PHYSICS OF COMPACT OBJECTS IN GENERAL RELATIVITY AND BEYOND LECTURES 4-5

A. The QNMs of a BH

When a BH is excited by a non-radial perturbation, it oscillates in its proper modes, i.e. the modes of free oscillations. They are called **quasi-normal modes** because, in contrast to the normal modes of Newtonian gravity, they are *damped*: the oscillating BH loses energy through GW emission. Therefore, the frequencies of the QNMs are complex:

$$\omega = \omega_R + i\,\omega_I\,;$$

any quantity with a time dependence $\sim e^{-i\omega t}$ with $\omega_I < 0$ describes a damped oscillation

$$e^{-i\omega t} = e^{-i\omega_R t} e^{\omega_I t}\,,$$

where the *oscillation frequency* of the mode is $\nu = \omega_R/(2\pi)$, and the corresponding e-folding time, the *damping time*, is $\tau = -\frac{1}{\omega_I}$.

The QNMs of a Kerr BH are a discrete set of complex frequencies ω_{nlm}, where lm are the harmonic index of the corresponding metric perturbation (see below), and for each value of lm the different QNMs are labelled by the integer number $n = 0, 1, \dots$; the $n = 0$ mode is the *fundamental mode*, while those with $n > 0$ are the *overtones*, typically more difficult to excite, and then generally less relevant.

The perturbation of a stationary BH can be of any kind, even a stone thrown toward the BH, but very few perturbations can excite the QNMs with amplitude large enough to allow the detection of the corresponding GWs. Only two possible sources are believed to be strong enough, and both are associated with the *birth* of the BH: a gravitational collapse associated with a supernova explosion, or the coalescence of two compact objects (other BHs, or NSs).

The SN explosion is one of the target sources for ground-based interferometer, but we must be lucky, or patient, to see it: either a galactic SN explodes while the 2nd generation detectors are taking data (the rate for galactic SNae is one, or very few, per century); or we have to wait for 3rd generation detectors, to see SNae in the surrounding of our galaxy. The compact binary coalescence GW signal is much stronger, and indeed we have already seen it: the stronger BBH coalescence we have seen (which happened to be also the first one), GW150914, allowed to observe, although marginally, with a very low SNR and very large error bands, the fundamental mode ($n = 0$) with $l = m = 2$ of the BH born from the coalescence, with a frequency of about $\nu = 250$ Hz and a damping time of about $4ms$, consistent with the theoretical prediction.

When, especially with next generation detectors, we will be able to measure not only the most excited mode, but also other modes - in general the signal is a linear combination of several modes, but $n = 0$, $l = m = 2$ is by far the most excited - this set of numbers will provide a formidable test of the Kerr nature of the BH spacetime, that is a test of the no-hair theorem, and of GR.

Let us briefly discuss the theoretical modelling of BH QNMs, and the approach to compute them. For who is interested to further details, several reviews exist on the subject, such as: Kokkotas, Schmidt arXiv:gr-qc/9909058; Ferrari, Gualtieri arXiv:0709.0657; Berti, Cardoso, Starinets arXiv:0905.2975.

Let us consider a perturbation of a BH spacetime. For simplicity I will consider the Schwarzschild spacetime,

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \qquad \text{with} \quad f = 1 - \frac{2M}{r} \,,$$

but the same, although more complicate, applies to the Kerr spacetime.

Before discussing the perturbations of Schwarzschild spacetime, it is instructive to consider a much simpler problem, the *dynamics of a scalar field* $\psi(t, r, \theta, \varphi)$ on the Schwarzschild background, assuming that it is a "test" field, with amplitude so small that we can neglect the terms $O(\psi^2)$, and then - since its stress-energy tensor is $O(\psi^2)$, we neglect the effect of the scalar field on the metric. We have then just the Klein-Gordon equation on a fixed, curved background:

$$\Box\psi = \nabla_\mu \nabla^\mu \psi = 0 \,,$$

where ∇_μ are the covariant derivatives. By computing the covariant derivatives, it is easy - I leave it to you as exercise - to show that this equation can be written as

$$-f^{-1}\frac{\partial^2\psi}{\partial t^2} + f\frac{\partial^2\psi}{\partial r^2} + \left(f_{,r} + \frac{2f}{r}\right)\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\left[\frac{\partial^2\psi}{\partial\theta^2} + \cot\theta\frac{\partial\psi}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}\right] = 0 \,.$$

