PHYSICS OF COMPACT OBJECTS IN GENERAL RELATIVITY AND BEYOND
LECTURES 6-7

WHAT DOES GR TELLS US ABOUT NSS

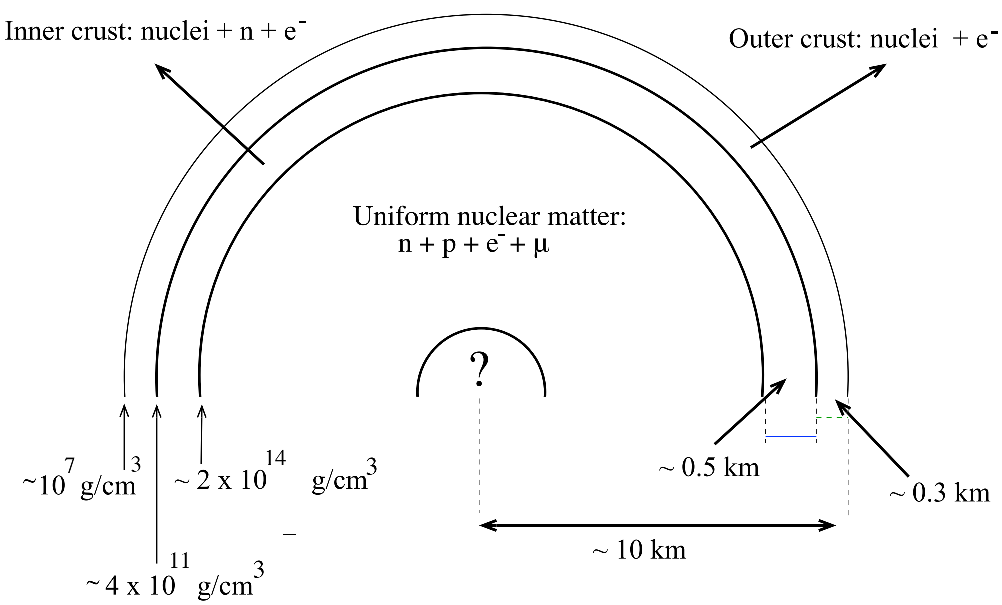
NSs are extremely compact stars formed in the aftermath of supernova explosions. The violent shock produced by the explosion leaves a core with an average density of the order of $\bar{\rho} \sim 10^{14} - 10^{15} g/cm^3$, which is of the order of the nuclear density. So, in these stars neutrons are packed together as in a nucleus, and most protons and electrons have been transformed to neutrons through the inverse β-decay

$$e^- + p \to n + \nu_e \, .$$

NSs have masses between $\sim 1.2 M_\odot$ and $\sim 2 M_\odot$, and radii of the order of $10 - 15$ km.

The structure of a NS

Here is a schematic structure of a NS.



From the exterior to the interior: there is an outer crust, with a lattice of heavy nucleons immersed in an electron gas; then, at the neutron drip density

$$\rho_d = 4 \times 10^{11} \text{ g/cm}^3$$

neutrons start leaking out from the nuclei, and the inner crust starts, where there are two mixed phases: neutron rich matter and a neutron gas. At larger densities, the nuclei merge forming an uniform fluid, and from the nuclear density

$$\rho_0 = 2.6 \times 10^{14} \text{ g/cm}^3$$

the outer core starts. The outer core is formed by a fluid of nuclear matter, consisting in neutrons, protons and electrons. When the density is large enough, the chemical potential of the electrons becomes larger than the rest mass of the muon, $m_\mu = 105\,MeV$, and then it is energetically favoured to form muons. Finally, at densities $\rho \sim 10^{15}$ g/cm^3, the inner core starts, where it is energetically favoured to form other, heavier particles.

There is a quite general consensus on the behaviour of matter in the outer and inner crust, and in the outer part of the core, because at these densities the properties of matter are constrained by experimental data on neutron-rich nuclei. In particular, the **equation of state** (EoS) of matter, i.e. the energy density as a function of the pressure $\epsilon(p)$, is well understood. Conversely, our understanding of the inner core is much more limited: we do not really know the EoS there; nuclear physicists have computed different possible EoSs, which differ in the particle content and in the approach to describe the interactions.

