CP}}} \\ 0 & 0 & 0 \\ s_{13} e^{i\delta_{\mathrm{CP}}} & 0 & 0 \end{pmatrix} \longrightarrow \mathcal{O}(\kappa^{1/2})$$

$$\widetilde{\mathcal{H}}^{(b)} = \Delta_{31} \begin{pmatrix} r_\Delta s_{12}^2 + s_{13}^2 & r_\Delta c_{12} s_{12} & 0 \\ r_\Delta c_{12} s_{12} & r_\Delta c_{12}^2 & 0 \\ 0 & 0 & -s_{13}^2 \end{pmatrix} \longrightarrow \mathcal{O}(\kappa^1)$$

K. Asano and H. Minakata. JHEP 06 (2011) 022 [1103.4387].

$$\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}^{(0)} + \widetilde{\mathcal{H}}^{(a)} + \widetilde{\mathcal{H}}^{(b)} + \widetilde{\mathcal{H}}^{(c)} + \widetilde{\mathcal{H}}^{(d)} \longrightarrow block - diagonal$$

$$\frac{-\widetilde{\mathcal{H}}^{(c)}}{\Delta_{31}} = \begin{pmatrix} 0 & 0 & (r_\Delta s_{12}^2 + \frac{1}{2} s_{13}^2) s_{13} e^{-i\delta_{\rm CP}} \\ 0 & 0 & r_\Delta s_{12} c_{12} s_{13} e^{-i\delta_{\rm CP}} \\ (r_\Delta s_{12}^2 + \frac{1}{2} s_{13}^2) s_{13} e^{i\delta_{\rm CP}} & r_\Delta s_{12} c_{12} s_{13} e^{i\delta_{\rm CP}} & 0 \end{pmatrix}$$

$$\longrightarrow \mathcal{O}(\kappa^{3/2})$$

and

$$\widetilde{\mathcal{H}}^{(d)} = -\Delta_{31} r_\Delta \begin{pmatrix} s_{12}^2 s_{13}^2 & \frac{1}{2} c_{12} s_{12} s_{13}^2 & 0 \\ \frac{1}{2} c_{12} s_{12} s_{13}^2 & 0 & 0 \\ 0 & 0 & -s_{12}^2 s_{13}^2 \end{pmatrix} \longrightarrow \mathcal{O}(\kappa^2)$$

K. Asano and H. Minakata. JHEP 06 (2011) 022 [1103.4387].

The allowed space for the NSI parameters is not that small!

This fact is not widely known!

$$\epsilon_{\alpha\beta}(\mathrm{min}:\mathrm{max}) \leq \begin{pmatrix} -0.65 : 1.4 & -0.19 : 0.16 & -1.10 : 0.43 \\ -0.19 : 0.16 & -- & -0.05 : 0.04 \\ -1.10 : 0.43 & -0.05 : 0.04 & -0.02 : 0.50 \end{pmatrix}$$

$\longrightarrow$ The constraints are very asymmetrical!

The bounds are combination of **global analysis of oscillation experiments** and the **COHERENT** results from P. Coloma *et. al.* JHEP 04 (2017) 116 [1701.04828]

We acknowledged the table of $\Delta\chi^2\times$ NSI parameters from Ref.[69] from *M. C. Gonzalez-Garcia* and *M. Maltoni*.

To include NSI in the perturbation theory

$\longrightarrow$ One *MUST* assume an order for the NSI parameters

Our GOAL is to create an *hierarchy of* $\widetilde{\epsilon}_{\alpha\beta}$ in $\widetilde{H}_{NSI}$

We *assume* that $\widetilde{\epsilon}_{\alpha\beta}$ *have the same order of magnitude as the corresponging SO term in* $\widetilde{H}$

