PHYSICS OF COMPACT OBJECTS IN GENERAL RELATIVITY AND BEYOND
LECTURE 8

Scalar-tensor theories of gravity

Scalar-tensor theories are theories in which gravity is described, besides the metric tensor, by one or more (I will assume one for simplicity) scalar field.

Let us consider a theory with a modification from GR, parametrized by some *coupling* parameters α_i,

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\,[R + \alpha_1(\dots) + \alpha_2(\dots) + \dots] + S_{matter}$$

where the first action is the "gravitational sector" of the theory, while S_{matter} is the "matter sector" of the theory, describing the non-gravitational interactions. We can consider this action in two possible frameworks.

- We can consider this as the "full action", requiring that the field equations for the gravitational field (of fields), and the field equations of the other fields, are "well behaved", i.e. do not give rise to ghosts of other instabilities. One example of these pathologies is the so-called *Ostrogradsky instability*, which occurs if the field equations are of order lager than two in time. All these problems should be taken into account when trying to modify the Einstein-Hilbert action. Also, we should know that proving the mathematical consistency of a theory is a very complex task, and requires further steps - such as verifying the possibility of a well-posed evolution - which have not been done for most of the theories proposed.
- Otherwise, we can consider the action as an "effective action", assuming that there are other terms, of higher order in the couplings α_i, not relevant at the energy scales we are going to consider. In this case we do not have to require that the theory is mathematically consistent, because any problem may be cured by the extra unknown terms. Of course, in this way the corrections will be small, and then difficult to detect, by definition.

Let us now consider the different possible theories with scalar fields.

Bergmann-Wagoner scalar-tensor theories

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\,[\phi R - B(\phi)\partial^\mu\phi\partial_\mu\phi - U(\phi)] + S_{matter}(\psi_m, g_{\mu\nu})$$

In these theories the scalar field has a non-minimal coupling with the Ricci scalar, ϕR. The simplest of them is Brans-Dicke theory, in which $B(\phi) = constant/\phi$, first proposed about half a century ago. The B-W theories do not have any pathology: their field equations are second order, and these are the unique modified gravity theories for which it has been demonstrated that a well-posed evolution formulation exists.

The dynamics of B-W theories is relatively simple, and it has been shown that **the no-hair theorem holds** for them. Moreover, the BH dynamics is also the same of that of GR (I will discuss below the link between the no-hair theory and the dynamics); so, for instance, the BBH waveform is the same as in GR, and the QNMs are also the same. Therefore, in order to test GR against these theories we have to look at the motion of **neutron stars**. In particular, binary NSs emit dipole radiation, which affects their orbit in a possibly observable way.

$f(R)$ gravity

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} f(R) + S_{matter}(\psi_m, g_{\mu\nu})$$

where $f(R)$ is a function of the curvature scalar. For instance, $f(R) = R + aR^2$. Apparently, this is not a scalar-tensor theory, but it has been shown that these theories can be reformulated as particular cases of Bergmann-Wagoner scalar-tensor theories. So, what we said for them holds for $f(R)$ theories; in particular, **the no-hair theorem holds**.

Horndeski gravity

This is the most general scalar-tensor theory with consistent (2nd order) field equations. It contains very complicate combinations of derivatives of the scalar field, some of which coupled with the Ricci scalar or with the Einstein tensor:

$$\begin{aligned} S = \int & d^4x \sqrt{-g} \{K(\phi, X) - G_3(\phi, x)\Box\phi + G_4(\phi, X)R \\ & + G_{4,X} \left((\Box\phi)^2 - (\nabla_\mu \nabla_\nu \phi)(\nabla^\mu \nabla^\nu \phi)\right) \\ & + G_5 G_{\mu\nu} \nabla^\mu \nabla^\nu \phi - G_{5,X} \frac{1}{6} \left[(\Box\phi)^3 \right. \\ & \left. - 3\Box\phi(\nabla_\mu \nabla_\nu \phi)(\nabla^\mu \nabla^\nu \phi) + 2(\nabla_\mu \nabla_\rho \phi)(\nabla^\mu \nabla_\sigma \phi)(\nabla^\rho \nabla^\sigma \phi)\right]\} \\ & + S_{matter}(\psi_m, g_{\mu\nu}) \end{aligned}$$

