

Relativistic hydrodynamics with spin

F. B., F. Piccinini, Ann. Phys. 323, 2452 (2008).

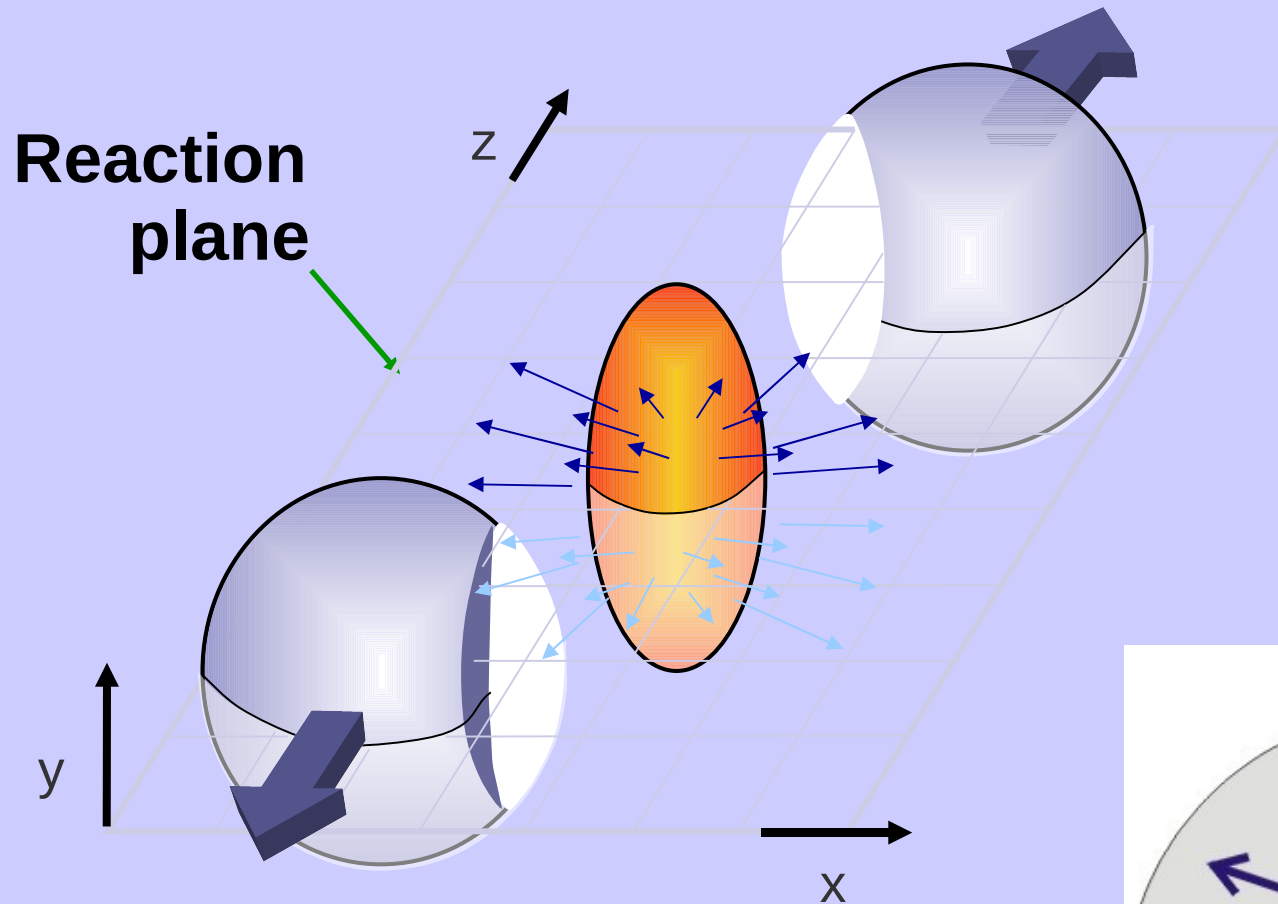
F.B., L. Tinti, arXiv:0911.0864, submitted to Ann. Phys.

F.B., J. Bechi, L. Tinti, in progress

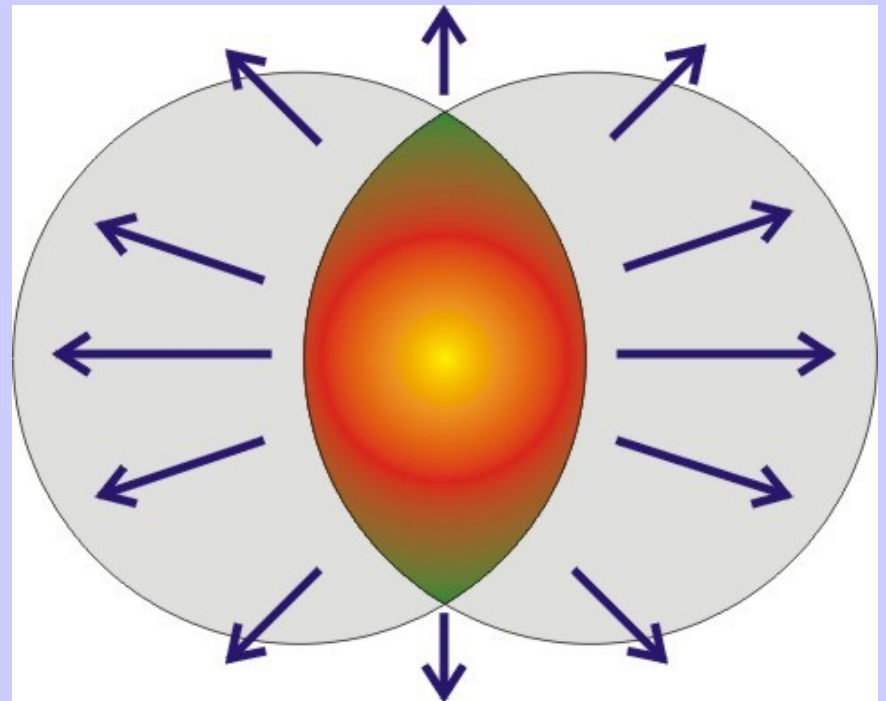
OUTLINE

- Motivations – QGP as a (quasi) ideal fluid
- Brief historical summary
- Relativistic fluids at thermodynamical equilibrium
- Ongoing investigations

Motivations: relativistic heavy ion collisions



Success of a (almost) ideal hydrodynamical description



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RHIC Scientists Serve Up "Perfect" Liquid

New state of matter more remarkable than predicted -- raising many new questions

April 18, 2005

TAMPA, FL -- The four detector groups conducting research at the [Relativistic Heavy Ion Collider](#) (RHIC) -- a giant atom "smasher" located at the U.S. Department of Energy's Brookhaven National Laboratory -- say they've created a new state of hot, dense matter out of the quarks and gluons that are the basic particles of atomic nuclei, but it is a state quite different and even more remarkable than had been predicted. In [peer-reviewed papers](#) summarizing the first three years of RHIC findings, the scientists say that instead of behaving like a gas of free quarks and gluons, as was expected, the matter created in RHIC's heavy ion collisions appears to be more like a *liquid*.

