PHYSICS OF COMPACT OBJECTS IN GENERAL RELATIVITY AND BEYOND LECTURES 1-2

With *compact objects* we mean astrophysical objects which have a *non-negligible compactness*, i.e.

$$C = \frac{M}{R} \lesssim 1$$

(geometrized units $G = c = 1$).

Near compact objects there is the *strong-field regime of gravity*, in which the GR effects are significant, and the **non-linearity** of Einstein's equations also becomes relevant.

Since this regime has been mostly impossible to observe until very recently, with the detection of GWs, GR in this regime is very poorly constrained by observational data; and thus, we can not exclude that **deviations** from GR are present in this regime.

To our present knowledge, only two kinds of compact objects exist:

- Neutron stars (possibly, quark stars) [NSs]
- Black holes [BHs]

This statement relies on our understanding of stellar evolution, of nuclear physics, of fundamental physics. Not only we understand, to some extent, the *structure* of these objects: we also know their *formation mechanism*, and thus we can estimate how many of them are in the universe, their distribution, their electromagnetic emission (on which we have plenty of observational data), their GW emission (on which we are starting to collect data). All this makes a coherent picture - with some holes and question marks, of course, but still a coherent picture.

Of course this does not exclude that different kinds of compact objects exist: either those that we observe may be different from what we think, or other compact objects still unobserved may be out there. The so-called “exotic compact objects”, ECOs, are still a possibility which deserves to be tested. In these lectures I will discuss BHs and NSs.

Some references: classic books: Misner, Thorne, Wheeler *Gravitation*, Weinberg *Gravitation and Cosmology*, Wald *General Relativity*, Chandrasekhar *The Gravitational theory of BHs*, and more recently: Poisson and Will *Gravity.* For some computations I also followed: Ferrari, Gualtieri, Pani *General Relativity and its Applications.*

Some of the main observables:

For BHs:

- Accretion disks around BHs. EM emission, mainly observed by X-ray telescopes. In the near future, precision measurements from large area X-ray telescopes as Athena, eXTP.
- BH shadow. EM emission observed by EHT (an array of radio-telescopes), the so-called "picture" of a BH.
- Motion of stars orbiting in the gravitational field of a large BH. EM emission, observed by infrared telescopes.
- BBH coalescences $\sim 5-100\,M_\odot$. GW emission, observed by ground-based detectors: now 2nd generation (LIGO, Virgo, soon Kagra), in the near future 3rd generation detectors with much larger sensitivity (ET, Cosmic Explorer).
- BBH coalescence $10^6 - 10^7\,M_\odot$. GW emission, to be observed in the near future by space-based detector LISA.
- EMRI (compact star or $\sim 5-100\,M_\odot$ around $10^6-10^9\,M_\odot$ BH). GW emission, to be observed in the near future by space-based detector LISA, and by PTA (GW affect motion of pulsars observed with radio-telescopes, see Sesana's lecture)
- Stochastic background of GWs, possibly observed in the near future by space-based detector LISA.

For NSs:

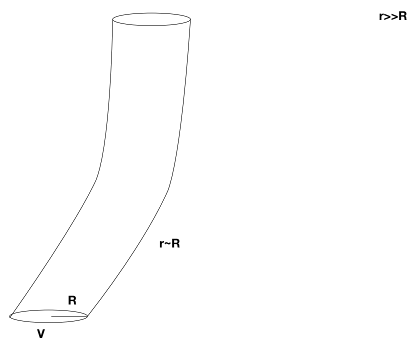
- binary pulsars. EM emission (but indirect effect of GWs), observed by radiotelescopes
- Accretion disks around NSs. EM emission, mainly observed by X-ray telescopes. In the near future, precision measurements from large area X-ray telescopes as Athena, eXTP.
- BNS coalescences. GW emission, observed by ground generation detectors: now 2nd generation (LIGO, Virgo), in the near future 3rd generation detectors with much larger sensitivity.
- Pulsars as continuous GW sources, to be observed in the near future by 3rd generation detectors.

In the following I will discuss BHs and NSs, first in GR and then beyond GR, in order to understand what do we know of their structure, what can we learn from them, and which fetures can help characterizing and understanding their phenomenology.