Since the background is spherically symmetric, to simplify this equation it is very convenient to perform a spherical harmonic decomposition of the scalar field, i.e. to expand the scalar field $\psi(t, r, \theta, \varphi)$ in a basis of complex functions of the angular variables, the spherical harmonics $\{Y^{lm}(\theta, \varphi)\}$, which are the eigenfunctions of the operator in square parentheses (which is also the angular part of the Laplacian operator in polar coordinates):

$$\frac{\partial^2 Y^{lm}}{\partial\theta^2} + \cot\theta\frac{\partial Y^{lm}}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 Y^{lm}}{\partial\varphi^2} = -l(l+1)Y^{lm}$$

with $l = 0, 1, \ldots$ and $m = -l, -l+1, \ldots, l$. Due to the orthogonality property

$$\int Y^{lm\,*} Y^{l'm'} \; d\Omega = \delta_{ll'}\delta_{mm'}$$

they form a complete basis: we can expand the scalar field in spherical harmonics:

$$\psi(t, r, \theta, \varphi) = \sum_{lm} R_{lm}(t, r) Y^{lm}(\theta, \varphi) \,.$$

and the coefficients R_{lm} can be obtained from the orthogonality relation:

$$R_{lm} = \int d\Omega \psi(t, r, \theta, \varphi) Y^{lm\,*}(\theta, \varphi) \,.$$

The Klein-Gordon equation then becomes:

$$\sum_{lm}\left[-f^{-1}\frac{\partial^2 R_{lm}}{\partial t^2}+f\frac{\partial^2 R_{lm}}{\partial r^2}+\left(f_{,r}+\frac{2f}{r}\right)\frac{\partial R_{lm}}{\partial r}-\frac{l(l+1)}{r^2}R_{lm}\right]Y^{lm}=0\,.$$

Multiplying this equation by $Y^{l'm'\,*}$, integrating over the solid angle and using the orthogonality condition, we obtain an infinite set of *decoupled* equations for the functions $R_{l'm'}(t,r)$ only:

$$f\frac{\partial^2 R_{l'm'}}{\partial r^2}-f^{-1}\frac{\partial^2 R_{l'm'}}{\partial t^2}+\left(f_{,r}+\frac{2f}{r}\right)\frac{\partial R_{l'm'}}{\partial r}-\frac{l(l+1)}{r^2}R_{l'm'}=0\,.$$

The fact that we get decoupled equations (i.e., the equations for R_{lm} with different l's are not mixed and can be solved separately) is a consequence of the fact that the background is spherically symmetric; in the case of Kerr background, we would get equations that couple perturbations R_{lm} with different l's; this problem can be overcome by using a different basis of angular functions, the spheroidal harmonics, with which the equations with different l's are decoupled.

Now, we redefine the scalar field as

$$R_{lm}(t,r)=\frac{\psi_{lm}(t,r)}{r}$$

and we define the *tortoise coordinate*

$$r_*=r+2M\log\left|\frac{r}{2M}-1\right| \qquad \text{which satisfies} \qquad \frac{dr}{dr_*}=f\,,$$

the KG equation can be written in the form of a one-dimensional wave equation

$$\frac{\partial^2\psi_{lm}(t,r)}{\partial r_*^2}-\frac{\partial^2\psi_{lm}(t,r)}{\partial t^2}-V_l^{\rm scalar}(r)\psi_{lm}(t,r)=0\,,$$

where

$$V_l^{\rm scalar}(r)=\left(1-\frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2}+\frac{2M}{r^3}\right)$$

is the effective potential, which is due to the spacetime curvature (in flat spacetime [$M=0$] $V_l^{\rm scalar}(r)=l(l+1)/r^2$, the usual centrifugal term, a monotonically decreasing function, while for $M\neq 0$ it is a potential barrier as in the figure below). Thus he scalar field propagates in the Schwarzschild background as a wave scattered by the effective potential.

By Fourier transforming

$$\psi_{lm}(t,r)=\int_{-\infty}^{+\infty}d\omega\,\tilde\psi_{lm}(\omega,r)e^{-i\omega t}\,,$$

we get the equation in the frequency domain, an ODE in the radial function $\tilde\psi_{lm}(\omega,r)$:

$$\frac{\partial^2\tilde\psi_{lm}(\omega,r)}{\partial r_*^2}+\left[\omega^2-V_l^{\rm scalar}(r)\right]\tilde\psi_{lm}(\omega,r)=0\,.$$

Some remarks about this wave equation. First of all, the **tortoise coordinate** r_*; it is very useful to describe the physics near the horizon, and has the property

$$\frac{dr_*}{dr} = \frac{1}{f}\,.$$

For a radial lightlike geodesic (θ, ϕ constant)

$$ds^2 = -f dt^2 + f^{-1} dr^2 = -f(dt^2 - dr_*^2) = -f(dt - dr_*)(dt + dr_*) = 0$$

and thus radial massless particles have $t - r_* = const$ (outgoing) or $t + r_* = const$ (ingoing). When $r \gg M$, $r_* \simeq r$, but as $r \to 2M$, $r_* \to -\infty$; thus, $r_* \in [-\infty, \infty]$ describes the region outside the horizon, $r \in [2M, \infty]$. It owes its name to the famous Zeno's tortoise: the coordinate r_* "never" reaches the horizon $r = 2M$, but approaches it logarithmically.