where $X \equiv \partial_\mu \phi \partial^\mu \phi$. It is extremely complex to study theoretically these theories, but some results have been found. In particular, for some subcases the no-hair theorem holds, for some others it is violated, and thus hairy BHs exist.

Quadratic gravity

It has been shown that GR may become renormalizable if the Einstein-Hilbert Lagrangian is extended to include quadratic terms in the curvature tensor. These terms could be considered as a truncation of a polynomial expansion in the curvature. The general action with quadratic terms in the curvature would be:

$$\mathcal{L} = R + \lambda_1 R^2 + \lambda_2 R_{\mu\nu} R^{\mu\nu} + \lambda_3 R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + \lambda_4 R R^*.$$

where $RR^* = \epsilon_{\mu\nu}{}^{\rho\sigma} R^{\mu\nu\alpha\beta} R_{\rho\sigma\alpha\beta}$. They can be rearranged as

$$\mathcal{L} = R + \alpha_1 R^2 + \alpha_2 R_{\mu\nu} R^{\mu\nu} + \alpha_3 R_{GB} + \alpha_4 RR^*$$

where

$$R_{GB} = R^4 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \,.$$

The latter is called **Gauss-Bonnet term** and is a total derivative. Therefore, its contribution to the action is vanishing. The term RR^* is also a total derivative. The remaining two terms are quadratic combinations of the Ricci tensor and of the Ricci scalar; therefore, the field equations are proportional to $R_{\mu\nu}$ and to R. The effects of such modification on BH solutions (which are in vacuum, where $R = R_{\mu\nu} = 0$) would then be negligible: the field equations, expanded around a (vacuum) GR background, would necessarily be the same as in GR. Such theory would then be very difficult to test.

Conversely, if we couple these terms to functions of a scalar field, we have the so-called *quadratic scalar-tensor theories of gravity*:

$$\mathcal{L} = R + \alpha_1 f_1(\phi) R^2 + \alpha_2 f_2(\phi) R_{\mu\nu} R^{\mu\nu} + \alpha_3 f_3(\phi) R_{GB} + \alpha_4 f_4(\phi) RR^* \,.$$

Again, the terms with α_1 and α_2 would not affect the BH spacetimes, but the last two would: the total derivative, once integrated by parts, give derivatives of the scalar field. These two terms lead to two different theories, both allowing for violations of the no-hair theorem:

- Scalar Gauss-Bonnet gravity $\mathcal{L} = R + \alpha_{GB} f(\phi) R_{GB}$.

  This theory admits **hairy BHs**. It belongs to the Horndesky class, therefore it has field equations of second order in time, and thus does not have instabilities or ghosts: it could be acceptable as a "full theory". The coupling has dimensions of mass squared, and for a BH solution the dimensionless coupling α/M^2 can be as large as ~ 0.6.

  If $f(\phi) = e^{\phi}$, this theory is called Einstein-dilaton Gauss-Bonnet gravity, and can be shown to originate from string theory solutions.

- Dynamical Chern-Simons gravity $\mathcal{L} = R + \alpha_{CS} f(\phi) RR^*$.

  This theory admits **hairy BHs**. Moreover, the Levi-Civita tensor leads to parity violation; in practice, in this theory there is *birefrangence* of GWs: different polarizations have amplitudes decaying in different ways, a "smoking-gun effect" conceptually simple to test. However, this theory has third-order time derivatives in the field equations and thus it is not acceptable as a full theory, only as a truncated version of some more fundamental theory; thus, one has to assume, for a BH solution, $\alpha_{CS}/M^2 \ll 1$.