Of course there are so many features to discuss, there is definitely no time to discuss all of them, so I will choose some of them which I think are interesting.

STATIONARY, ISOLATED BODIES IN GR

Let us consider a **stationary, isolated body**. The support of its stress-energy tensor - the spacetime region in which it is non-vanishing is the worldvolume of the body, i.e. a certain *worldtube*; outside this region, $T_{\mu\nu} = 0$. Since the body is isolated, the spacetime



generated by the body as *asymptotically flat*. There is a precise mathematical definition of asymptotic flatness, based on the concept of conformal rescalings introduced by Roger Penrose, but we do not have the time to discuss this here; we only need to know that when a spacetime is asymptotically flat, it is possible to define a spacelike coordinate r such that

$$\lim_{r\to\infty} g_{\mu\nu} = \eta_{\mu\nu}$$

and then the metric can be expanded in powers of $1/r$. If R is the lengthscale of the object, there is a *near region* in which r is comparable to R, which - if the body is a compact object ($C \lesssim 1$) - is a *strong-field* region. And there is the **far-field region** $r \gg R$, which is always a weak-field region, and thus where

$$g_{\mu\nu} = \eta_{\mu\nu} + O\left(\frac{1}{r}\right) = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{with} \quad |h_{\mu\nu}| \ll 1\,, \quad |h'_{\mu\nu}| \ll 1/R\,.$$

A. Far-field metric of a stationary, isolated body

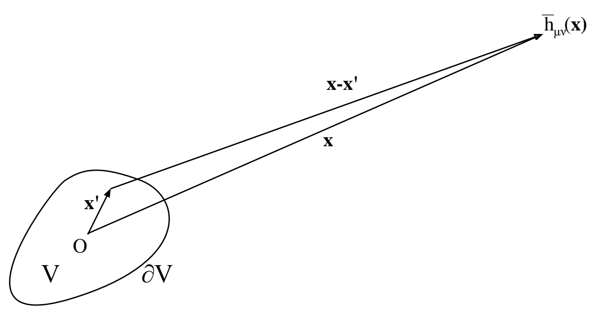
The **far-field** metric is a mine of precious information on the compact object. The leading-order terms of the $1/r$ expansion can tell us the **mass** M and **angular momentum** J of the central body: the metric of a stationary, isolated object in the far-field region can always be written as:

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1+\frac{2M}{r}\right)dr^2 + r^2(d\theta^2+\sin^2\theta d\varphi^2) - \frac{4J}{r}\sin^2\theta dt d\varphi + \text{ higher-order terms in } 1/r\,. \quad (1)$$

Let us first consider the case in which the field is weak, i.e.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{with} \;\; |h_{\mu\nu}| \ll 1\,, \;\; |h'_{\mu\nu}| \ll 1/R$$

not only far away from the body, but also throughout the body itself. Solving Einstein's equations linearized on flat space for a *stationary spacetime*



$$\Box_F \bar{h}_{\mu\nu} = -\nabla^2 \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}$$
$$\bar{h}^{\mu}{}_{\nu,\mu} = \bar{h}^{i}{}_{\nu,i} = 0\,,$$

where

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^{\alpha}{}_{\alpha}$$

can be solved as:

$$\bar{h}_{\mu\nu}(\vec{x}) = 4\int_V \frac{T_{\mu\nu}(\vec{x'})}{|\vec{x}-\vec{x'}|}\, d^3x'$$

By Taylor expanding $1/|\vec{x}-\vec{x}'|$ around $|\vec{x}'|=0$ we find the so-called *multipolar expansion*:

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r} + \frac{x^i x'^i}{r^3} + O\left(\frac{1}{r^3}\right)\,,$$

where $r = |\vec{x}|$. By replacing this expansion,

$$\bar{h}_{\mu\nu}(\vec{x}) = \frac{4}{r}\int_V T_{\mu\nu} d^3x' + \frac{4x^i}{r^3}\int_V T_{\mu\nu} x'^i d^3x' + O\left(\frac{1}{r^3}\right) .$$