Both when $r_* \to -\infty$, i.e. $r \to 2M$ and when $r_* \simeq r \to \infty$, the effective potential vanishes, and the equation becomes a simple wave equation whose solutions are $e^{i\omega r_*}$ and $e^{-i\omega r_*}$. Therefore, the asymptotic solutions of the wave equations for $r_* \to \pm\infty$ can be written as

$$\tilde{\psi}_{lm}(\omega, r) \simeq A_{in} e^{-i\omega r_*} + A_{out} e^{i\omega r_*}\,.$$

and, making the inverse fourier transform,

$$\psi_{lm}(t, r) \simeq \frac{1}{2\pi} \int d\omega [A_{in} e^{-i\omega(r_* + t)} + A_{out} e^{i\omega(r_* - t)}]\,.$$

Here we have considered $\psi_{lm}(t, r)$ a general function of time, and ω is just the parameter of the Fourier transform, so it is necessarily a real quantity. Let us, instead, look for a particular kind of solution, one that describes a *damped oscillator* $\sim e^{i\omega t}$, where now ω is in general a **complex quantity**, to be determined. This means that we make an *ansats*:

$$\psi_{lm}(t, r) = \tilde{\psi}_{lm}(r) e^{-i\omega t} = \tilde{\psi}_{lm}(r) e^{-i\omega_R t} e^{i\omega_I t} = \tilde{\psi}_{lm}(r) e^{-i\omega_R t} e^{-it/\tau}\,.$$

Then the asymptotic solution, either at $r_* \to \infty$ (infinity) or at $r_* \to -\infty$ (horizon), is:

$$\psi_{lm}(t, r) \simeq A_{in} e^{-i\omega(r_* + t)} + A_{out} e^{i\omega(r_* - t)}\,.$$

The first term describes an *ingoing* wave, moving towards smaller values of r_*, and the second describes an outgoing wave, moving towards larger values of r_*.

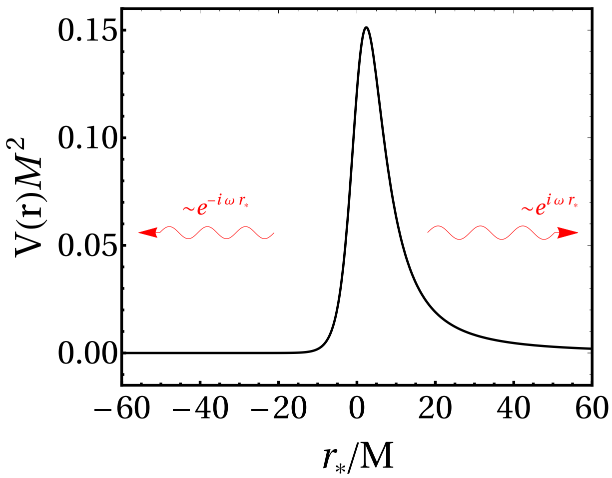
If we are interested in the free oscillations of the black hole, we have to impose that no wave is incoming from infinity, i.e.

$$\psi_{lm}(r) \propto e^{i\omega r_*} \qquad r_* \to \infty\,.$$

As a consequence, $A_{\rm in} = 0$ as $r_* \to \infty$. In addition, since nothing can escape from a black hole horizon, only ingoing waves are allowed when $r_* \to -\infty$, i.e.

$$\psi_{lm}(r) \propto e^{-i\omega r_*} \qquad r_* \to -\infty\,,$$

therefore, $A_{\rm out} = 0$ as $r_* \to -\infty$. We have then a second-order ODE for $\psi_{lm}(r)$ - the wave



equation, a Schroedinger - like equation with an effective potential - and two boundary conditions to be satisfied at $r_* \to \pm\infty$. Similarly to the case of the Schroedinger equation, the boundary conditions can be satisfied only for a **discrete set** of eigenfrequencies ω.

Let us now consider **gravitational perturbations of the metric**

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$$

where $g^{(0)}_{\mu\nu}$ is Schwarzschild's metric, and $h_{\mu\nu}(t, r, \theta, \phi)$ is a small perturbation of the metric. You have already seen, when studying GWs, the perturbations of *flat spacetime*, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$; in this case, near a BH (but the same occurs near a NS), the background is in the strong-field regime, and we can not expand the metric around flat space. We shall assume that the perturbation, and its derivatives, are small, thus neglecting terms quadratic in the perturbation, which we will denote as $O(h^2)$.

Replacing the expansion in Einstein's equations and dropping $O(h^2)$ yields a *linear differential equation* in $h_{\mu\nu}$; on other words, we linearize Einstein's equations.