"Once again, the physics research sponsored by the Department of Energy is producing historic results," said Secretary of Energy Samuel Bodman, a trained chemical engineer. "The DOE is the principal federal funder of basic research in the physical sciences, including nuclear and high-energy physics. With today's announcement we see that investment paying off."

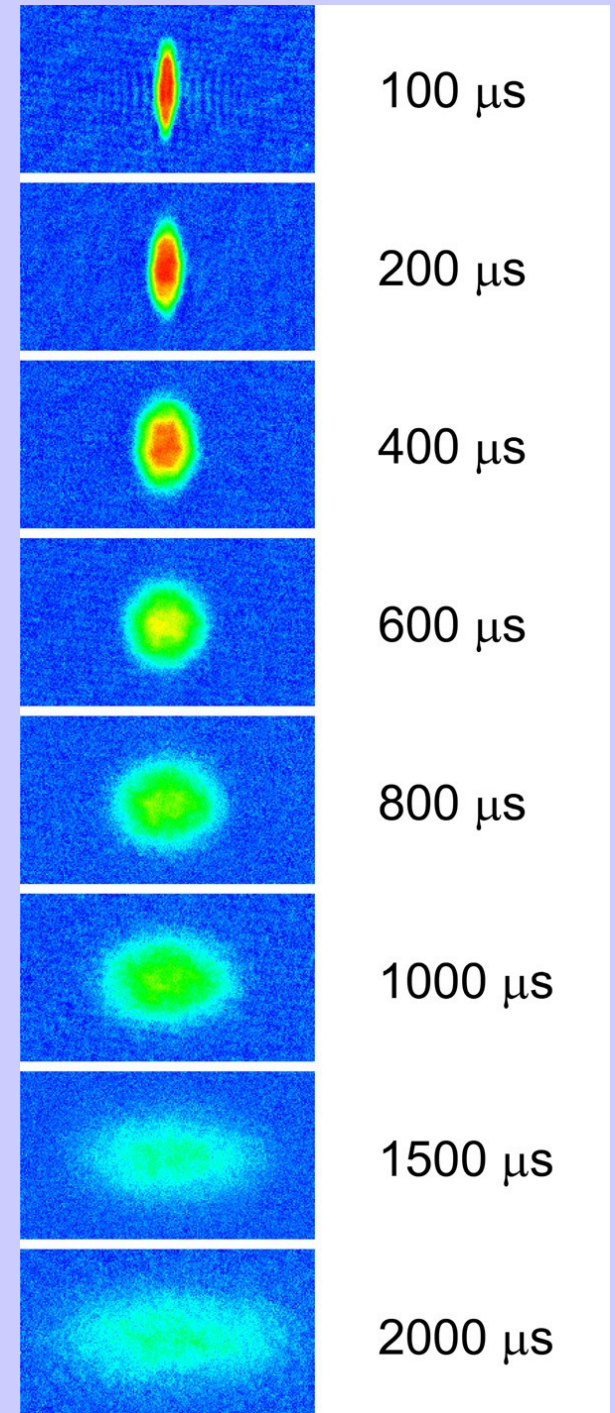
"The truly stunning finding at RHIC that the new state of matter created in the collisions of gold ions is more like a liquid than a gas gives us a profound insight into the earliest moments of the universe," said Dr. Raymond L. Orbach, Director of the DOE Office of Science.

Also of great interest to many following progress at RHIC is the emerging connection between the collider's results and calculations using the methods of string theory, an approach that attempts to explain fundamental properties of the universe using 10 dimensions instead of the usual three spatial dimensions plus time.



Secretary of Energy
Samuel Bodman

Fenomeno analogo osservato in fisica atomica:
atomi “freddi” di Litio
accoppiati fortemente in trappola ottica



Viscosita' nelle teorie di campo fortemente interagenti dalla fisica dei buchi neri

Kovtun, Son, Starinets PRL 94, 111601 (2005)

Spin-off (?) della comunita' delle stringhe

Congettura di Maldacena (1998):

teoria di gauge supersimmetrica di Yang-Mills $N=4$ e' equivalente (duale)
ad una teoria di stringa su un background di Anti de Sitter $\times S_5$

AdS/CFT correspondance

Dualita': quando l'accoppiamento nella teoria di gauge e' grande, allora il calcolo corrispondente di teoria delle stringhe diventa “semplice” supergravita' classica e puo' essere svolto analiticamente

CAVEAT: nelle teorie conformi non c'e' running della costante di accoppiamento e quindi questa dualita' non riguarda esattamente la QCD

Altri esempi di dualita' sono stati costruiti, ma in tutte le teorie di campo nel limite di accoppiamento grande:

$$\frac{\eta}{s} = \frac{\hbar}{4\pi}$$

Congettura: si tratta di un limite quantico universale che viene raggiunto per accoppiamento infinito

Non sono stati in grado di dimostrarlo in termini piu' semplici

$$\eta \sim \rho v \ell, \quad s \sim n = \frac{\rho}{m}$$

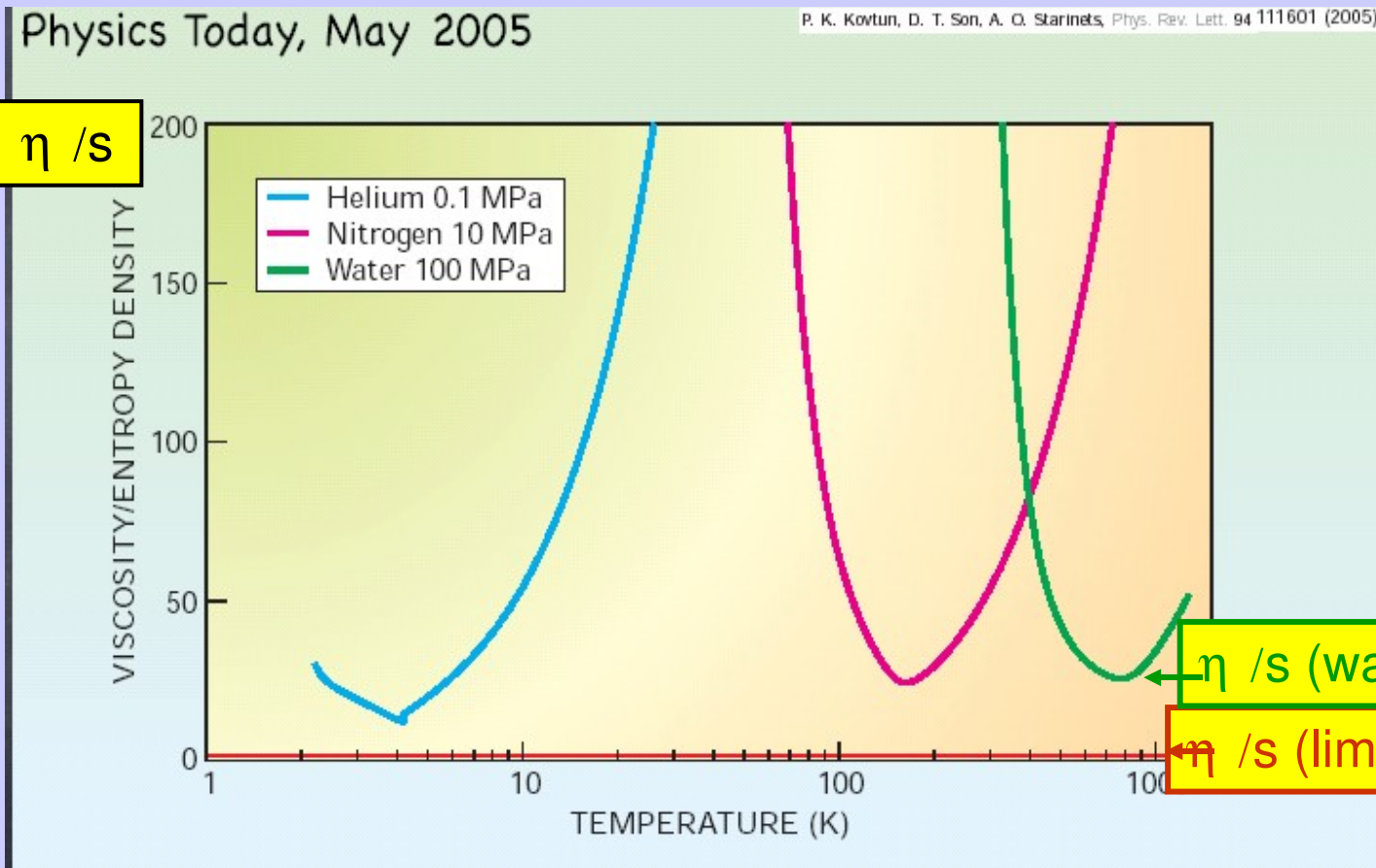
$$\frac{\eta}{s} \sim m v \ell \sim \hbar \frac{\text{mean free path}}{\text{de Broglie wavelength}}$$