Our choice for the NSI magnitude implies that $\widetilde{H} = \widetilde{\mathcal{H}} + \widetilde{H}_{NSI}$

$$\widetilde{H} = \widetilde{H}^{(0)} + \widetilde{H}^{(a)} + \widetilde{H}^{(b)} + \widetilde{H}^{(c)} + \widetilde{H}^{(d)} \longrightarrow \textit{block-diagonal form !!}$$

$$\widetilde{H}^{(0)} = \Delta_{31} \begin{pmatrix} r_A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \Delta_{31} r_A \begin{pmatrix} \widetilde{\epsilon}_{\rm ee} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \widetilde{\epsilon}_{\tau\tau} \end{pmatrix} \longrightarrow \mathcal{O}(\kappa^0)$$

$$\widetilde{H}^{(a)} = \Delta_{31} \begin{pmatrix} 0 & 0 & s_{13} e^{-i\delta_{\rm CP}} \\ 0 & 0 & 0 \\ s_{13} e^{i\delta_{\rm CP}} & 0 & 0 \end{pmatrix} + \Delta_{31} r_A \begin{pmatrix} 0 & 0 & \widetilde{\epsilon}_{e\tau} \\ 0 & 0 & 0 \\ \widetilde{\epsilon}_{\tau e} & 0 & 0 \end{pmatrix} \longrightarrow \mathcal{O}(\kappa^{1/2})$$

$$\widetilde{H}^{(b)} = \Delta_{31} \begin{pmatrix} r_\Delta s_{12}^2 + s_{13}^2 & r_\Delta c_{12} s_{12} & 0 \\ r_\Delta c_{12} s_{12} & r_\Delta c_{12}^2 & 0 \\ 0 & 0 & -s_{13}^2 \end{pmatrix} + \Delta_{31} r_A \begin{pmatrix} 0 & \widetilde{\epsilon}_{e\mu} & 0 \\ \widetilde{\epsilon}_{\mu e} & 0 & \widetilde{\epsilon}_{\mu\tau} \\ 0 & \widetilde{\epsilon}_{\tau\mu} & 0 \end{pmatrix}$$

$$\longrightarrow \mathcal{O}(\kappa^1)$$

Our choice for the NSI magnitude:

NSI	3σ limit on $\epsilon_{\alpha\beta}$	3σ limit translated to $\widetilde{\epsilon}_{\alpha\beta}$	Our typical $\widetilde{\epsilon}_{\alpha\beta}$
ϵ_{ee}	-0.65 : 1.40	-0.65 : 1.40	1.0
$\epsilon_{\tau\tau}$	-0.02 : 0.50	-0.06 : 0.29	1.0
$\epsilon_{e\tau}$	-1.10 : 0.43	-0.64 : 0.29	0.17
$\epsilon_{e\mu}$	-0.19 : 0.16	-0.14 : 0.46	0.03
$\epsilon_{\mu\tau}$	-0.05 : 0.04	-0.25 : 0.01	0.03

- $\widetilde{\epsilon}_{ee}$ and $\widetilde{\epsilon}_{\tau\tau}$: our assumption covers all the allowed domain.
- $\widetilde{\epsilon}_{e\tau}$: it is in the same order of magnitude than the current experimental bound at 3σ.
- $\widetilde{\epsilon}_{e\mu}$ and $\widetilde{\epsilon}_{\mu\tau}$: our assumption is only one order of magnitude below the current limit at 3σ.

Our resulting oscillation formulas:

Let us define the quantities:

$$
\begin{aligned}
\Sigma &= |\Sigma| e^{i\phi_\Sigma} \equiv s_{13} e^{-i\delta_{\mathrm{CP}}} + r_A \widetilde{\epsilon}_{e\tau} \\
\Omega &= |\Omega| e^{i\phi_\Omega} \equiv r_\Delta c_{12} s_{12} + r_A \widetilde{\epsilon}_{e\mu} \\
\Lambda &\equiv \frac{1}{r_A} + \widetilde{\epsilon}_{\tau\tau} \\
\Gamma &\equiv (1 + \widetilde{\epsilon}_{ee}) \\
\eta &\equiv \Lambda - \Gamma
\end{aligned}
$$

Take home message: The final expressions are given in terms of these *special combinations* of SO and NSI parameters

Our resulting oscillation formulas:

Let us define the quantities:

$$
\begin{aligned}
\Sigma &= |\Sigma| e^{i\phi_\Sigma} \equiv s_{13} e^{-i\delta_{\mathrm{CP}}} + r_A \widetilde{\epsilon}_{e\tau} \\
\Omega &= |\Omega| e^{i\phi_\Omega} \equiv r_\Delta c_{12} s_{12} + r_A \widetilde{\epsilon}_{e\mu} \\
\Lambda &\equiv \frac{1}{r_A} + \widetilde{\epsilon}_{\tau\tau} \\
\Gamma &\equiv (1 + \widetilde{\epsilon}_{ee}) \\
\eta &\equiv \Lambda - \Gamma
\end{aligned}
$$

Take home message: The appearence of these *special combinations* of SO and NSI parameters is a direct consequence of *our choice* for the NSI intensity hierarchy

These are the quantities that oscillation experiments can effectively constrain!!