It is important to note that since $\Gamma^{\mu}_{\alpha\beta} = \Gamma^{(0)\mu}_{\alpha\beta}(g^{(0)}) + \delta\Gamma^{\mu}_{\alpha\beta}(g^{(0)}, h)$, the first term is the Christoffel computed in the background, the second is its correction, linear in h (and we neglect $O(h^2)$ terms). Then,

$$h_{\mu\nu;\alpha} = h_{\mu\nu,\alpha} - \Gamma^{(0)\lambda}_{\mu\alpha} h_{\lambda\nu} - \delta\Gamma^{\lambda}_{\mu\alpha} h_{\lambda\nu} + (\mu \leftrightarrow \nu)$$

but the latter term is $O(h^2)$ and is neglected. Therefore, in the following we shall perform the covariant derivatives on the metric perturbation using the background metric only. For the same reason, we shall raise its indices with the background metric:

$$h^{\mu}{}_{\nu} = g^{(0)\mu\alpha} h_{\alpha\nu} + O(h^2)\,.$$

This means that we can treat $h_{\mu\nu}$ as it is just a field **living on the background spacetime**. And we can consider its field equation, i.e. the linearized Einstein's equation, as an equation on the Schwarzschild spacetime, as we have done for the scalar field.

A simple but a bit tedious computation (the steps of it are done of course in the original papers of Regge, Wheeler and Zerilli, but also in books such as Chandrasekhar's book *The Mathematical Theory of BHs*, and also more recently in other books as e.g. Ferrari et al. *GR and its applications*) shows that

$$2\delta R_{\mu\nu} = h^{\rho}{}_{\nu;\mu\rho} + h^{\rho}{}_{\mu;\nu\rho} - h^{\rho}{}_{\rho;\mu\nu} - g^{0\,\sigma\rho} h_{\mu\nu;\sigma\rho}\,.$$

The linearized Einstein's equations in vacuum are just $\delta R_{\mu\nu} = 0$.

In order to solve this equation, we expand the field $h_{\mu\nu}(t, r, \theta, \varphi)$, defined on Schwarschild's spacetime, in **tensor spherical harmonics**. Being spherically symmetric, Schwarschild's spacetime M_4 is the product of two two-dimensional manifolds,

$$M_4 = M_2 \times S^2\,.$$

The two-sphere S^2 is described by the coordinates $y^a = (\theta, \varphi)$, while M_2 is described by the coordinates $z^A = (t, r)$:

$$x^\mu = (z^A, y^a)\,.$$

The metric on the submanifold S^2 is $\gamma_{ab} = \mathrm{diag}(1, \sin^2\theta)$, and we denote the covariant derivative with respect to this metric with a colon.

Any tensor can be decomposed as a tensor on M_2 times a tensor on S^2. With this decompositon, a vector field is split as $V^\mu = (V^A, V^a)$, and the components of a rank-two symmetric tensor field $X_{\mu\nu}$ are split as

$$X_{\mu\nu} = \begin{pmatrix} X_{AB} & X_{Aa} \\ X_{aA} & X_{ab} \end{pmatrix}\,.$$

Note that with this decomposition, X_{AB} are *scalar* with respect to the submanifold S^2, while X_{aA} are vectors, and X_{ab} are rank-two symmetric tensors with respect to the same manifold.

Now, we have seen that a scalar field on the two-sphere can be decomposed in the complete basis of scalar spherical harmonics $Y^{lm}(\theta, \phi)$. Similarly, it can be shown that any vector field on the two-sphere, V_a, can be decomposed in the complete basis of **vector spherical harmonics**, formed by the *polar vector harmonics*

$$Y^{lm}_a \equiv Y^{lm}_{:a} = \left(Y^{lm}_{,\theta}, Y^{lm}_{,\varphi}\right)$$

and the *axial vector harmonics*

$$S^{lm}_a \equiv -\varepsilon_a{}^b Y^{lm}_{:b} = \left(-\frac{1}{\sin\theta} Y^{lm}_{,\varphi}, \sin\theta Y^{lm}_{,\theta}\right)\ :$$

$$V_a = \sum_{lm} [V^{lm}_{pol} Y^{lm}_a + V^{lm}_{ax} Y^{lm}_a]\,.$$

Similarly, a symmetric tensor X_{ab} can be decomposed in a trace part $\gamma_{ab}Y^{lm}$, and in the complete basis of traceless rank-two tensor harmonics, formed by the *polar rank-two tensor harmonics*