Quasiparticles: de Broglie wavelength \lesssim mean free path

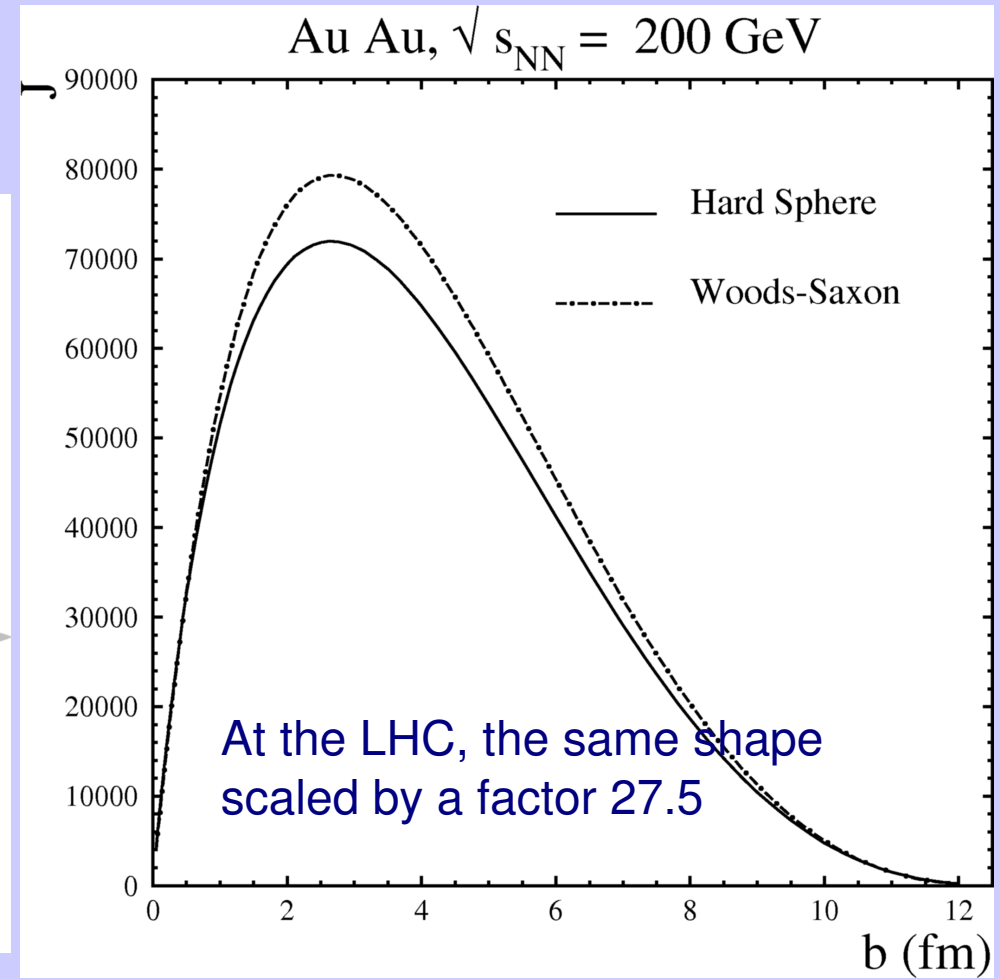
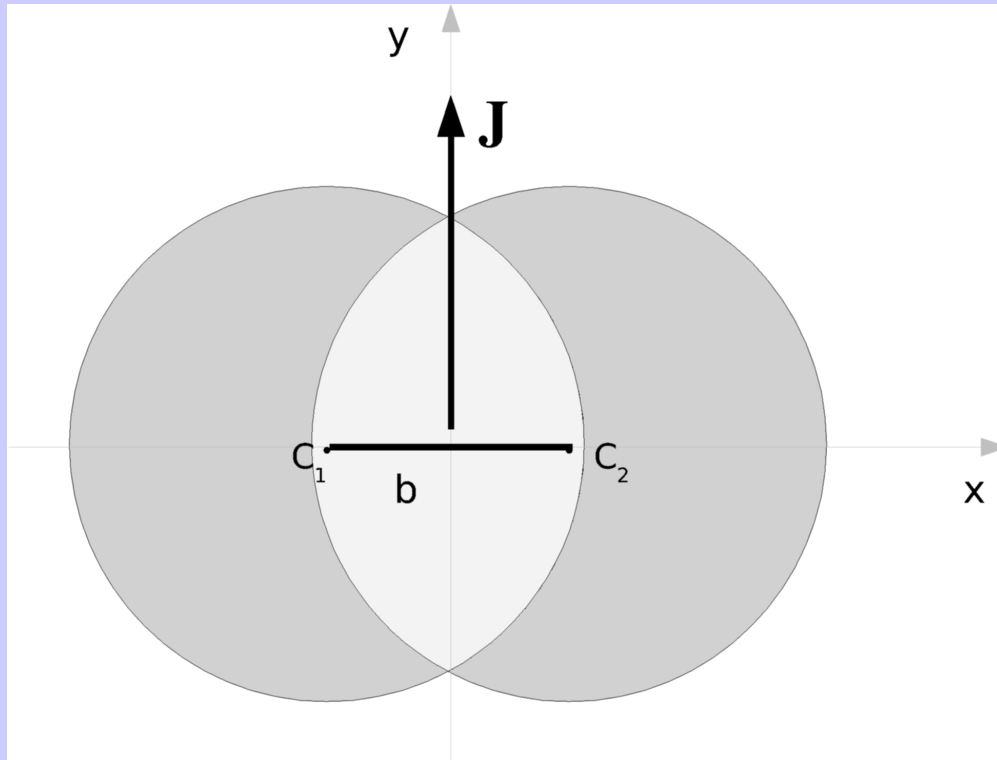
Therefore $\eta/s \gtrsim \hbar$

Il limite universale sembra funzionare...

$$4\pi \cdot \eta / s$$



Peripheral collisions: large angular momentum



See also: Z. T. Liang and X. N. Wang, Phys. Rev. Lett. 94, 102301 (2005)

Can part of this large angular momentum turn into polarization of particles?