There are two reason for this. The first is that the densities in the inner core can not be reproduced in the laboratory: we do not have experimental constraints. The second is that, since hadronic matter at these densities is described by the non-perturbative regime of quantum chromo-dynamics, our theoretical understanding of its behaviour is also limited. As I said, we do not even know the particle content: there may be hyperons, a Bose condensate of mesons, or even deconfined quark matter. Astrophysical observations and, even more, GW observations, are our best hope to constrain these models, and eventually understand the behaviour of matter in the inner core of NSs.

A. Thermodynamics of perfect fluids in GR

I will briefly recall the basic concepts of thermodynamics in GR, for a **perfect fluid**, i.e. a non-viscous fluid without heat flux, with fixed chemical composition and in thermodynamic equilibrium.

We can define a four-velocity field $u^\mu(x)$, i.e. the four-velocity of an infinitesimal fluid element at the spacetime event (x^0, x^1, x^2, x^3) (which I denote collectively as x). We can define, at each x, a set of quantities describing the thermodynamical state of the fluid element in x:

- the number of baryons per unit of volume, or *baryon number density* n;
- the *energy density* ϵ, which includes the rest-mass energy and the potential energy of the interactions;
- the *pressure* p;
- the *temperature* T;
- the *entropy per baryon* s.

All these quantities are evaluated in the LIF comoving with the fluid element at the event x.

Since the baryon number is always conserved, the baryon number density satisfies the conservation law

$$(nu^{\alpha})_{;\alpha} = 0 \quad \text{(continuity equation)}\,.$$

The first law of thermodynamics can be cast as

$$d\epsilon = \frac{\epsilon + p}{n} dn + nT ds\,.$$

The stress-energy tensor of the fluid can be written as:

$$T^{\mu\nu} = (\epsilon + p)u^{\mu}u^{\nu} + pg^{\mu\nu}\,.$$

The thermodynamical state of the fluid is determined by two of these variables; then, the others can be obtained from the EoS, a relation - determined by the microphysics of the fluid - which gives one of these variables in terms of two others, for instance

$$\epsilon = \epsilon(p, s)\,.$$

However, a NS (unless it is newly born) has a temperature which, although apparently large (at most $T \sim 10^9$ K), is much smaller than the Fermi temperature of the fluid ($T_F \sim 10^{11}$ K); therefore, a NS behaves as a zero-temperature fluid; the thermodynamical state is determined by one variable only, and the EoS is **barotropic**, i.e. it is a function of one variable only:

$$\epsilon = \epsilon(p)\,.$$

B. The TOV equations

Let us consider a static, spherically symmetric star made of a perfect fluid, with barotropic equation of state $\epsilon = \epsilon(p)$. It can be shown that the general form of the metric of a spherically symmetric spacetime can be written as:

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,,$$

where $\nu(r)$ and $\lambda(r)$ are functions to be determined. It is worth noting that there is a theorem, due to **Birkhoff**, stating that *the unique asymptotically flat solution of Einstein's equations in vacuum which is spherically symmetric is the Schwarzschild solution.* This is a very powerful result: the *exterior* of any spherically symmetric body is described by Schwarzshild's metric. The same is not valid for rotating stars, whose exterior is described by a metric which depends on the EoS of the star, and is anyway different from Kerr's metric.

So, for $r > R$ we know that

$$e^{2\nu(r)} = 1 - \frac{2M}{r} \qquad \text{and} \qquad e^{2\lambda(r)} = \frac{1}{1 - \frac{2M}{r}}\,.$$

In order to find the functions $\nu(r)$ and $\lambda(r)$, we solve Einstein's equations, with this metric and with the stress-energy tensor

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + p g^{\mu\nu} \,.$$

We introduce a new function $m(r)$ defined as

$$m(r) \equiv \frac{1}{2} r \left(1 - e^{-2\lambda(r)}\right) \qquad \to \qquad e^{-2\lambda(r)} = 1 - \frac{2m(r)}{r} \,.$$