$$Z^{lm}_{ab} \equiv Y^{lm}_{:ab} + \frac{l(l+1)}{2}\gamma_{ab}Y^{lm}$$

and the axial rank-two tensor harmonics

$$S^{lm}_{ab} \equiv \frac{1}{2}(S^{lm}_{a:b} + S^{lm}_{b:a}) \,.$$

These are just 2×2 matrices of combinations of derivatives of the scalar spherical harmonics, combined with sines and cosines of θ. Note that there are two kinds of vector and tensor harmonics, called polar and axial, or even and odd: for a parity transformation $\theta \to \pi-\theta$, $\varphi \to \varphi+2\pi$, a polar (i.e. even) spherical harmonic transforms as $(-1)^l$ (like the scalar harmonics Y^{lm}), while an axial (i.e. odd) spherical harmonic transforms as $(-1)^{l+1}$. As I said, these harmonics form a complete basis, and harmonics of the same rank but with different values of l, m are orthogonal; axial and polar harmonics are orthogonal, too.

So, we expand the metric perturbation in tensor spherical harmonics:

$$\begin{aligned}
h_{AB}(t,r,\theta,\varphi) &= \sum_{lm} \bar{h}_{AB\,lm}(t,r)Y^{lm}(\theta,\varphi)\,, \\
h_{aA}(t,r,\theta,\varphi) &= \sum_{lm}\left[\bar{h}^{\rm pol}_{A\,lm}(t,r)Y^{lm}_a(\theta,\varphi) + \bar{h}^{\rm ax}_{A\,lm}(t,r)S^{lm}_a(\theta,\varphi)\right]\,, \\
h_{ab}(t,r,\theta,\varphi) &= \sum_{lm}\left[r^2\left(K_{lm}(t,r)\gamma_{ab}Y^{lm}(\theta,\varphi) + G_{lm}(t,r)Z^{lm}_{ab}(\theta,\varphi)\right)\right. \\
&\quad \left. +\bar{h}_{lm}(t,r)S^{lm}_{ab}(\theta,\varphi)\right]\,.
\end{aligned}$$

We have the freedom to choose the gauge i.e. to make a coordinate transformation

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x)$$

where we require the expansion $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$ to be preserved, with $g^{(0)}_{\mu\nu}$ Schwarzschild and $h_{\mu\nu}$ small, and thus it has to be $\epsilon^\mu = O(h)$. With such transformation, it can be shown that the metric perturbation changes as:

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + \epsilon_{\mu;\nu} + \epsilon_{\nu;\mu}\,.$$

It is possible to show that choosing carefully the functions $\epsilon^\mu(t,r,\theta,\varphi)$, it is possible to set to zero some of the terms of the expansion above:

$$\bar{h}^{\rm pol}_{0\,lm}(t,r) = \bar{h}^{\rm pol}_{1\,lm}(t,r) = G_{lm}(t,r) = \bar{h}_{lm}(t,r) = 0\,.$$

This is the so-called **Regge-Wheeler gauge**. Then, we call

$$\bar{h}_{AB\,lm}(t,r) = \begin{pmatrix} f(r)H_{0\,lm}(t,r) & H_{1\,lm}(t,r) \\ H_{1\,lm}(t,r) & f(r)^{-1}H_{2\,lm}(t,r) \end{pmatrix}$$

and we choose the ansats of a damped oscillating solution, depending on the complex frequency ω, to be determined. Then, we can write

$$h_{\mu\nu}(t,r,\theta,\varphi) = h^{\rm pol}_{\mu\nu}(t,r,\theta,\varphi) + h^{\rm ax}_{\mu\nu}(t,r,\theta,\varphi)$$

with

$$h^{\rm pol}_{\mu\nu}(t,r,\theta,\varphi) = \begin{pmatrix} fH_{0\,lm}(r) & H_{1\,lm}(r) & 0 & 0 \\ * & f^{-1}H_{2\,lm}(r) & 0 & 0 \\ * & * & r^2K_{lm}(r) & 0 \\ * & * & * & r^2\sin^2\theta K_{lm}(r) \end{pmatrix} Y^{lm}e^{-i\omega t}\,,$$

and

$$h^{\rm ax}_{\mu\nu}(t,r,\theta,\varphi) = \begin{pmatrix} 0 & 0 & -h_{0\,lm}(r)\frac{1}{\sin\theta}Y^{lm}_{,\varphi} & h_{0\,lm}(r)\sin\theta Y^{lm}_{,\theta} \\ * & 0 & -h^{lm}_1(r)\frac{1}{\sin\theta}Y^{lm}_{,\varphi} & h_{1\,lm}(t)\sin\theta Y^{lm}_{,\theta} \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix} e^{-i\omega t}\,.$$

The perturbation then depends, for each l, m, on a set of polar functions $\{H_{0\,lm}(r), H_{1\,lm}(r), H_{2\,lm}(r), K_{lm}(r)\}$, and a set of axial functions $\{h_{0\,lm}(r), h_{1\,lm}(r)\}$. One has to replace this expansion in the field equations for $h_{\mu\nu}$,

$$2\delta R_{\mu\nu} = h^{\rho}{}_{\nu;\mu\rho} + h^{\rho}{}_{\mu;\nu\rho} - h^{\rho}{}_{\rho;\mu\nu} - g^{0\,\sigma\rho}h_{\mu\nu;\sigma\rho}\,,$$

and decompose $\delta R_{\mu\nu}$ in tensor spherical harmonics in the same way. Then, one is left, for each l, m, with a system of ODEs in r for the polar and axial pertubation functions.