If the fluid is almost ideal, also spin degrees of freedom should locally equilibrate very quickly and we could end up with a fluid with finite spin density

Relation between angular momentum and polarization pointed out by

Z. T. Liang and X. N. Wang, Phys. Rev. Lett. 94, 102301 (2005). PQCD CALCULATION

The description of the fluid should involve a spin tensor $\mathcal{S}^{\lambda,\mu\nu}$ besides the stress-energy tensor $T^{\mu\nu}$

$$\begin{aligned}\partial_\mu T^{\mu\nu} &= 0 \\ \partial_\lambda \mathcal{S}^{\lambda,\mu\nu} &= T^{\nu\mu} - T^{\mu\nu}\end{aligned}$$

Many questions arise at this point

Does the spin tensor have a direct physical meaning?

Definition of macroscopic tensors

$$T^{\mu\nu}(x) = \text{tr}[\hat{\rho}\hat{T}^{\mu\nu}(x)]$$

In classical and quantum field theory, these tensors are somewhat arbitrary and can be changed, provided that they fulfill the same conservation equations and that they yield the same energy, momentum and angular momentum

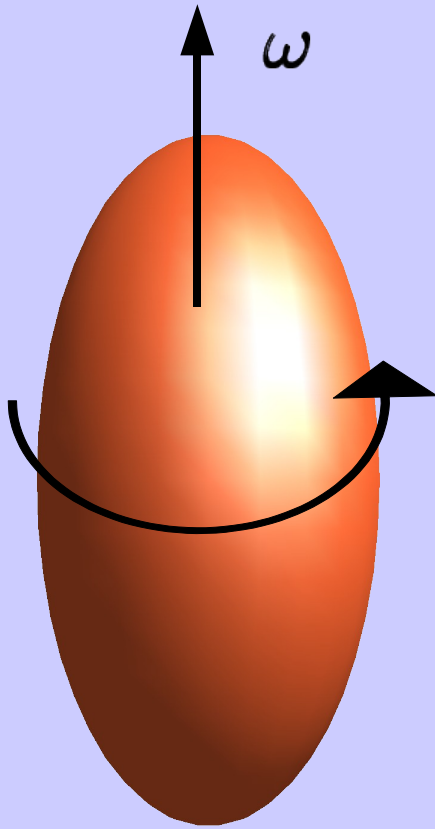
$$\begin{aligned}T'^{\mu\nu} &= T^{\mu\nu} - \frac{1}{2}\partial_\lambda(\Phi^{\lambda,\mu\nu} - \Phi^{\mu,\lambda\nu} - \Phi^{\nu,\lambda\mu}) \\ \mathcal{S}'^{\lambda,\mu\nu} &= \mathcal{S}^{\lambda,\mu\nu} + \Phi^{\lambda,\mu\nu}\end{aligned}$$

However, there is also thermodynamics, which dictates a definite relation between mechanical (pressure, energy, related to stress-energy tensor) and statistical quantities (entropy, i.e. the number of degrees of freedom, related to the discrete nature of matter). This can help to single out one couple, or at least to constrain it.

EXPERIMENTAL EVIDENCE: Barnett effect

S. J. Barnett, Magnetization by Rotation, Phys. Rev.. 6, 239–270 (1915).

Spontaneous magnetization of an uncharged body when spun around its axis



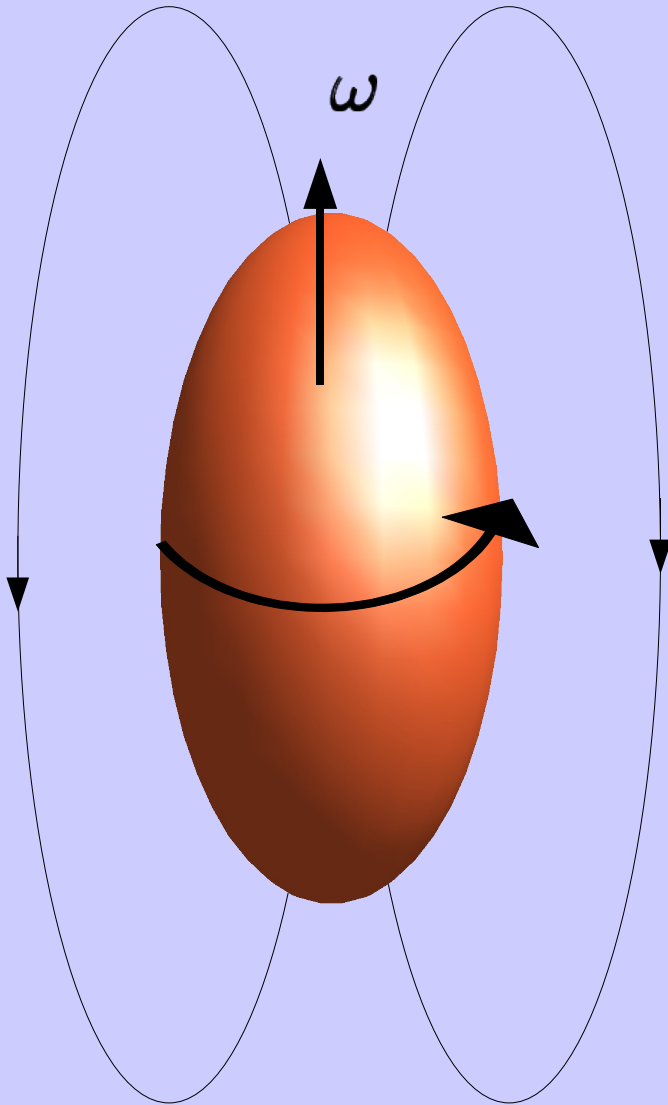
$$M = \frac{\chi}{g} \omega$$

It is a dissipative transformation of the orbital angular momentum into spin of the constituents. The angular velocity decreases and a small magnetic field appears; this phenomenon is accompanied by a heating of the sample. Requires a spin-orbit coupling.

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EXPERIMENTAL EVIDENCE: Einstein-De Haas effect

A. Einstein, W. J. de Haas, Koninklijke Akademie van Wetenschappen te Amsterdam, Proceedings, 18 I, 696-711 (1915)



Rotation of a ferromagnet originally at rest when put into an external H field

An effect of angular momentum conservation:

spins gets aligned with H (irreversibly) and this must be compensated by a on overall orbital angular momentum

A short history of fluid with spin

- M. Mathisson, Acta Phys. Pol. 6, 163 (1937).