Thus, it is possible to express $\bar{h}_{\mu\nu}$ in terms of integral over the source at fixed time, in particular

$$\int_V T_{00} d^3x' = M$$

and

$$J^i = \epsilon_{ijk}\int_V x'^j T^{0k} d^3x' .$$

Note that in the weak field regime we can apply the laws of SR, and since in this case there are no velocities, we can apply Newtonian physics, where $T_{00} = \rho$ mass density, and $T^{0k} = \mathcal{P}^k$ momentum density. Then, these quantities are the well-known **mass** and **angular momentum** of the source; for the latter, note that if $\vec{x} = (x^1, x^2, x^3)$ and $\vec{\mathcal{P}} = (\mathcal{P}^1, \mathcal{P}^2, \mathcal{P}^3)$, then $\epsilon_{ijk} x^j T^{0k}$ are the components of

$$\vec{x'} \times \vec{\mathcal{P}}$$

density of the angular momentum.

A straightforward computation shows then that the far-field limit metric is given by:

$$\begin{aligned} h_{00} &= \frac{2M}{r} + O\left(\frac{1}{r^3}\right) \\ h_{0i} &= \frac{2}{r^3}\epsilon_{ijk} x^j J^k + O\left(\frac{1}{r^3}\right) \\ h_{ij} &= \frac{2M}{r}\delta_{ij} + O\left(\frac{1}{r^3}\right) . \end{aligned} \tag{2}$$

Then, by carefully choosing the coordinate system, one finds the line element (1): at leading order in the $1/r$ expansion, the metric only depends on M and on J.

If we drop the simplifying assumption of weak field *on the source*, we can not rely on the Newtonian definitions of mass and angular momentum. We can just solve the field equations in vacuum,

$$\begin{aligned} \nabla^2 \bar{h}_{\mu\nu} &= 0 \\ \bar{h}^i{}_{\nu,i} &= 0 , \end{aligned}$$

in the far field limit only. This is a differential equation, and the general solution will depend on some integration constant. Choosing appropriately the coordinate system, the general solution is again (2), but now M and J are **just integration constants**.

Still, it is possible to give a physical interpretation of these constants. You know that it is impossible to define, in a general spacetime, a conserved energy density and a conserved momentum density, because the stress-energy tensor does not take into account the energy density associated to the gravitational field. However, it is possible to define a quantity, the *stress-energy pseudo-tensor*

$$(-g)(t^{\mu\nu} + T^{\mu\nu}) = \frac{1}{16\pi}\frac{\partial^2}{\partial x^\alpha \partial x^\beta}(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta})$$

which **does** take into account the energy density associated to the gravitational field, and is conserved. This definition cannot be applied locally, in a small neighboroud, but it can be applied if integrated in a large enough region of spacetime. We can define the total four-momentum of the spacetime as an integral in the entire three-volume V_{tot} at $t = const$, extending *up to the far field region.*

$$P^\mu = \int_{V_{tot}} d^3x(-g)(T^{0\mu} + t^{0\mu})$$

This quantity which is *constant in time.*

Remarkably, this quantity is a total divergence, and thus it can be written - by Gauss' theorem - as an integral on the surface of V_{tot}. To compute this integral, then, we only need to know the metric in the far-field limit. If we replace the metric (2), we find that

$$P^0 = M$$

and similarly by defining the angular momentum

$$\epsilon_{ijk}\int_{V_{tot}} d^3x(-g)x^j(T^{0k} + t^{0k}) = J\ :$$

the integration constants of the solution (1) can be interpreted as the mass and the angular momentum not only of the body, but also of the gravitational field.

The fact that the features of the body show up in the far field metric is important because it has phenomenological consequences: we can **measure** the mass and the angular momentum of the body, by looking at the motion of a test body in the far-field region of the central body. In practice:

- We can measure the mass by looking at the orbits of a test body along geodesics; indeed, in the weak-field limit the geodesics equation reduces to Newtonian gravity and then to Kepler's laws; by fitting the orbits of a test body we can measure the mass of the central object. This has been done, for instance, with the observation of the motion of stars around SGR A*, measuring the mass of the supermassive BH at the center of our galaxy to be $M = 4.1 \times 10^6\, M_\odot$.