Thus $m(r) = M$ for $r > R$. Replacing this in Einstein's equations, with the assumptions of staticity and spherical symmetry, we obtain the following set of ODEs in r, known as the **Tolman-Oppenheimer-Volkoff (TOV) equations**:

$$\begin{cases} \dfrac{dm}{dr} = 4\pi r^2\,\epsilon \\[2ex] \dfrac{dp}{dr} = -\dfrac{(\epsilon + p)[m(r) + 4\pi r^3\,p]}{r[r - 2m(r)]} \end{cases} ,$$

with the further equation

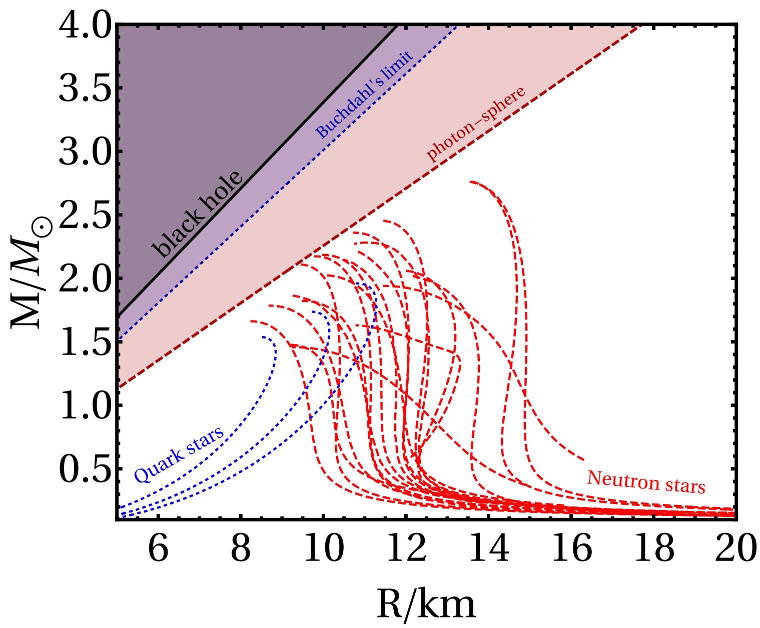
$$\nu_{,r} = \frac{m + 4\pi r^3 p}{r\,(r - 2m)} \,.$$

The above system is closed if we assign a barotropic EoS of the form $p = p(\epsilon)$ (or $\epsilon = \epsilon(p)$)

The first equation tells us that $m(r)$ is the total mass-energy enclosed in a sphere of radius r; the second is the GR generalization of the hydrostatic equation. These two (supplemented with the EoS) can be solved separately for $m(r)$ and $p(r)$, and once they are solved, it is possible to determine $\nu(r)$.

The appropriate boundary conditions of these equations are the matching with Schwarzschild solution at $r = R$, $m(0) = 0$, and $p(0) = p_c$ central pressure. They can easily be solved numerically, finding (given the EoS) a one-parameter family of solutions: one solution for each choice of the central pressure p_c (or, equivalently, for each value of the central energy density $\epsilon_c = \epsilon(p_c)$). Note that it is always $dp/dr < 0$: the pressure decreases moving outwards; by definition, the radius at which the pressure vanishes is the radius of the star; in other words, the radius R is defined by $p(R) = 0$. In this way, by solving the TOV equations it is possible (for each p_c) to find the radius R, and the mass $M = m(R)$. We can then get, for a given EoS, a *mass-radius diagram*: each point corresponds to a different value of p_c.

Each EoS leads to a different mass-radius diagram; thus, by measuring the mass and the radius of a NS, it is possible to discard some EoS, and in general put constraints on them. Actually, sometimes even the measurement of the mass only can give some limited information: few years ago, a NS with $M = 2M_\odot$ was observed; previosuly, all observed NSs were lighter than that; this simple observation was enough to discard some EoSs, since they predicted a maximum possible value of the mass which was smaller than $2M_\odot$! However, to obtain stronger constraints we need also to measure the radius; this can be difficult. Current measurements of R rely on the X-ray emission at the surface of the star



but this depends on very complex and not fully understood physics. In the near future, new large-area X-ray observers (Athena, eXTP) will give us new data to improve our models of the physics at the NS surface, and then to perform a reliable measurement of the NS radius.

To date, the best constraints on the EoS do not come from EM observations, but from GW observations, as I am going to discuss. These measurements also allow to estimate the NS radius and then can be an useful check of the modelling of the X-ray emission from the NS surface.

C. Tidal deformations of compact stars and Love numbers

We shall discuss the tidal deformation of neutron stars, becase, as we will see, they affect the GW signal from BNS coalescences, and carry information on the EoS.