Our resulting oscillation formulas:

$$\begin{aligned}
\Sigma &= |\Sigma| e^{i\phi_\Sigma} \equiv s_{13} e^{-i\delta_{\rm CP}} + r_A \widetilde{\epsilon}_{e\tau} \\
\Omega &= |\Omega| e^{i\phi_\Omega} \equiv r_\Delta c_{12} s_{12} + r_A \widetilde{\epsilon}_{e\mu} \\
\Lambda &\equiv \frac{1}{r_A} + \widetilde{\epsilon}_{\tau\tau} \; ; \qquad \Gamma \equiv (1 + \widetilde{\epsilon}_{ee}) \\
\eta &\equiv \Lambda - \Gamma
\end{aligned}$$

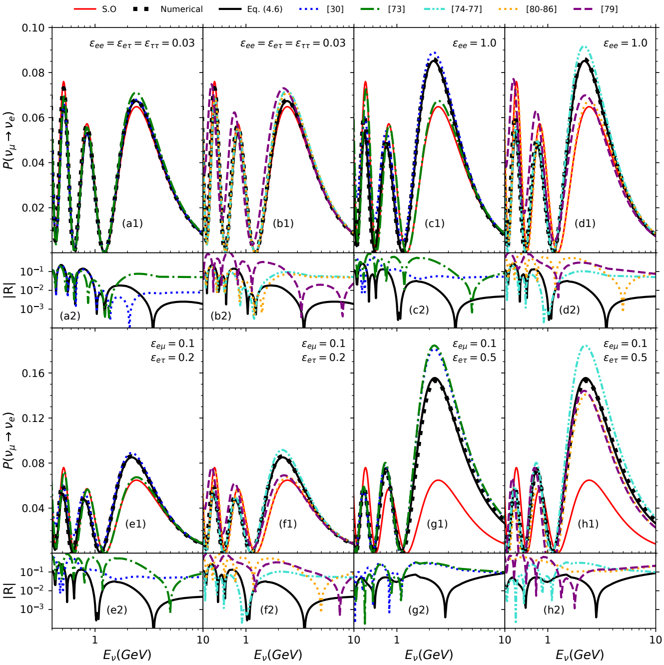
$$P^{(1)}(\nu_\mu \to \nu_e) = 4 \frac{|\Sigma|^2 s_{23}^2}{(r_A \eta)^2} \sin^2 \left(\frac{\Delta_{31} x}{2} r_A \eta \right)$$

Our resulting oscillation formulas:

$$
\begin{aligned}
\Sigma &= |\Sigma| e^{i\phi_\Sigma} \equiv s_{13} e^{-i\delta_{\rm CP}} + r_A \widetilde{\epsilon}_{e\tau} \\
\Omega &= |\Omega| e^{i\phi_\Omega} \equiv r_\Delta c_{12} s_{12} + r_A \widetilde{\epsilon}_{e\mu} \\
\Lambda &\equiv \frac{1}{r_A} + \widetilde{\epsilon}_{\tau\tau} \;; \qquad \Gamma \equiv (1 + \widetilde{\epsilon}_{ee}) \\
\eta &\equiv \Lambda - \Gamma
\end{aligned}
$$

$$
P^{(1)}(\nu_\mu \to \nu_e) = 4 \frac{|\Sigma|^2 s_{23}^2}{(r_A \eta)^2} \sin^2 \left(\frac{\Delta_{31} x}{2} r_A \eta \right)
$$

$$
\begin{aligned}
P^{(3/2)}(\nu_\mu \to \nu_e) &= \frac{8 c_{23} s_{23} |\Sigma| |\Omega| \sin\left(\frac{\Delta_{31} x}{2} r_A \Gamma\right) \sin\left(\frac{\Delta_{31} x}{2} r_A \eta\right)}{r_A^2 \Gamma \eta} \\
&\times \cos\left(\frac{\Delta_{31} x}{2} r_A \Lambda - \phi_\Sigma + \phi_\Omega \right)
\end{aligned}
$$