- We can measure the angular momentum by measuring the precession of gyroscopes orbiting around the central body. Note that in general different effects are present. If the gyroscope moves along a geodesic, its *intrinsic spin vector* S^μ satisfies

$$u^\alpha S^\mu{}_{;\alpha} = 0$$
$$S_\mu u^\mu = 0$$

i.e. it does not have time component in a comoving frame, and it parallely transported. This determines a **precession** of the space components of the spin, $\vec{S} = (S^i)$: by neclecting the effects of the orbital motion

$$\frac{dS^i}{d\tau} = u^0 S^i{}_{,0} = -u^0 \Gamma^i_{0j} S^j = -\frac{1}{2}(h_{0i,j} - h_{0j,i})S^j = \epsilon_{ijk}\omega^j S^k$$

where

$$\omega^k = \frac{1}{2}\epsilon^{kij} h_{0i,j} \, .$$

By replacing the far-field metric we find

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{\rm LT} \equiv \frac{1}{r^3}\left(-\vec{J} + 3\frac{\vec{J}\cdot\vec{x}}{r^2}\vec{x}\right) \, .$$

This is the so-called Lense-Thirring precession. Moreover, there is another - generally larger - precession, the geodetic precession, due to the coupling between the orbital angular momentum and the spin of the gyroscope..

This effect has been observed in the gravitational field of the Earth, in different ways: by a dedicated space experiment, Gravity Probe B, in which a gyroscope was sent on a satellite orbiting around the Earth, and also by studying the relative motion of different satellites. We didn't observe yet the Lense-Thirring precession in compact stars, but we expect to see it soon by analyzing the motion of binary pulsars.

B. Multipole moments

The far-field metric of a stationary, isolated object contains much more information than that. Indeed, the higher-order terms in the multipole expansion depend on a set of constants, the **multipole moments**, which characterize the structure of the body. Multipole expansions are a powerful tool to extract physical content from a (gravitational or electrostatic) potential, or from a spacetime metric, as long as they can be treated, at least approximately, as stationary. Possible references for this part: Poisson, Will *Gravity*; Cardoso, and Gualtieri, review article, arXiv:1607.03133

Multipole expansion were first introduced in Newtonian mechanics to describe the gravitational (or elecrostatic) potential generated by a distribution of masses (of charges) in terms of a set of scalar quantities, the **multipole moments**; then, they have been extended to GR, to describe the spacetime metric of a stationary, isolate object.

Let us start with Newtonian gravity. The Poisson equation is $\nabla^2\Phi = 4\pi G\rho$ and we have already seen that its solution

$$\Phi(t,\vec{x}) = -\int \frac{\rho(t,\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x'$$

can be expressed through a multipolar expansion of $1/|\vec{x}-\vec{x}'|$ around $\vec{x}'=\vec{0}$. If we also expand in spherical harmonics $Y^{lm}(\theta,\phi)$ we can write:

$$\Phi(t,\vec{x}) = -\sum_{lm}\frac{1}{r^{l+1}}\frac{4\pi}{2l+1}I_{lm}(t)Y_{lm}(\theta,\phi)\,.$$

where

$$I_{lm}(t) = \int \rho(t,\vec{x})r^l Y_{lm}(\theta,\phi)d^3x\,.$$

are the *multipole moments* of the body.

This expansion can also be expressed in terms of symmetric-trace-free (STF) tensors:

$$\Phi(t,\vec{x}) = -\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\frac{(2l-1)!!}{l!}I^{<i_1\cdots i_l>}n^{i_1}\cdots n^{i_l}\,,$$

where $(2l-1)!! = (2l-1)(2l-3)(2l-5)\cdots 1$, and

$$n^i = x^i/r = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\,;$$

the brackets $<\cdots>$ denote the *symmetric and trace-free* part of a tensor, and

$$I^{<i_1\cdots i_l>}(t) = \int \rho(t,\vec{x}')(x^{i_1}\cdots x^{i_l} - \text{trace parts})d^3x$$

are the multipole moments.

The components of the STF product of l vectors $\hat{n}$, $n^{<i_1}\cdots n^{i_l>}$, are linear combinations of the spherical harmonics with that value of l. For instance,

$$n^{<i}n^{j>} = n^i n^j - \frac{1}{3}\delta^{ij}$$

is a 3×3 matrix whose components are combinations of sines and cosines of θ, φ; it turns out that there are five independent components, and they are linear combinations of the five spherical harmonics Y_{2m}. Similarly, $n^{<i}n^jn^{k>} = n^in^jn^k$ minus complicate terms with Kronecker deltas, are combinations of the seven Y_{3m}, and so on.