In order to understand how tidal deformations affect a NS, let us consider a static, spherically symmetric star with mass M, placed in a static quadrupolar external field. Let us first consider the problem in Newtonian physics; let Φ_{ext} by the external potential. By expanding the potential around the center of mass of the body,

$$\Phi_{ext} = const + \frac{\partial \Phi_{ext}}{\partial x^i} x^i + \frac{1}{2} E_{ij} x^i x^j + O(r^3)$$

where

$$E_{ij} = \frac{\partial^2 \Phi_{ext}}{\partial x^i \partial x^j}$$

is the **tidal tensor**. That constant term is irrelevant and the term $\Phi_{ext,i} x^i$ gives the force acting on the origin (the center of mass of the body), but since we assume it is a tidal field, it vanishes, i.e.:

$$\Phi_{ext} = \frac{1}{2} E_{ij} x^i x^j + O(r^3)\,.$$

Since the external gravitational potential has no sources at the origin, $\nabla^2 \Phi_{ext} = 0$ and then E_{ij} is traceless, i.e. it is a STF tensor $E_{ij} x^i x^j = E_{ij} x^{<i} x^{j>} = E_{ij} r^2 n^{<i} n^{j>}$. I recall that the STF functions $n^{<i} n^{j>}$ are combinations with the spherical harmonics with $l = 2$; in other words, the leading term in this expansion is a **quadrupolar** term.

The total gravitational potential is

$$\Phi = -\frac{M}{r} - \frac{3}{2}\frac{1}{r^3} Q_{ij} n^{<i} n^{j>} + O\left(\frac{1}{r^4}\right) + \frac{1}{2} E_{ij} r^2 n^{<i} n^{j>} + O(r^3)\,.$$

Note that in this expansion there are terms divergent at infinity: it holds in a *buffer region*, not too close to the body, but not too far away, where there are the sources of the tidal field. Of course, Φ is not *really* divergent at infinity, because beyond the sources of the tidal field it falls off again, although such behaviour is not captured by this expansion.

The body is spherically symmetric *per se*, but it is placed in a quadrupolar field, and thus it acquires a quadrupolar deformation: the quadrupole moment Q_{ij} is *induced* by the external tidal field E_{ij}. By solving the perturbed Newtonian equation, it follows that these tensors are indeed proportional:

$$Q_{ij} = -\lambda E_{ij}$$

where λ is called **tidal deformability** of the body, and depends on its structure (and in particular on its EoS). It has dimensions $[L]^5$ and, since it grows very roughly as R^5, one defines the dimensionless **tidal, quadrupolar Love number** of the star:

$$k_2 = \frac{3}{2}\frac{\lambda}{R^5}\,.$$

Other similar dimensionless Love numbers characterize the deformation properties of the star.

This can be extended to GR. This is discussed extensively in several articles, for instance Hinderer, arXiv:0711.2420, but for more details you should read Binnington and Poisson, arXiv:0906.1366; Damour and Nagar, arXiv:0906.0096.

It can be shown that the 00 component of the metric of a static, spherically symmetric star placed in an external tidal field can be expanded, similarly to the Newtonian case, as

$$g_{00} = -1 + \frac{2M}{r} + \frac{3}{r^3} Q_{ij} n^{<i} n^{j>} + O\left(\frac{1}{r^3}\right) - E_{ij} r^2 n^{<i} n^{j>} + O(r^3)$$

where now the tidal tensor is defined in terms of the Riemann tensor:

$$E_{ij} = u^\mu u^\nu R_{\mu i \nu j}\,.$$

By solving the perturbed Einstein's equations one finds that, as in the Newtonian case,

$$Q_{ij} = -\lambda E_{ij}$$

and it is then possible to compute the tidal deformability λ for a given stellar model, i.e. for a given EoS and for a given mass. Let us see the main steps of this computation.