- J. Weyssenhoff, A. Raabe, Acta Phys. Pol. 9, 7 (1947).

$$\mathcal{S}^{\lambda, \mu\nu} = \sigma^{\mu\nu} u^\lambda$$

Frenkel condition $t^\mu = \sigma^{\mu\nu} u_\nu = 0$

- D. Bohm and J.P. Vigier, Phys. Rev. **109** 1882 (1958)

Critique of the Frenkel condition

- F. Halbwachs, Theorie relativiste des fluids `a spin, Gauthier-Villars, Paris, (1960)

Variational lagrangian theory of ideal fluids with spin, still with Frenkel condition

From 1960 onwards: many papers with extension and possible applications of Halbwach's theory to general relativity, Einstein-Cartan theory and relevant cosmologies

The simplest fluid: rigidly rotating relativistic ideal Boltzmann gas

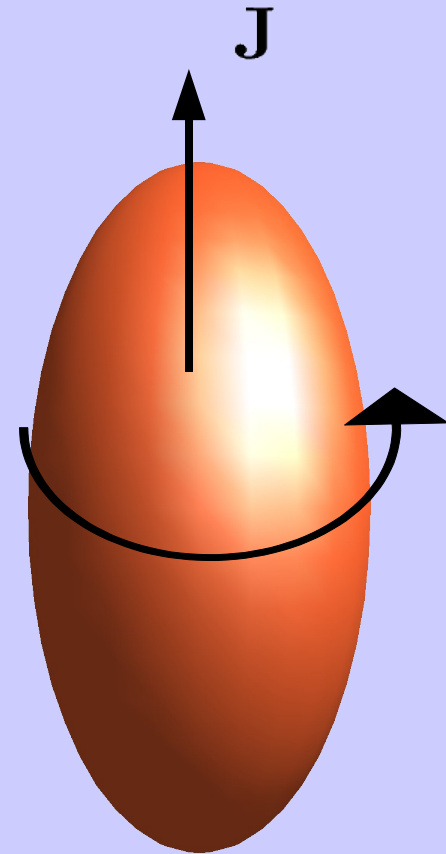
This is a system at complete thermodynamical equilibrium and can be used to learn much about the hydrodynamics of fluids with spin without begging the question, i.e. without introducing any prior stress-energy and spin tensor. The only assumption is statistical equilibrium.

For large enough systems:

$$\hat{\rho} = \frac{1}{Z_{\omega}} e^{-\hat{H}/T + \omega \cdot \hat{\mathbf{J}}/T} \mathbf{P}_V$$

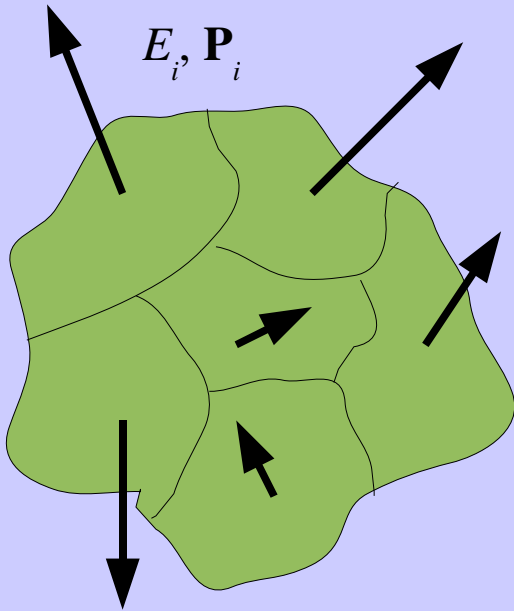
$$Z_{\omega} = \text{tr}[e^{-\hat{H}/T + \omega \cdot \hat{\mathbf{J}}/T} \mathbf{P}_V]$$

Grand-canonical rotational
partition function



which can be obtained by maximizing $S = -\text{tr}[\hat{\rho} \log \hat{\rho}]$ with the constraints of energy and angular-momentum conservation

Why is it rigidly rotating? Landau's argument

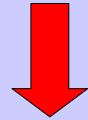


$$S = \sum_i S_i(\sqrt{E_i^2 - \mathbf{P}_i^2})$$

$$\frac{\partial S_i}{\partial E_i} = \frac{E_i}{M_i} \frac{\partial S_i}{\partial M_i} = \frac{\gamma_i}{T_i}$$

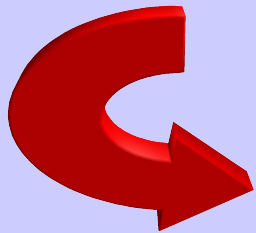
Maximize entropy with constraints

$$\sum_i S_i - \frac{\beta}{T} \cdot \sum_i \mathbf{P}_i - \frac{1}{T} (\sum_i E_i - E_0) - \frac{\omega}{T} \cdot (\sum_i \mathbf{x}_i \times \mathbf{P}_i - \mathbf{J})$$



$$\frac{\gamma_i}{T_i} = \frac{1}{T} \quad \forall i$$

$$\beta_i = \omega \times \mathbf{x}_i \quad \forall i$$



$$T_i = \frac{T}{\sqrt{1 - (\omega \times \mathbf{x}_i)^2}}$$

Local temperature

Rotating fluid at thermodynamical equilibrium: entropy

As a consequence of $\hat{\rho} = \frac{1}{Z_\omega} e^{-\hat{H}/T + \boldsymbol{\omega} \cdot \hat{\mathbf{J}}/T} P_V$

with

$$P_V = \sum_h |h_V\rangle \langle h_V|$$

for an ideal Boltzmann gas

$$Z_\omega = \exp \left[\sum_j \frac{\lambda_j}{(2\pi)^3} \int d^3x \int d^3p \operatorname{tr} D^{S_j}(\mathbf{R}_{\hat{\mathbf{J}}}(i\omega/T)) e^{-\varepsilon_j/T} e^{\boldsymbol{\omega} \cdot (\mathbf{x} \times \mathbf{p})/T} \right]$$

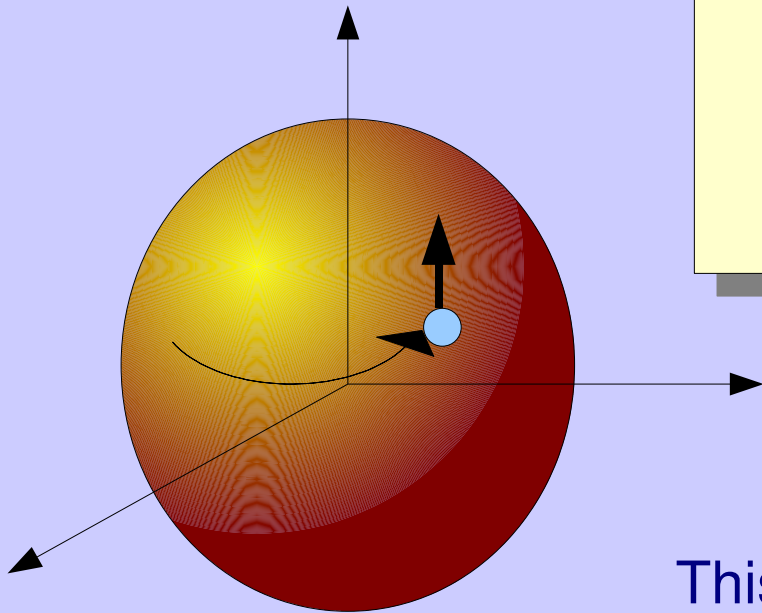
and:

$$S = \frac{U}{T} - \frac{\boldsymbol{\omega} \cdot \mathbf{J}}{T} + \log Z_\omega - \frac{\sum_i \mu_i Q_i}{T}$$