The equations for the perturbations with different values of l, m are compeltely **decoupled**, then we can solve separately each harmonic contribution. As in the case of the scalar field, this is a consequence of the spherical symmetry of the background; in the Kerr case, the same result can be obtained using a different set of harmonic functions, the spin-weigthed spheroidal tensor harmonics, but in that case there is no separation in polar and axial harmonics.

By combining and manipulating these equations, one can find that they reduce to two wave equations, one for axial perturbations, one for polar perturbations, and that these wave equations have the same structure as the scalar field equations we have discussed before.

Let us consider axial perturbtions $\{h_{0\,lm}(r), h_{1\,lm}(r)\}$. By defining the **Regge-Wheeler function** Q_{lm},

$$Q_{lm} \equiv f\frac{h_{1\,lm}}{r}\,.$$

one of the equations implies that

$$h_{0\,lm} = \frac{if}{\omega}\left(Q_{lm}r\right)'\,.$$

The other equations, combined, yield the **Regge-Wheeler equation**

$$\frac{d^2 Q_{lm}}{dr_*^2} + \left(\omega^2 - V_l^{\rm axial}\right) Q_{lm} = 0$$

where

$$V_l^{\rm axial}(r) = \left(1 - \frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} - \frac{6M}{r^3}\right] .$$

In the case of the polar perturbations $\{H_{0\,lm}(r), H_{1\,lm}(r), H_{2\,lm}(r), K_{lm}(r)\}$, we can define the **Zerilli function**, Z_{lm}, as a certain combination of K_{lm} and $H_{1\,lm}$ and their first derivatives; then, some of the equations give $H_{0\,lm}$ and $H_{2\,lm}$ as algebraic combinations of them, and the other equations, combined, yield the **Zerilli equation**

$$\frac{d^2 Z_{lm}}{dr_*^2} + \left(\omega^2 - V_l^{\rm polar}\right) Z_{lm} = 0$$

where

$$V_l^{\rm polar}(r) = \left(1 - \frac{2M}{r}\right)\left[\frac{(l-1)(l+2)}{3}\left(\frac{1}{r^2} + \frac{2(l-1)(l+2)(l^2+l+1)}{\left(6M + r(l-1)(l+2)\right)^2}\right) + \frac{2M}{r^3}\right] .$$

These potentials behave like the scalar potential discussed before, and the same results apply. The axial perturbations

$$Q_{lm} e^{-i\omega t}$$

behave, at infinity and at the horizon, as a combination of ingoing and outgoing waves; by imposing the boundary conditions of free oscillations, i.e. purely outgoing wave at infinity and purely ingoing wave at the horizon, the equation has solution for a **discrete set** of (complex) values of ω. In the same way, the polar perturbations

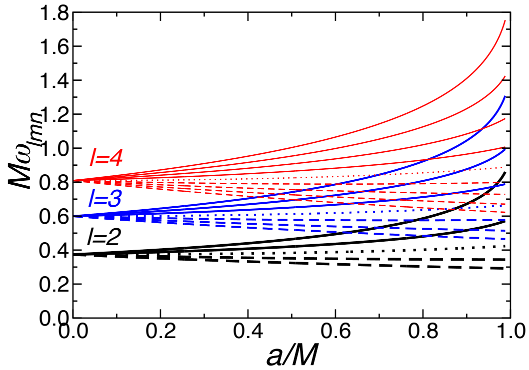
$$Z_{lm} e^{-i\omega t}$$

behave, at infinity and at the horizon, as a combination of ingoing and outgoing waves, and by imposing the same boundary conditions we find a discrete set of comples frequencies.

These frequencies are the **quasi-normal modes of the Schwarzschild BH**. Some remarks on these modes.