We consider the metric above as a perturbation of the static spherically symmetric spacetime solution of the TOV equations:

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$$

where $g^{(0)}_{\mu\nu} = \mathrm{diag}(-e^{2\nu(r)}, e^{2\lambda(r)}, r^2, r^2\sin^2\theta)$. We expand the perturbation $h_{\mu\nu}$ in tensor spherical harmonics but, differently from the case of QNMs, here the perturbation is **static**. Moreover, it is purely quadrupolar, because the SFT terms $n^{<i}n^{j>}$ with $n^i = (\sin\theta\sin\varphi, \sin\theta\cos\varphi, \cos\theta)$ are combinations of the shperical harmonics Y_{2m} with $m = -2, \ldots, 2$. Moreover, it is purely polar, and it can be shown that for static perturbations the contribution tr, $H_1(r)$, identically vanishes; therefore (leaving implicit the superscript $2m$ on the perturbation functions)

$$h_{\mu\nu} = \begin{pmatrix} -e^{2\nu(r)}H_0(r) & 0 & 0 & 0 \\ 0 & e^{2\lambda(r)}H_2(r) & 0 & 0 \\ 0 & 0 & r^2K(r) & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta K(r) \end{pmatrix} Y_{2m}(\theta,\varphi)\,.$$

The stress-energy tensor is perturbed as well: $T^{\mu\nu} = T^{(0)\mu\nu} + \delta T^{\mu\nu}$, where $u^\mu = u^{(0)\mu} + \delta u^\mu$, $\epsilon = \epsilon^{(0)} + \delta\epsilon$, $p = p^{(0)} + \delta\epsilon$,

$$\begin{aligned} T_{(0)\mu\nu} &= (\epsilon^{(0)} + p^{(0)})u^{(0)}_\mu u^{(0)}_\nu + p^{(0)}g^{(0)}_{\mu\nu} \\ \delta T_{\mu\nu} &= (\epsilon^{(0)} + p^{(0)})(u^{(0)}_\mu \delta u_\nu + u^{(0)}_\nu \delta u_\mu) + (\delta\epsilon + \delta p)u^{(0)}_\mu u^{(0)}_\nu + p^{(0)}h_{\mu\nu} + \delta p g^{(0)}_{\mu\nu} \end{aligned}$$

and then the background Einstein's equations give the TOV equations, while the perturbed Einstein's equations

$$\delta R_{\mu\nu} - \frac{1}{2}g^{(0)}_{\mu\nu}\delta R - \frac{1}{2}h_{\mu\nu}R^{(0)} = 8\pi\delta T_{\mu\nu}$$

give equations for the perturbations $\{H_0(r), H_2(r), K(r), \delta\epsilon, \delta p\}$. Note that δu^μ has only the time component, being the perturbation static, which is given by the normalization $u^\mu u_\mu = -1$.

The EoS gives $\delta\epsilon$ in terms of δp; by combining the perturbation equations it is possible to simplify δp. Moreover, some of the Einstein's equations give H_2 and K in terms of H_0, so we end up with a single ODE in the quantity $H_0(r)$. By neglecting the superscript (0) on the background quantities,

$$\begin{aligned} &H_0(r)'' + \left\{\frac{2}{r} + e^{2\lambda}\left[\frac{2m}{r^2} + 4\pi r(p-\epsilon)\right]\right\}H_0' \\ &+ \left[-\frac{6e^{2\lambda}}{r^2} + 4\pi e^{2\lambda}\left(5\epsilon + 9p + \frac{\epsilon+p}{dp/d\epsilon}\right) - 4(d\nu/dr)^2\right]H_0 = 0\,. \end{aligned}$$

In the exterior of the star this equation simplifies, and becomes

$$H'' + 2\left(\frac{1}{r} - \lambda'\right)H' - \left(\frac{6e^{2\lambda}}{r^2} + 4(\lambda')^2\right)H = 0$$

where $e^{-2\lambda} = 1 - 2M/r$. The general solution in the exterior can be written in terms of associated Legendre functions:

$$H(r) = c_1 {Q_2}^2\left(\frac{r}{M} - 1\right) + c_2 {P_2}^2\left(\frac{r}{M} - 1\right)$$

where c_1 and c_2 are arbitrary constants. In the limit $r \gg M$, ${Q_2}^2$ goes like $1/r^3$ and ${P_2}^2$ goes like r^2:

$$H = \frac{8}{5}c_1\left(\frac{M}{r}\right)^3 + O\left(\left(\frac{M}{r}\right)^4\right) + 3c_2\left(\frac{r}{M}\right)^2 + O\left(\left(\frac{r}{M}\right)^3\right).$$

If we compare this expression with the metric expansion

$$g_{00} = -1 + \frac{2M}{r} + \frac{3}{r^3}Q_{ij}n^{<i}n^{j>} + O\left(\frac{1}{r^3}\right) - E_{ij}r^2n^{<i}n^{j>} + O(r^3)$$

we see that the term proportional to c_1, falling off asymptotically, corresponds to the quadrupole contribution, while the term proportional to c_2, divergent asymptotically, corresponds to the tidal field contribution. The ratio between the constants c_1 and c_2, then, times some numerical coefficient, gives the ratio between the quadrupole $Q_{ij} = -\lambda E_{ij}$ and the tidal field E_{ij}, i.e. it gives the tidal deformability λ.