Rotating fluid at thermodynamical equilibrium: polarization

F. B., F. Piccinini, Ann. Phys. 323, 2452 (2008).

$$\Pi^\mu = \text{tr}(\hat{W}^\mu \hat{\rho}) \qquad \hat{W}^\mu = -\frac{1}{2m} \epsilon^{\mu\nu\rho\sigma} \hat{J}_{\nu\rho} \hat{p}_\sigma$$



in the inertial frame

$$\Pi = \frac{\sum_{n=-S}^S n e^{n\omega/T}}{\sum_{n=-S}^S e^{n\omega/T}} \left(\frac{\hat{\omega} \cdot \hat{\mathbf{p}}}{m}, \frac{\varepsilon}{m} \hat{\omega} \right)$$

$$\Pi = \mathcal{O}(\hbar\omega/KT)$$

This polarization is very small for usual systems but it is just what causes Barnett effect

Crucial steps in the derivation

To obtain the polarization expression, the key ingredient is the matrix element

$$\langle p, \tau | \hat{R}_{\hat{\omega}}(\phi) P_V | p, \sigma \rangle = \int_V d^3x \, e^{i\mathbf{x} \cdot (\mathbf{p} - R_{\hat{\omega}}(\phi)^{-1}(\mathbf{p}))} \, \frac{1}{2} \left(D^S([p]^{-1} R_{\hat{\omega}}(\phi)[p]) + D^S([p]^\dagger R_{\hat{\omega}}(\phi)[p]^\dagger{}^{-1}) \right)_{\tau\sigma}$$

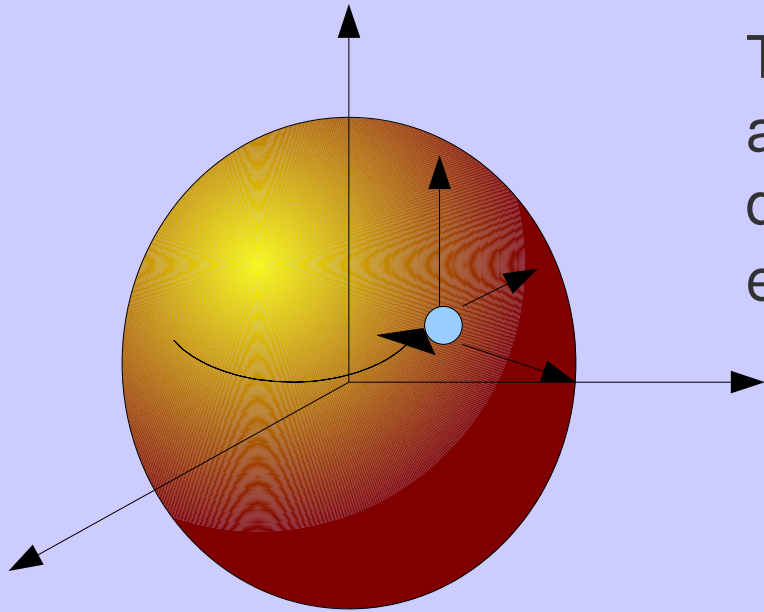
which in turn stems from an ansatz

$$\langle p', \sigma' | P_V | p, \sigma \rangle \propto \left(D^S([p']^{-1}[p]) + D^S([p']^\dagger[p]^\dagger{}^{-1}) \right)_{\sigma'\sigma}$$

motivated by the enforcement of

$$[P_V, \hat{R}_{\hat{\mathbf{n}}}(\phi)] = 0$$

From global to local: hydrodynamical view



The spinning ideal gas, if large, can be seen as a fluid and local quantities, like proper energy density or pressure can be inferred from the expression of Z_ω

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}$$

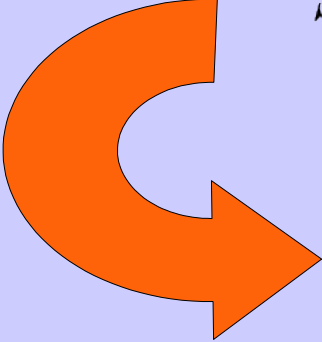
$$p = T \frac{d \log Z_\omega}{d^3 x} \quad \rho = \frac{1}{\gamma} \frac{d \langle E \rangle}{d^3 x}$$

Since particles are polarized, there must exist also a spin tensor:

$$\mathcal{S}^{\lambda, \mu\nu} = \sigma^{\mu\nu} u^\lambda$$

$$\sigma = \frac{1}{\gamma} \frac{d \langle \mathbf{S} \rangle}{d^3 x}$$

Local thermodynamical relation


$$S = \frac{U}{T} - \frac{\boldsymbol{\omega} \cdot \mathbf{J}}{T} + \log Z_{\omega} - \frac{\sum_i \mu_i Q_i}{T}$$

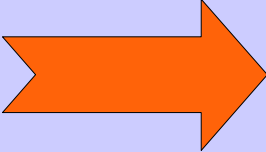
$$T_0 s + \mu n = \rho + p - \gamma \boldsymbol{\omega} \cdot \boldsymbol{\sigma}$$

To make it covariant the angular velocity is to be replaced with the acceleration tensor constructed with the Frenet-Serret tetrad

$$\Omega_{\mu\nu} = \sum_{i=0}^3 e_{\mu}^i \dot{e}_{i\nu}$$

See F. Halbwachs

“Theorie relativiste des fluids a spin”, Paris, 1960


$$T_0 s + \mu n = \rho + p + \frac{1}{2} \Omega_{\mu\nu} \sigma^{\mu\nu}$$

$$T_0 s + \mu n = \rho + p + \frac{1}{2} \Omega_{\mu\nu} \sigma^{\mu\nu}$$

Already been used in literature in GR and Einstein-Cartan theory.
It should be stressed that the last term is a QUANTUM
correction to thermodynamics because it involves a spin density tensor

For the ideal Boltzman gas, one can derive the expression of the phase-space
distribution (F.B., L. Tinti, arXiv:0911.0864)

$$\beta = \frac{1}{T_0} u$$

$$f(\mathbf{x}, \mathbf{p})_{\sigma\rho} = \lambda e^{-\beta \cdot p} \frac{1}{2} \left[D^S([p]^{-1} R_{\hat{\mathbf{J}}}(i\omega/T)) [p] + D^S([p]^\dagger R_{\hat{\mathbf{J}}}(i\omega/T) [p]^\dagger{}^{-1}) \right]_{\sigma\rho}$$

In Dirac spinor's notation

$$f(\mathbf{x}, \mathbf{p})_{\sigma,\rho} = \lambda e^{-\beta \cdot p} \frac{1}{2} \bar{u}_\sigma(p) D^{(1/2,1/2)}(R(i\omega/T)) u_\rho(p)$$

Spin density tensor for an ideal Boltzmann gas

$$\sigma^{\mu\nu} = \iota \Omega^{\mu\nu}$$

$$\iota = \frac{n}{T} \frac{S(S+1)}{3}$$

This implies that the Frenkel condition is VIOLATED

$$t^\mu = \sigma^{\mu\nu} u_\nu = \iota \Omega^{\mu\nu} u_\nu = \iota A^\mu \neq 0$$

even for the simplest fluid with spin!