$$R = \frac{P^{\text{model}} - P^{\text{numerical}}}{P^{\text{numerical}}}$$

$$P^{\text{model}} = P^{(1+3/2+2)}_{(\nu_\mu \to \nu_e)}$$

Conclusions

- We developed a perturbative approach where $\widetilde{\epsilon}_{\alpha\alpha}$ are kept non-perturbative
- Perturbative parameters: r_Δ, $\sin\theta_{13}$ and $\widetilde{\epsilon}_{\alpha\beta}$; $\quad \alpha \neq \beta$
- NSI hierarchy: $\widetilde{\epsilon}_{e\tau} \longrightarrow \mathcal{O}(\kappa^{1/2})$; $\quad \widetilde{\epsilon}_{e\mu}$ and $\widetilde{\epsilon}_{\mu\tau} \longrightarrow \mathcal{O}(\kappa)$
- $\widetilde{H}$ is block-diagonal $\longrightarrow P_{(\nu_\mu \to \nu_e)}(\Lambda,\ \Gamma,\ \Sigma,\ \Omega)$
- The resulting formulas are applicable for a wide range in the allowed space for the NSI
- We have the previous literature as limit cases of our formalism

Bonus Tracks:

The impact of NSI until first order in perturbation theory

In the SO case:

$$P^{(1)}_{(\nu_\mu\to\nu_e)}(\mathrm{SO}) = A^{(1)}_{\mathrm{SO}} \times \sin^2\left(\Phi^{(1)}_{\mathrm{SO}}\right)$$

The impact of NSI:

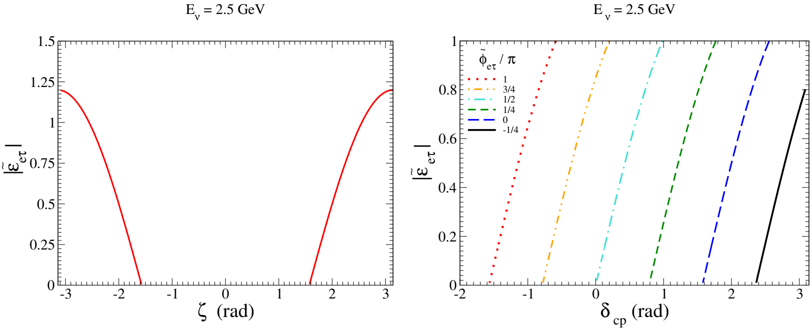
$$A^{(1)}_{(\mathrm{NSI})} = \frac{1 + a\left(a + 2\cos(\zeta)\right)}{(1+\gamma)^2} \times A^{(1)}_{\mathrm{SO}}\ ; \qquad \Phi^{(1)}_{(\mathrm{NSI})} = (1+\gamma)\Phi^{(1)}_{\mathrm{SO}}$$

where

$$a \equiv \frac{r_A}{s_{13}}|\widetilde{\epsilon}_{e\tau}|, \qquad \gamma \equiv \frac{r_A\left(\widetilde{\epsilon}_{\tau\tau} - \widetilde{\epsilon}_{ee}\right)}{1 - r_A}\ ; \qquad \zeta \equiv \delta_{\mathrm{CP}} + \widetilde{\phi}_{e\tau} = \phi_{\Sigma}$$

Study of degeneracies: If $\left\{ \widetilde{\epsilon}_{\tau\tau} = \widetilde{\epsilon}_{ee} ; |\widetilde{\epsilon}_{e\tau}| = -2\frac{s_{13}}{r_A}\cos\zeta \right\}$

$\longrightarrow A^{(1)}_{(\mathrm{NSI})} = A^{(1)}_{(\mathrm{SO})}$ and $\Phi^{(1)}_{(\mathrm{NSI})} = \Phi^{(1)}_{(\mathrm{SO})}$



Left (right) panel:combinations of ζ and $|\widetilde{\epsilon}_{e\tau}|$ (δ_{CP} and $|\widetilde{\epsilon}_{e\tau}|$) that lead to $\left[P^{(1)}(\nu_\mu \to \nu_e)\right]^{(\mathrm{SO})} = \left[P^{(1)}(\nu_\mu \to \nu_e)\right]^{(\mathrm{NSI})}$