Scalar charge and dipolar emission

In order to understand the effects of GR modifications in the motion of compact objects for the wide class of scalar-tensor theories, we should consider a crucial quantity: the **scalar charge** of a compact object.

In the stationary solution describing the compact object, the scalar field has generally the form (at a sufficient distance from the body):

$$\phi = const + \frac{\alpha}{r} + O\left(\frac{1}{r^2}\right)$$

where α is the *scalar charge* of the body. This quantity is not really a charge in the Noether sense - it is not conserved - but still it is a feature which determines how the scalar field affects the phenomenology of the body.

In particular, in compact binary inspirals in scalar-tensor theory, there is a **dipole emission** correction:

$$\dot{E}_{GW} = \dot{E}_{GR}\left[1 + \frac{5}{96}\Delta\alpha^2\left(\frac{GM}{r}\right)^{-1}\right]$$

where $\Delta\alpha$ is the difference between the scalar charges of the bodies. As you can see, *the earlier is the inspiral, the larger is the effect!*

Similarly, if a compact body inspirals around a supermassive BH in an EMRI (extreme mass-ratio inspiral, i.e. an inspiral with $m_1 \ll m_2$, a target source for LISA), the waveform has a correction which is - in its leading term - independent of the theory, only proportional to the scalar charge of the body.

Therefore, we can can devise the measure of a quantity - the scalar charge - without having to choose a particular theory, as long as it is a gravity theory with a scalar field. This allows for powerful and practical tests of GR.

But which is the scalar charge of compact objects in different theories? Well, in general, the scalar charge of a BH is vanishing if the no-hair theorem applies, because the scalar field can not "anchor" on the BH; conversely, NSs in general have a non-vanishing scalar charge.

- In B-W theories, the no-hair theorem applies. BHs have vanishing scalar charge. That's why the BBH inspiral is the same as in GR. NSs, instead, have a scalar charge depending on their EoS; more compact NSs have larger scalar charge. All other less gravitating bodies (e.g. white dwarfs) have negligible scalar charge.

  It is worth mentioning a phenomenom discovered by Damour and Esposito-Farese: in a certain region of the parameter space of B-W theories, compact stars show the phenomenom of *spontaneous scalarization*: they dynamically develop a non-trivial scalar field profile, with a significant scalar charge. In this case, even if the corrections to GR are tiny, the "scalarized" stars are significantly different from the GR one. However, if this occurs, we would probably already have seen it in the observations of binary pulsars. So, today this phenomenom is believed to be unlikely.

- In scalar Gauss-Bonnet gravity, the no-hair theorem is violated; BHs have a non-trivial scalar field profile, and a scalar charge given by the coupling of the theory: $\alpha = \alpha_{GB}/M$. Some of these theories allow for a spontaneous scalarization of BHs: a Kerr BH, changing its mass or angular momentum, can become instable to scalar field growth, until it becomes a different stationary solution, with a non-trivial scalar field profile. NSs, instead, have always vanishing scalar charge.
- In Horndesky theories, it has been shown that, at least for a subclass of the theories, those with shift simmetry, i.e. those invariant for $\phi \to \phi + const$, the only theory which can violate the no-hair theorem is scalar Gauss-Bonnet gravity, which belongs to the Horndesky family.

Quasi-normal modes

Finally, a comment on QNMs. In scalar-tensor theories, with the usual exception of Bergmann-Wagoner theories, the QNMs spectrum of BHs is modified. The general pattern of the modification (same for all theories, and also with theories with more scalar fields, or with different kinds of extra fields) is:

- new classes of modes in the GW spectrum, due to coupling to extra fields;
- a (small) shift in the modes predicted by GR.

This has been studied in some theories, such as scalar GB and CS. It was found that the new classes of modes are likely to be poorly excited in BH coalescences, while the shifts in the “old” modes could be detectable, if the amplitude of the signal is large enough.