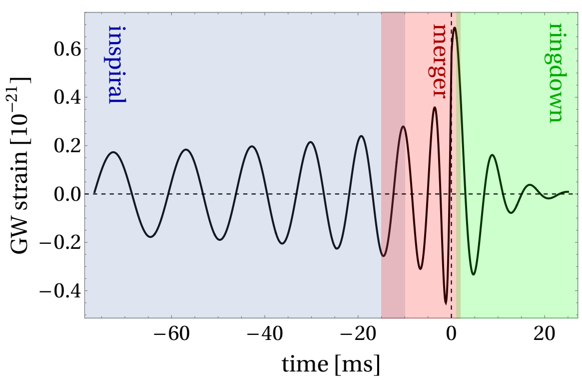
- A remarkable property of the QNMs of Schwarzshild spacetime, is that polar and axial QNMs are identical: they are isospectral. This is due to a deep and highly non-trivial symmetry property of the perturbation equations, which has been studied in detail by Chandrasekhar. For Kerr BHs, instead, the perturbations can not be separated in polar and axial since they have no definite parity.
- Another property, which in this case is a simple consequence of the spherical symmetry of the background, is that the perturbation equations depend on l but not on m, and thus the QNMs do not depend on the harmonic index m. For Kerr BHs, instead, the rotation induces a sort of "Zeeman splitting" of the QNM frequences which depends on m.

- For each value of l, there is a set of modes, denoted by the integer number $n = 0, 1, \dots$. The $n = 0$ mode is the fundamental mode and the easiest to excite in a physical setup.
- Differently from normal modes in Newtonian physics, the QNMs do not form a complete basis of the perturbation; indeed, when a BH is perturbed, after a transient the metric perturbation (and then the GW signal) is described with good approximation by a combination of quasi-normal modes, but at late times it also includes a *power-law tail* $\sim t^{-2l+3}$.
- In geometric units the mode frequency has dimensions of inverse length (i.e. inverse mass), and is proportional to the unique dimensionful scale of the system: the inverse BH mass. So, for instance, for $l = 2$, $n = 0$
$$M\omega \simeq 0.3736 - i0.0890\,,$$
for $l = 2$, $n = 1$ $M\omega \simeq 0.3467 - i0.2739$ and so on. So, the larger the mass of the BH, the smaller the frequency of the mode, and the larger the damping time. In physical units, for the fundamental $l = 2$ mode,
$$\nu \simeq 12\frac{M_\odot}{M}\,\text{kHz} \quad \tau = 5.5\frac{M}{M_\odot} \times 10^{-5}\,\text{s}\,.$$
So, for a $M = 10M_\odot$ BH the frequency of the fundamental QNM is ~ 1 kHz and the damping time is ~ 0.5 ms, while for a supermassive BH like that at the center of our galaxy, with $M \sim 10^6 M_\odot$, $\nu \sim 10^{-2}$ Hz and $\tau \sim 50$s.
- In general, the BH obtained from a coalescence is rapidly spinning; most of them have $a \sim 0.7M$, which what is expected when the spin of the initial BHs are small, and comes from the orbital angular momentum of the binary, due to angular momentum conservation. So, the QNMs of Schwarzschild spacetime are not appropriate, we should consider the modes of Kerr, which are more complicate but possible to compute. The result is a set of functions of a:
$$(M\omega_{nlm})(a)\,.$$
- It turns out that all QNMs of Schwarzschild, and all QNMs of Kerr, are complex number with *negative imaginary part*. This is the prove of the stability of the Schwarzschild and Kerr solution: even a single mode with $\omega_I > 0$ would mean that there is an exponentially growing solution, and a tiny excitation of this solution would necessarily explode, but this is not the case, all the modes are exponentially decreasing.
- Most importantly, the values of the QNMs can be measured from the observed GW emission, and they are an incredibly powerful probe to test the dynamics of GR, and of the BH solution. While the multipole moments only depend on the stationary solution of the equations, the QNMs carry the imprint of the dynamics of the theory, and would then be affected by those modifications of GR which do not affect stationary solutions.



B. Some remarks on BH observation by GW emission

BBH coalescence has been first observed in 2015; today the ground-based interferometers LIGO and Virgo (and very soon Kagra) routinely observe this process.



The gravitational waveform emitted in a BBH coalescences is composed of three stages: the **inspiral**, when the two bodies approach each other as they lose energy through GW emission; the **merger**, when they merge to form a single BH; and finally the **ringdown**, when the distorted BH formed as a result of the merger oscillates in a combination of its QNMs, until it reaches a stationary state.

These stages contain different information, and they are modeled theoretically using different approaches.

- For the inspiral, the most appropriate approach to study the motion of the binary, and for finding the emitted gravitational waveform, is the **post-Newtonian expansion**. For this approach I refer the reader to the review article of Blanchet, arXiv:1310.1528, or to the book of Poisson and Will, *Gravity*. The idea is to expand the motion, and eventually the metric and the gravitational waveform, in powers of (I use physical units with c and G)

$$x \equiv \left(\frac{v}{c}\right)^2 = \frac{1}{c^2}[GM\pi\nu_{GW}]^{2/3}$$

where $M = m_1 + m_2$, v is the velocity of the bodies, with masses m_1 and m_2, and ν_{GW} is the frequency of the emitted GW. The metric is expanded as a perturbation of flat spacetime, but the perturbation is not simply defined as $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$: it is defined as

$$h^{\mu\nu} = \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}\,;$$

in this way it can be shown that Einstein's equation (without approximations) can be cast in the form

$$\Box_F h^{\mu\nu} = \frac{16\pi G}{c^4}|g|T^{\mu\nu} + \text{ terms of higher order in } h^{\mu\nu}\,.$$