In order to determine the constants c_1 and c_2, we have to integrate the equations inside the star, from the center to the surface $r = R$. By imposing regularity at the center one gets $H(r) \simeq c_0 r^2 + O(r^4)\ldots$; the constant c_0 is arbitrary because so is the amplitude if the perturbation: it depends on the amplitude of the external tidal field, and is an overall multiplicative constant which cancels from the ratio c_1/c_2. Then, at $r = R$ we impose continuity and regularity, matching $H(R)$ and $H'(R)$ with the exterior analytical solution. In this way, we find c_1, c_2 and then the tidal deformability λ.

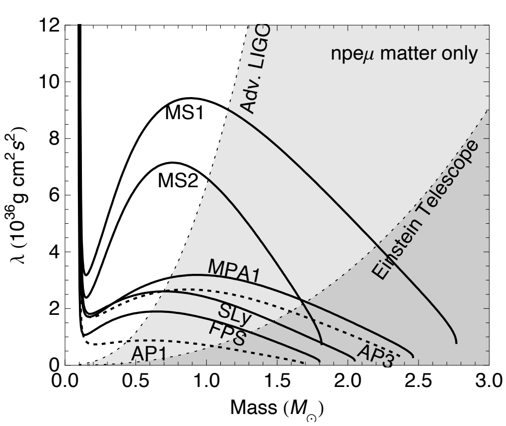


FIG. 1. Credits: arXiv:0911.3535

By drawing the tidal deformability as a function of the mass for different choices of the EoS, one finds that is is very sensitive of the EoS: knowing the mass and λ, it is possible to exclude some EoSs, and more generally to constrain it.

Measure of λ

We have seen that the PN waveform of inspiralling compact binaries has the form, in the frequency domain, $h(x) = Ae^{\phi}$ with $x = (M\pi\nu_{GW})^{2/3}$ and

$$\phi = \phi_{PP} = \frac{3x^{-5/2}}{128\eta}\left[1 + \frac{20}{9}\left(\frac{743}{336} + \frac{11}{4}\eta\right)x - 16\pi x^{3/2} + \dots\right].$$

This is the point-particle contribution, since the BHs of the binary can be treated, in the inspiral, as point particles. When the bodies are NSs, in the early part of the inspiral they can still be treated as point particles, and thus the waveform is exactly the same as for BHs; but in the *late inspiral*, the **finite-size effects** become non-negligible. They appear as a correction to the waveform:

$$\phi = \phi_{PP} + \phi_T \quad \text{where} \quad \phi_T = \frac{3x^{-5/2}}{128\eta}\left(-\frac{39}{2}\tilde{\Lambda}x^5\right)$$

where the tidal deformabilities of the two stars appear in dimensionless form $\Lambda_A = \lambda_A/M^5$ ($M = m_1 + m_2$), $A = 1, 2$, and $\tilde{\Lambda}$ is a combination of the two tidal deformabilities:

$$\tilde{\Lambda} = \frac{16}{13}\frac{(m_1 + 12m_2)m_1^5\Lambda_1 + (m_2 + 12m_2)m_2^4\Lambda_2}{M^5}.$$

Therefore, the (combined) tidal deformability of the two stars can be *directly measured* from the waveform. And indeed, it **has been measured**, or at least some bounds have been set, when the BNS coalescence GW170817 has been measured.

This observation allowed to determine that, assuming that the two stars have the same EoS, a $M = 1.4M_{\odot}$ satisfies

$$\Lambda < 800 \quad \text{at 90\% condfidence level}$$

excluding the largest values of tidal deformability, i.e. excluding the most deformable EoSs.

BEYOND GENERAL RELATIVITY

For this part, I suggest to read the book: Will, *Theory and Experiment in Gravitational Physics*, or one of the various review articles on the subject, such e.g. Berti et al., arXiv:1501.07274.

General relativity has passed all tests in the century after its formulations, so why should we bother at studying possible deviations? There are several answers to this question.