The first terms in the expansion above are the monopole moment, which is the **mass** M, and the quadrupole ($l = 2$) moment, which is the *quadrupole moment*

$$Q^{ij} = I^{<ij>} = \int \rho(t, \vec{x}') \left(x^i x^j - \frac{1}{3}\delta^{ij} r^2 \right) d^3x \,.$$

Choosing the origin of the reference frame in the center of mass, the dipole ($l = 1$) component identically vanishes. Thus, the first terms in the expansion are:

$$\Phi(t, \vec{x}) = -\frac{M}{r} - \frac{3}{2}\frac{1}{r^3} Q^{ij} n^{<i} n^{j>} + \dots$$

In the case of a *stationary, axisymmetric* body with symmetry axis $\hat{k} = (0, 0, 1)$, the only non-vanishing moments are I_{l0} and, defining

$$M_l \equiv \sqrt{\frac{4\pi}{2l+1}} I_{l0} = \int \rho(\vec{x}) r^l P_l(\cos\theta) d^3x \,,$$

the expansion reduces to

$$\Phi(\vec{x}) = -\sum_{l=0}^{\infty} \frac{M_l}{r^{l+1}} P_l(\cos\theta) \,,$$

where, $M_0 = M$, $M_1 = 0$ in the center-of-mass frame, $M_2 = Q$ quadrupole moment. With the further assumption that the body (and then, the gravitational potential) is *reflection-symmetric across the equatorial plane*, i.e., symmetric for $\theta \to \pi - \theta$, then $M_{2l+1} = 0$: the only non-vanishing multipoles are those with even values of l.

In STF notation, axisymmetry with respect to $\hat{k}$ implies that $I^{<i_1 \cdots i_l>} \propto k^{<i_1} \cdots k^{i_l>}$. Using normalization properties of STF tensors, it can be shown that

$$I^{<i_1 \cdots i_l>} = M_l k^{<i_1} \cdots k^{i_l>} \,.$$

For instance, the quadrupole STF tensor is (calling $Q = M_2$):

$$Q^{ij} = I^{<ij>} = M_2(k^i k^j - \delta^{ij}/3) = Q \,\mathrm{diag}(-1/3, -1/3, 2/3) \,.$$

The theory of multipole moments has been extended in GR. Thorne has shown that the far-field metric of a stationary, isolated object can be expressed, in a class of coordinate systems called "asymptotically Cartesian mass-centered" (ACMC) in terms of an expansion in multipole moments:

$$g_{00} = -1 + \frac{2M}{r} + \sum_{l\geq 2} \frac{1}{r^{l+1}} \left(2\frac{(2l-1)!!}{l!} M^{a_1\cdots a_l} n^{<a_1} \cdots n^{a_l>} + \dots \right)$$
$$g_{0j} = -2\sum_{l\geq 1} \frac{1}{r^{l+1}} \left(\frac{2l(2l-1)!!}{(l+1)!} \epsilon^{jka_l} S^{<ka_1\dots a_{l-1}>} n^{<a_1} \cdots n^{a_l>} + \dots \right)$$

The moments $M^{<a_1\cdots a_l>}$ are the **mass multipoles** and reduce in the classical limits to the multipoles of Newtonian gravity. Indeed, in the weak field limit the Newtonian potential is given by the 00 component of the metric:

$$\Phi \simeq \frac{1-g_{00}}{2} = -\frac{M}{r} - \sum_{l\geq 2} \frac{1}{r^{l+1}} \left(\frac{(2l-1)!!}{l!} M^{a_1\cdots a_l} n^{<a_1} \cdots n^{a_l>} + \dots \right) \tag{3}$$
$$= -\frac{M}{r} - \frac{3}{2}\frac{Q^{ij}}{r^3} n^{<i} n^{j>} + \dots \tag{4}$$