Degeneracy between NSI and θ_{23}

$$\left[P^{(1)}(\nu_\mu \to \nu_e)(\widetilde{\epsilon}_{\alpha\beta}, \boldsymbol{\theta_{23}})\right]^{\rm (NSI)} = \left[P^{(1)}(\nu_\mu \to \nu_e)(\widetilde{\epsilon}_{\alpha\beta} = 0, \theta_{23})\right]^{\rm SO}$$

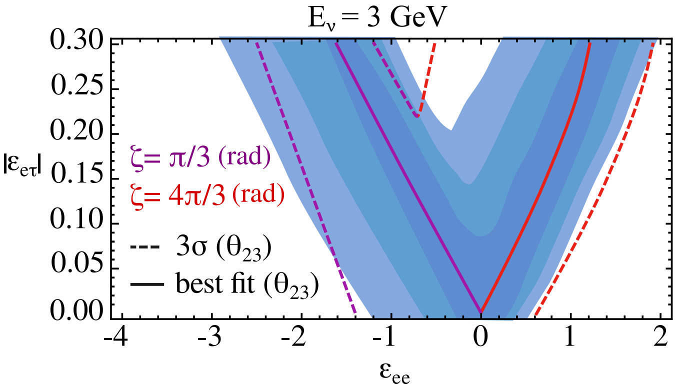
$$\frac{r_A^2}{s_{13}^2}|\widetilde{\epsilon}_{e\tau}|^2 + 2\frac{r_A}{s_{13}}|\widetilde{\epsilon}_{e\tau}|\cos(\zeta) + 1 = \left(\frac{s_{23}}{\boldsymbol{s_{23}}}\right)^2 \left(\frac{B_1}{B_1 + B_2 r_A(\widetilde{\epsilon}_{ee} - \widetilde{\epsilon}_{\tau\tau})}\right)^2$$

where

$$B_1 = \text{sinc}\left(\frac{\Delta_{31}x}{2}(1 - r_A)\right)$$

$$B_2 = \left(\frac{\Delta_{31}x}{2}\right)\frac{\cos\left(\frac{\Delta_{31}x}{2}(1 - r_A)\right) - \text{sinc}\left(\frac{\Delta_{31}x}{2}(1 - r_A)\right)}{\left(\frac{\Delta_{31}x}{2}(1 - r_A)\right)}$$

Degeneracy between NSI and θ_{23}



Superposition of our iso-probabilities (lines) with the allowed region (shaded region) reported by P. Coloma *et. al.* for the DUNE case. Solid (dashed) red curves are generated using the best-fit point for $\boldsymbol{\theta_{23}}$, $\boldsymbol{s^2_{23}} = 0.441$ and for combined phase $\zeta = 4\pi/3$ (rad) (the 3σ values $\boldsymbol{s^2_{23}} = 0.385 \to 0.635$). The solid (dashed) magenta curves have the same respective $\boldsymbol{\theta_{23}}$ values but the phase ζ is equal to $\zeta = \pi/3$ (rad).

Backup slides:

$H_{\rm NSI}$ is function of the effective matter potentials $\epsilon'_{\alpha\beta}$:

$$\epsilon'_{\alpha\beta} = \sum_{\rm f=e,u,d} Y_f(x)\epsilon^{\rm f\,V}_{\alpha\beta}, \qquad \alpha,\beta = e,\mu,\tau$$

Here, $Y_f(x) = n_f(x)/n_e(x)$, and $\epsilon^{\rm f\,V}_{\alpha\beta} = \epsilon^{\rm f\,\it L}_{\alpha\beta} + \epsilon^{\rm f\,\it R}_{\alpha\beta}$ are the coupling parameters in the non-standard interaction effective Lagrangian

$$-\mathcal{L}^{\rm eff}_{\rm NSI} = \sum_{\rm f=e,u,d}\sum_{X=L,R} \epsilon^{\rm f\,\it P}_{\alpha\beta} 2\sqrt{2} G_F (\overline{\nu}_\alpha \gamma_\rho L \nu_\beta)(\overline{f}\gamma^\rho X f)$$

Here $X = (L,R) = (1-\gamma^5, 1+\gamma^5)/\sqrt{2}$, and $\epsilon^{\rm f\,\it P}_{\alpha\beta} \equiv \frac{G_X}{G_f}$ are the neutrino couplings with electrons and quarks due the X exchange.