- We know that GR is not renormalizable, and more generally it can not be reconciled with quantum field theory. Note that there are regimes in which both quantum effects and GR effects are relevant: at a scale comparable with the Planck length, 10^{-33} cm, or at the corresponding energies $\sim 10^{19}$ GeV; we do not have a theory describing these regimes. So, GR has to be the low-energy limit of some other theory.
- GR has pathologies, like the prediction of sigularity as the generic outcome of gravitational collapse, or the information loss problem.
- The current paradigm of cosmology predicts the existence od DM and DE, but we do not really understand them; what if the cosmological observations are instead due to modifications of GR? This is a possiblity worth exploring.
- Until 2015, all tests of GR have been tests of the weak-field limit of GR. Now the new regime of strong-field gravity has been opened to observations, and it is natural to ask if GR describes correctly the gravitational interaction also in this scale.
- Even if we think that GR is correct, in order to devise significant tests of GR we have to think to possible modifications.

Two main approaches are possible to model gravity deviations: bottom-up and top-down. Both have plus and minus sides, so both are worth being pursued.

Bottom-up approach

In the bottom-up approach we choose the **phenomenology** to be studied, and the **quantities** most appropriate to describe this phenomenology; devise a **parametrization** of these quantities; typically, each parameter is associated to the violation/modification of some GR property; we compute **observables** in terms of the parameters; we perform observations/experiments, setting **bounds** to the parameters.

Examples are:

- Parametrizations of the metric generated by an isolated body. The first example was the PPN metric (which is a development of early suggestions by Eddington just after the formulation of GR); it has been used to parametrize deviations of the solar system metric, and have been used for solar system tests. More recently, parametrizations of the metric of BHs have been proposed, such as the Johansen-Psaltis metric, the Konoplya-Rezzolla-Zhidenko metric, and others. They can be used, for instance, for tests on the BH shadow, for tests of the motion of stars near BHs, for tests of the EM emission from accretion disks.

- Parametrizations of the motion of compact binaries, such as the PPK expansion, used to study the motion of binary pulsars.
- Parametrizations of the waveform emitted by a BBH coalescence. There are two examples, both have been used to perform tests of GR using the GW150914 signal. In the first, the PN formula

$$\phi = \sum_{n=0}^{\infty} \alpha_n x^{n-5/2}$$

has been generalized, by considering possible deviations of the parameters α_n, fitting these with the signal, and obtaining deviations consistent with zero. The second, more sophisticated, is the PPE expansion, in which the waveform is parametrized as:

$$h(x) = A_{GR}(x)(1 + \alpha x^a) e^{i\phi_{GR} + i\beta x^b} .$$

GR is recoverd for $\alpha = \beta = 0$; the waveform depends on the parameters α, β, a, b. They have been fitted with the signal, finding, again, values consistent with zero.

- Parametrizations of the modifications of QNMs. Several parametrization exists, describing the shifts in frequencies and damping times, as functions of the GR deviation parameters and of the BH spin.
- There are also parametrizations of cosmological deviations, the so-called PPF formalism.

Top-down approach

In the top-down approach, one considers **GR modifications**, possibly inspired by fundamental physics considerations; work out observational **consequences**, which typically depend on parameters describing the amplitude of the modification; and compare with observations, setting **bounds** on the parameters.

Some remarks:

- in most cases we are looking to tiny modifications (parameters small due to existing data)
- often difficult to disentangle a truly from poorly known “standard” physics effects (BHs better than NSs)
- best (when possible) would be to find new effects (the so-called “smoking-guns”).

There are several ways to modify GR. They can be classidied using the Lovelock theorem, stating that in $D = 4$ the only symmetric, rank-two divergenceless tensor diffeomorphisms-invariant is the Einstein tensor. Dropping one or another of these hypotheses one has possible modifications:

- consider higher dimensions;

- allow non-divergenceless tensor, i.e. allow for violations of the weak equivalence principle (but it has been tested with incredible precision, up to $\sim 10^{-15}$);
- drop diffeomorphism invariance, such as in the Lorentz-violating theories, or in massive gravity;
- include extra fiels (scalar fields, vector fields, etc.)

In practice, most of these theories can be reformulated as theories with extra fields, in particular theories with a scalar field in the gravitational sector: the so-called "scalar-tensor theories". Then, in the next lecture I will focus on these theories.