But there is a new set of multipoles, the **current multipoles** $S^{<a_1\cdots a_l>}$. For an axisymmetric body, there is one mass moment M_l and one current moment S_l for each l:

$$M^{<i_1\cdots i_l>} = M_l k^{<i_1} \cdots k^{i_l>}$$
$$S^{<i_1\cdots i_l>} = \frac{l+1}{2l} S_l k^{<i_1} \cdots k^{i_l>}\,.$$

Note that the current moments exist also in Newtonian gravity: in the weak field limit

$$M_l = \int \rho r^l P_l(\cos\theta) d^3x$$
$$S_l = \frac{2}{l+1} \int \rho v^\phi r^l P_l'(\cos\theta) \sin^2\theta d^3x\,,$$

but *in Newtonian gravity they do not appear in the gravitational potential.* In particular, $S_1 = J$, it is the angular momentum of the body.

If the body is symmetric with respect to reflection with respect to the equatorial plane, the mass moments have only *even* l, and the current moments have only *odd* l; thus, the multipole moments with lowest values of l (typically, the most important) are the mass $M = M_0$, the angular momentum $J = S_1$, the quadrupole $Q = M_2$.

The multipole moments give a detailed description of the body. Their definition can be extended to non-stationary bodies, and the time derivative of the multipole moments **characterize the gravitational wave emission**. In particular, you all know that for a slowly varying source the leading contribution to the GW emission comes from the time derivatives of the quadrupole moment, through the **quadrupole formula** (in physical units):

$$\bar{h}^{ij}(t,r) = \frac{2G}{rc^4}\frac{d^2}{dt^2} Q^{ij}\left(t - \frac{r}{c}\right)\,.$$

Subleading contributions to the gravitational waveform depend on time derivatives of higher-order mass moments, and of current moments. Note that the quadrupole contribution is the most important because (in GR) only moments with $l \geq 2$ can contribute:

monopole and dipole moments do not radiate; and higher-order multipoles have more time derivatives, leading (since we assume the motion of the source to be $v \ll c$) to contributions suppressed by orders of v/c. As we shall see, there are deviations of GR which admit dipole $l = 1$ radiation; even a tiny correction to the equations of the theory may lead to an observable effect, because the dipole radiation is *enhanced* by a negative power of v/c.

Other remarks:

- Thorne's definition of multipole moments is unique for all ACMC coordinate system; still, it depends on the choice of requiring that the coordinates are ACMC, they are not really defined as gauge-invariant quantities. A gauge-invariant definition of multipole moments of a general stationary, asymptotically flat spacetime exists; it is due to Geroch and Hansen, which uses the techinques - first introduced by Penrose - to treat rigorously the "surfaces at infinity" of the spacetime through conformal rescalings of the metric. In this way it is possible to define rigorously a set of tensor quantities at infinity, the multipole moments $\{M^{<i_1\cdots i_l>}, S^{<i_1\cdots i_l>}\}_{l=0,1,\ldots}$. These quantities **gauge invariant**. Remarkably, it has been later shown that the GH multipoles coincide with the Thorne multipoles, modulo normalization constants!

- A practical way to measure the multipole moments has been found by Ryan in the nineties: the geodesic motion of a test body in a stationary, asmyptotically flat spacetime depends on quantities which can be expressed in terms of the GH moments. Let us consider circular, equatorial geodesics; they are characterized by a constant energy (constant of motion associated with the timelike Killing vector) E; let $\Omega = d\phi/dt$ the angular velocity, and $v = (M\Omega)^{1/3}$ the velocity of the test body. Then, by defining

$$\Delta E = -\Omega \frac{\partial E}{\partial \Omega} \,,$$

  its expansion in v/c can be expressed in terms of the GR moments:

$$\Delta E = \frac{v^3}{3} - \frac{v^4}{2} + \frac{20}{9}\frac{S_1}{M_0^2} v^5 - \left(\frac{27}{8} - \frac{M_2}{M_0^3}\right) v^6 + \ldots$$

  and the various terms contain the entire tower of multipole moments. Then, if - by observations of the motion of smaller bodies around a central, larger body - we can measure with great precision the function $E(v)$ (i.e., $E(\Omega)$), we can **map the spacetime of the compact object**. In practice, it is not clear if this program is feasible: we would need to know $E(v)$ with incredible precision. Still, this is an useful computational tool to find the gauge-invariant multipole moments of a metric given in any coordinate frame.