H_{NSI} is function of the effective matter potentials $\epsilon'_{\alpha\beta}$:

$$H_{\mathrm{NSI}} = \Delta_{31} r_A \epsilon' = \Delta_{31} r_A \begin{pmatrix} \epsilon'_{\mathrm{ee}} & \epsilon'_{e\mu} & \epsilon'_{e\tau} \\ \epsilon'_{\mu e} & \epsilon'_{\mu\mu} & \epsilon'_{\mu\tau} \\ \epsilon'_{\tau e} & \epsilon'_{\tau\mu} & \epsilon'_{\tau\tau} \end{pmatrix}$$

We remove the global phase $\epsilon_{\mu\mu}$ and redefine

$$\epsilon' \rightarrow \epsilon \equiv \Delta_{31} r_A \begin{pmatrix} \epsilon_{\mathrm{ee}} & \epsilon_{e\mu} & \epsilon_{e\tau} \\ \epsilon_{\mu e} & \epsilon_{\mu\mu} & \epsilon_{\mu\tau} \\ \epsilon_{\tau e} & \epsilon_{\tau\mu} & \epsilon_{\tau\tau} \end{pmatrix} = \Delta_{31} r_A \begin{pmatrix} \epsilon'_{\mathrm{ee}} - \epsilon'_{\mu\mu} & \epsilon'_{e\mu} & \epsilon'_{e\tau} \\ \epsilon'_{\mu e} & 0 & \epsilon'_{\mu\tau} \\ \epsilon'_{\tau e} & \epsilon'_{\tau\mu} & \epsilon'_{\tau\tau} - \epsilon'_{\mu\mu} \end{pmatrix}$$

We will work for now on with the variables $\epsilon_{\alpha\beta}$.

Our resulting oscillation formulas:

$$
\begin{aligned}
P^{(2)}(\nu_\mu \to \nu_e) &= \frac{4c_{23}^2|\Omega|^2 \sin^2\left(\frac{\Delta_{31}x}{2} r_A\Gamma\right)}{(r_A\Gamma)^2} \\
+ \quad & 2\left|\Sigma\right|^2 s_{23}^2 \left(\frac{2\left|\Sigma\right|^2}{r_A^3\eta^3} - \frac{r_\Delta s_{12}^2 + 2s_{13}^2}{r_A^2\eta^2}\right)(\Delta_{31}x)\sin\left(r_A\eta\Delta_{31}x\right) \\
- \quad & 4s_{23}^2 \left(\frac{4\left|\Sigma\right|^4}{r_A^4\eta^4} - \frac{2\left|\Sigma\right|^2\left(r_\Delta s_{12}^2 + 2s_{13}^2\right)}{r_A^3\eta^3} + \left|\Sigma\right| s_{13}\left(2r_\Delta s_{12}^2 + s_{13}^2\right)\right. \\
\times \quad & \left.\frac{\cos\left(\delta_{\rm CP} + \phi_\Sigma\right)}{r_A^2\eta^2}\right) \times \sin^2\left(\frac{\Delta_{31}\, x r_A\eta}{2}\right) + 4c_{23}\left|\widetilde{\epsilon}_{\mu\tau}\right|\left|\Sigma\right|^2 s_{23}\sin\left(\frac{\Delta_{31}x}{2} r_A\eta\right) \\
\times \quad & \left(\frac{\sin\left(\widetilde{\phi}_{\mu\tau} - \frac{\Delta_{31}x\, r_A(\Gamma+\Lambda)}{2}\right)}{r_A^2\eta\Gamma\Lambda} - \frac{\sin\left(\widetilde{\phi}_{\mu\tau} - \frac{\Delta_{31}x\, r_A\eta}{2}\right)}{r_A^2\eta^2\Gamma}\right. \\
- \quad & \left. + \frac{\sin\left(\tilde{\phi}_{\mu\tau} + \frac{\Delta_{31}x\, r_A\eta}{2}\right)}{r_A^2\eta^2\Lambda}\right)
\end{aligned}
$$