This deserves a little reflection. It means that we cannot describe the spin density of a fluid with ONE spacial vector (polarization), but we need TWO.

This is unfamiliar because a single quantum relativistic particle in a pure state has $t=0$ but a fluid is made of many particles with random polarization vectors

$$\langle t^\mu \rangle \propto \sum_i J_i^{\mu\nu} \sum_j u_{\nu j} \neq \sum_i t^n u_i = \sum_i J_i^{\mu\nu} u_{\nu i}$$

Is the spin tensor arbitrary?

From pure thermodynamics, we have inferred that there is a non-vanishing polarization of particles for the simplest fluid at equilibrium with finite angular momentum and this in turn implies that spin density is non-vanishing as well

$$\mathcal{S}^{\lambda, \mu\nu}(x) = \sigma^{\mu\nu} u^\lambda \neq 0$$

Therefore, if we want to keep the correspondance between microscopic (quantum) and macroscopic tensors

$$\mathcal{S}^{\lambda, \mu\nu}(x) = \text{tr}[\hat{\rho} \hat{\mathcal{S}}^{\lambda, \mu\nu}(x)]$$

the only possible conclusion is that NOT ALL transformation of stress-energy and spin tensors are allowed.

Particularly, the Belinfante symmetrization procedure is ruled out by thermodynamics

$$T'^{\mu\nu} = T^{\mu\nu} + \frac{1}{2}\partial_\lambda(\mathcal{S}^{\lambda,\mu\nu} - \mathcal{S}^{\mu,\lambda\nu} - \mathcal{S}^{\nu,\lambda\mu})$$
$$\mathcal{S}' = 0$$

From a local point of view, thermodynamics enforces a relation between the stress-energy tensor and a quantity which is beyond a classical relativistic theory: ENTROPY.

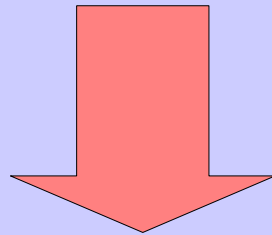
$$Ts + \mu n = \rho + p$$

$$\rho = T^{\mu\nu}u_\mu u_\nu \quad p = -\frac{1}{3}(T^\mu_\mu - \rho)$$

Entropy can only be introduced counting states, i.e. in a quantum field framework; it requires the existence of *particles*.

Ongoing studies

- Determination of a class of transformations of the stress-energy and spin tensor leaving entropy density (i.e. thermodynamics) invariant
- Determination of kinetic coefficients for spin transport
- Is it possible to transform macroscopic spin into orbital angular momentum reversibly (the problem of ideal fluid with spin)?

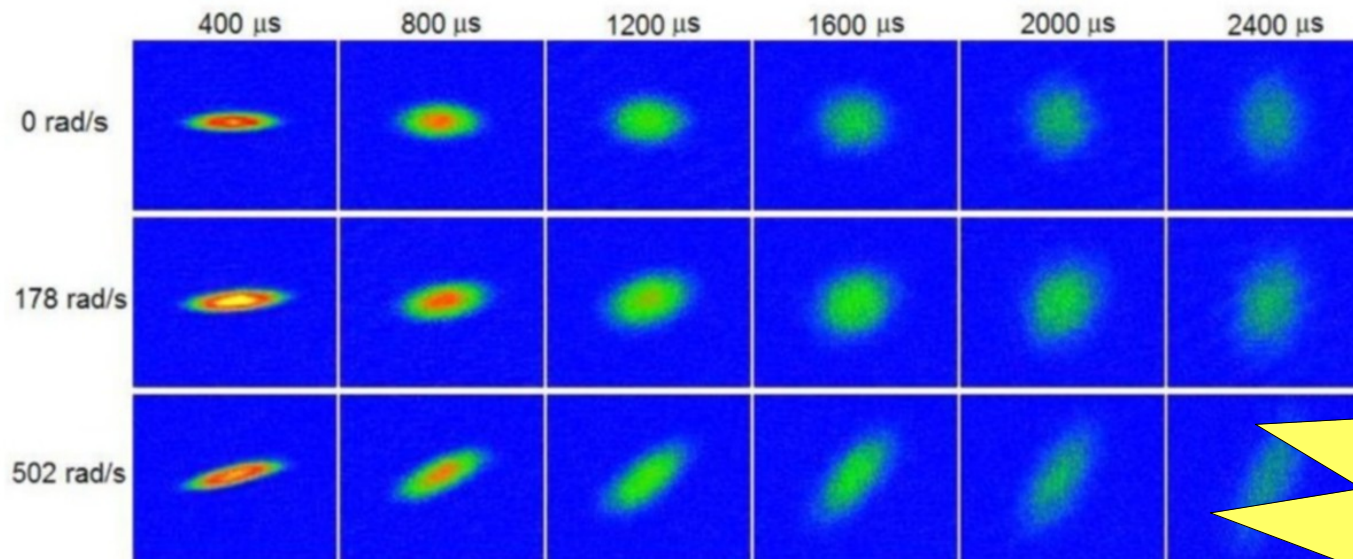
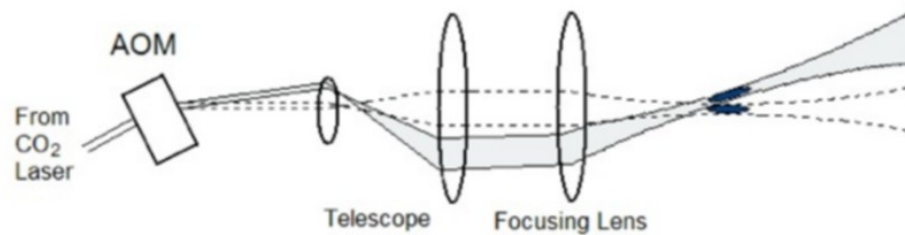


General formulation of hydrodynamics with spin

Possible application to cold atoms ?