These equations are solved iteratively, adding at each step higher orders in $h^{\mu\nu}$ in terms of integrals. In this way, one can write at each order the equations of motion in the form of the Newtonian equations, plus post-Newtonian corrections.

$$\vec{a}_1 = -\frac{Gm_2}{r_{12}^2}\vec{n}_{12} + \frac{1}{c^2}(\dots)$$

and from that one finds the emitted gravitational waveform:

$$h(x) = Ae^{i\phi}$$

where the phase (which is the quantity measured with greater precision, as a function of x, i.e. of the frequency) can be written as a sum over *half-integer numbers* n:

$$\begin{aligned}\phi &= \sum_{n=0}^{\infty}\alpha_n x^{n-5/2} \\ &= \frac{3x^{-5/2}}{128\eta}\left[1 + \frac{20}{9}\left(\frac{743}{336} + \frac{11}{4}\eta\right)x - 16\pi x^{3/2} + \dots\right]\,.\end{aligned}$$

Here $\eta = m_1 m_2/(m_1 + m_2)^2$ is the symmetric mass ratio and α_n are the **PN paramters**. The first term in the square parenthesis is the Newtonian contribution, while the others are the post-Newtonian corrections; the term of $O(x^n)$ is called n-PN correction, and corresponds to $(v/c)^{2n}$ terms; I do not write explicitly the higher-order PN corrections, which are expressions in terms of very large rational numbers. Today we know the motion of the binary fully up to 3.5-PN corrections. Note that this is the waveform in the frequency domain (x is the rescaled frequency at power 2/3), but the inspiral signal is a **chirp**, in which - like in the sound of a singing bird - the frequency increases with time (and the amplitude increases as well). So, in the early inspiral the main contribution to the signal comes from small x's, and only the first terms of the expansion are relevant, while as the inspiral proceed, higher and higher PN terms become relevant.

By comparing the inspiral part of the waveform with the PN expressions, it is possible to extract with great precision the masses of the bodies, and also (with smaller precision) other parameters as their spins, the eccentricity, precession, etc.

- The inspiral contains a wealth of information on the BHs and their motion, but it is not really a strong-field source. Then, the merger contains information complementary to that of the inspiral: it is in the merger that the non-linearity of Einstein's equations becomes important. However, there is no semianalytical approach such as the PN expansions which can model the merger, which is a truly strong-field, non-linear phenomenom. It has to be modelled using **numerical relativity**, i.e. by solving numerically the full Einstein's equations using parallel supercomputing. Technically, this is a very complex problem. Einstein's equations have to be formulated as an evolution problem, by choosing a time coordinate; by foliating the spacetime in family of spacelike surfaces, each corresponding to a different time; by defining consistent "initial data" in one of these surfaces; by formulating Einstein's equations as evolution equations, which, given the metric in one hypersurface, gives the metric in the next one; and finally, numerically integrating these equations. There are several references on the subject, see e.g the book of Alcubierre, *Introduction to* $3+1$ *numerical relativity.*

  This problem is particularly severe for BHs, because it is highly non-trivial how to treat the horizon, how to teach to the computer that a singularity is where the manifold itself is not defined, how to choose the gauge such that spurious effects due to numerical truncation errors do not spoil the validity of the entire simulation. It took decades to solve this problem, but finally in 2006, with the so-called "breakthrough", different groups, using different methods, succeeded at the same time.

  Today NR simulations, although numerically costly, are common, and are performed by different groups. They allow to predict the signal produced by a binary, whatever are their parameters, and are a fundamental tool for the data analysis of the GW signals.

  Between inspiral and merger there is the region of the **late inspiral**, in which the pure PN expansions are not accurate enough; some extensions of the PN expansions have been developed, such as the **phenomenological waveforms** and the **effective-one-body waveforms** which use sophisticated mathematical tools to extend the validity of the PN waveforms to late inspiral. They use some of the results of NR simulations to calibrate some parameters, which allow to describe very accurately the waveform in the late inspiral, and to some extent also in the merger, and being semi-analytical are also a fundamental tool for the data analysis for gravitaional waveforms for BBH coalescences.

- Finally, during the ringdown the final BH oscillates in its QNMs. I have already described how perturbation theory around a curved background is the most appropriate approach to study this stage. In the data analysis, the final part of the waveform is fitted with the main QNMs of the final BHs, finding further information on the BH parameters, and on the dynamics of the underlying theory.