Our resulting oscillation formulas:

To obtain the anti-neutrino oscillation probabilities we change $U \to U^*$, $r_A \to -r_A$ and $\epsilon \to \epsilon^*$, whose implies in the modifications in the effective parameters given as

$$
\begin{aligned}
\Gamma &\to \Gamma \\
\Sigma &\to \overline{\Sigma} = s_{13} e^{i\delta_{\rm CP}} - r_A (\widetilde{\epsilon}_{e\tau})^* \\
\Omega &\to \overline{\Omega} = r_\Delta c_{12} s_{12} - r_A (\widetilde{\epsilon}_{e\mu})^*
\end{aligned}
$$

Our resulting oscillation formulas for $P(\nu_\mu \to \nu_\mu)$:

$$P^{(0)}_{\nu_\mu\to\nu_\mu} = 1 - 4c_{23}^2 s_{23}^2 \sin^2\left(\frac{\Delta_{31} x r_A \Lambda}{2}\right),$$

$$\begin{aligned}
P^{(1)}_{\nu_\mu\to\nu_\mu} &= -\frac{4|\Sigma|^2 s_{23}^4}{r_A^2(\Gamma-\Lambda)^2}\sin^2\left(\frac{\Delta_{31} x r_A(\Gamma-\Lambda)}{2}\right) \\
&+ 2c_{23}^2 s_{23}^2\left(c_{12}^2 r_\Delta + \frac{|\Sigma|^2}{r_A(\Gamma-\Lambda)} + s_{13}^2\right)\sin(\Delta_{31} x r_A \Lambda)(\Delta_{31} x) \\
&+ \frac{2|\Sigma|^2 s_{23}^2 c_{23}^2}{r_A^2(\Gamma-\Lambda)^2}\left[\cos(\Delta_{31} x\, r_A \Gamma) - \cos(\Delta_{31} x\, r_A \Lambda)\right] \\
&- \frac{8|\widetilde{\epsilon}_{\mu\tau}| c_{23} s_{23}\left(c_{23}^2 - s_{23}^2\right)\cos(\widetilde{\phi}_{\mu\tau})}{\Lambda}\sin^2\left(\frac{\Delta_{31} x \Lambda r_A}{2}\right)
\end{aligned}$$

Our resulting oscillation formulas for $P(\nu_\mu \to \nu_\mu)$:

$$
\begin{aligned}
P^{(3/2)}_{\nu_\mu\to\nu_\mu} &= -\frac{8 r_\Delta c_{12} s_{12} c_{23} s_{23} s_{13}\left(c_{23}^2 - s_{23}^2\right)\cos(\delta_{\rm CP})}{r_A\Lambda}\sin^2\left(\frac{\Delta_{31} x\, r_A\Lambda}{2}\right)\\
&+ \frac{4|\Omega\Sigma| c_{23} s_{23}\cos\left(\phi_\Sigma\phi_\Omega\right)}{r_A^2\Gamma(\Gamma-\Lambda)}\left(c_{23}^2\cos(\Delta_{31} x\, r_A\Gamma)\right.\\
&- \left. c_{23}^2 + s_{23}^2\right)\cos(\Delta_{31} x\, r_A(\Gamma-\Lambda))\\
&- \frac{4|\Omega\Sigma| c_{23} s_{23}\cos\left(\phi_\Sigma - \phi_\Omega\right)}{\Lambda r_A^2(\Gamma-\Lambda)}\left(c_{23}^2\cos(\Delta_{31} x\, r_A\Lambda) + 1\right)\\
&+ \frac{4|\Omega\Sigma| c_{23} s_{23}\cos\left(\phi_\Sigma - \phi_\Omega\right)}{r_A^2\Gamma\Lambda}\left(s_{23}^2\cos\Delta_{31} x\, r_A\Lambda)\right)
\end{aligned}
$$

The muon neutrino probability it is

$$
\begin{aligned}
P^{\rm perturbative}(\nu_\mu\to\nu_\mu) &= P(\nu_\mu\to\nu_\mu)^{(0)} + P(\nu_\mu\to\nu_\mu)^{(1)}\\
&+ P(\nu_\mu\to\nu_\mu)^{(3/2)}
\end{aligned}
$$