J. Thomas, arXiv: 0907.0140

Measuring Viscosity from the expansion of a rotating gas



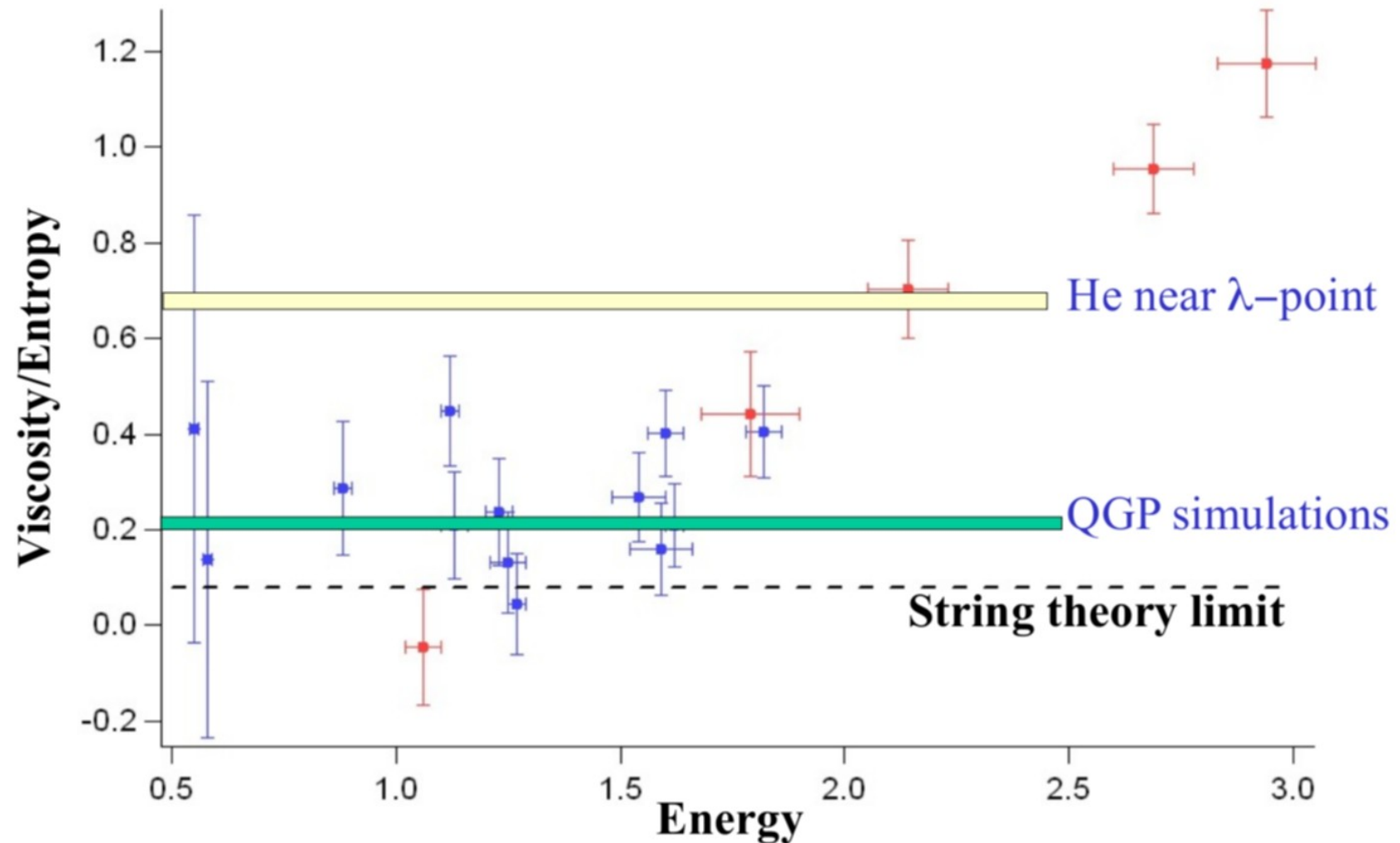
$$\hbar\omega / KT \sim 0.1$$



**Duke
Physics**

Atom Cooling and Trapping

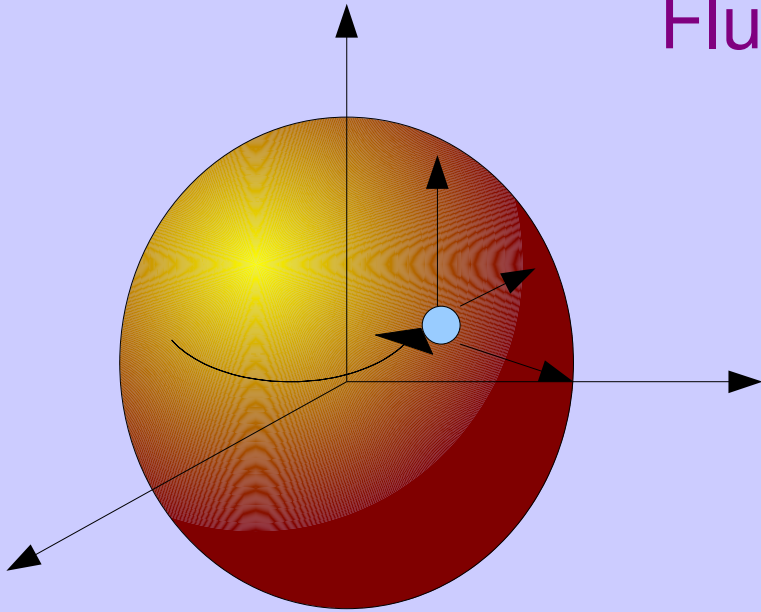
Viscosity/Entropy (natural units)



CONCLUSIONS

- We have calculated the exact expression of the polarization and spin density tensor in a rotating relativistic Boltzmann gas in full thermodynamical equilibrium
- Thermodynamics requires the inequivalence of stress-energy and spin tensors (same with gravity...)
- ◆ We look forward to have a general formulation of hydrodynamics with spin and test it in heavy ion collisions as well as in cold atoms experiments

Fluid cell point of view



$$T_0 s + \mu n = \rho + p + \frac{1}{2} \omega_{\mu\nu} \sigma^{\mu\nu}$$

The accelerated observer measures a polarization of particles. Therefore, there must be a general relation between polarization and acceleration which is contained in the distribution:

Local density matrix for an accelerated thermodynamical system at equilibrium

$$\hat{\rho} \propto \exp[-\beta(x) \cdot \hat{P} + \frac{1}{2} \sqrt{\beta^2} \omega(x) : \hat{S}]$$

Acceleration implies polarization

Acceleration effects depend on the energy scale

$$\frac{\hbar A}{c}$$

which is $3.4 \cdot 10^{-42}$ J for $a=g$ corresponding to $T = 2.5 \cdot 10^{-19}$ K

In heavy ions, average acceleration due to transverse expansion can be as high as

$$a = \frac{c}{\tau} \approx 10^{30} m/sec^2$$

corresponding to an energy scale $O(1 \text{ MeV})$.

This would imply polarization effects of the order of a percent or less.

Yet, their dependence on momentum could be such to allow their measurement.

The angular velocity tensor

In the inertial frame

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & \gamma^3 \mathbf{a} - \gamma^3 \boldsymbol{\omega} \times \mathbf{v} \\ -\gamma^3 \mathbf{a} - \gamma^3 \boldsymbol{\omega} \times \mathbf{v} & \gamma^3 \boldsymbol{\omega} + \gamma^3 \mathbf{a} \times \mathbf{v} - \gamma^3 \boldsymbol{\omega} \cdot \mathbf{v} \mathbf{v} \end{pmatrix}$$

In the comoving frame

$$\omega_{ij} = \begin{pmatrix} 0 & \mathbf{A} \\ -\mathbf{A} & \varepsilon_{ijk} \omega'^k \end{pmatrix}$$

Being $A = (0, \mathbf{A})$ the four-acceleration in the comoving frame