PHYSICS OF COMPACT OBJECTS IN GENERAL RELATIVITY AND BEYOND
LECTURE 3

WHAT DOES GR TELLS US ABOUT BHS

A **black hole** can be defined as **a region of spacetime surrounded by an event horizon**. Its name is due to the fact that no signal from this region can reach an observer outside the horizon.

A series of theorems developed in the seventies by Hawking, Israel, Carter and others prove that:

a stationary, asymptotically flat black hole is described by the Kerr solution

Note that axisymmetry *is not an hypothesis* of these theorems, it is just a consequence, following from the hypotheses of stationarity. If, instead, the spacetime is static, then it is spherically symmetric and is described by the **Schwarzschild solution**. If electic charge is present, the BH is described by a more complicate solution, the Kerr-Newman solution, but this is not relevant: astrophysical BHs are believed to be electrically neutral, because if they acquire a charge, they would lose it in a very short timescale due to charge neutralization by astrophysical plasmas and by quantum effects.

When a BH is **not** stationary, it rapidly settles down, through GW emission, to a stationary configuration in a so-called *dynamical timescale*, i.e. up to a small multiple of the characteristic time

$$t \sim M = \frac{M}{M_\odot} 3 \times 10^{-5}\, s\,.$$

For instance, for the BH formed by the coalescence of GW150914, $M \simeq 63 M_\odot$ and the dynamical timescale is of the order of milliseconds. When it was born it started oscillating, as we will discuss later, and after few tens of milliseconds it became, with very good approximation, a stationary black hole.

Thus, in the Universe we can see either

- a brief, highly-dynamical process involving BHs, such that the coalescence of a BBH or the birth of a BH in a gravitational collapse
- a stationary BH.

I should also mention that these theorems apply to BHs **in vacuum**. Still, as a first approximation we can neglect the effects of neighbouring matter on the spacetime, because this matter - organized, in astrophysical BHs, in *accretion disks* has a density far smaller that that of NSs, and thus it affects very marginally the BH spacetime.More massive matter configuration would not persist for a significant amount of time.

The effect of matter around BHs - the so-called **environmental effects** - has been studied in recent years, finding that they are small but not necessarily negligible; I refer who is interested to a review article on the subject: Barausse, Cardoso, Pani, arXiv:1404.7149.

Stationary BHs

As mentioned above, a stationary BH is described by the Kerr metric, which in Boyer-Lindquist coordinates reads:

$$ds^2 = -dt^2 + \Sigma\left(\frac{dr^2}{\Delta} + d\theta^2\right) + (r^2 + a^2)\sin^2\theta d\varphi^2$$
$$+\frac{2Mr}{\Sigma}(a\sin^2\theta d\varphi - dt)^2\,,$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 + a^2 - 2Mr$. It depends on two parameters, $M > 0$ and $a \in [-M, M]$. A comparison with the far-field limit metric tells us that M is the **mass** of the BH, and $J = Ma$ is its **angular momentum**.

This BH solution, similarly to Schwarzschild's solution, has a curvature singularity (which his $\Sigma = 0$) and a coordinate singularity $\Delta = 0$. Solving $\Delta = 0$ yields two surfaces, $r_\pm = M \pm \sqrt{M^2 - a^2}$. An analysis of the properties of the $r = const$ surfaces shows that

- the surfaces $r = const$ with $r > r_+$ and $r < r_-$ are *timelike hypersurfaces*, which can be crossed in both directions
- the surfaces $r = const$ with $r_- < r < r_+$ are *spacelike hypersurfaces*, which can be crossed in one direction only
- the surfaces $r = r_\pm$ are *null hypersurfaces*, which can be crossed in one direction only.

Thus, an observer or a signal crossing $r = r_+$ can not exit that surface anymore; $r = r_+$ is then the **event horizon** of the BH.

I do not have time in these lectures to discuss topics as the curvature singularities, the removal of the coordinate singularity, or more generally what happens at $r < r_+$: I will only focus on the phenomenologically relevant region, i.e. the observed region, which is $r > r_+$.

Which are the multipole moments of a Kerr BH? First of all, since the Kerr solution is equatorially symmetric,

$$M_{2l+1} = S_{2l} = 0\,.$$

Then, we have seen the first moments: $M_0 = M$, $S_1 = J = Ma$. By computing the multipole moment of the Kerr metric, it is simple to show that:

$$M_{2l} = (-1)^l Ma^{2l} \qquad\qquad S_{2l+1} = (-1)^l Ma^{2l+1}\,.$$

So, $Q = M_2 = -Ma^2$, the octupole is $S_3 = -Ma^3$, and so on. This simple property is related with what I am going to speak next.

A. No-hair theorem

We have seen that in GR a stationary, asymptotically flat black hole is described by the Kerr solution. This means that the structure of asymptotically flat, stationary BHs is completely determined by two parameters: the mass M and the angular momentum J, which are so-called *global charges*, since they refer to the entire spacetime (we have seen that they describe the mass-energy and the angular momentum of the entire spacetime, including the contribution associated with the gravitational field itself). For a static BH, there is only one global charge, the mass M, because it is described by the Schwarzschild solution.

The original theorems of Hawking, Israel, Carter etc. have the hypothesis of vacuum spacetime, $T_{\mu\nu} = 0$, but later they have been extended. Indeed, one may ask if it is possible to construct a stationary BH solution with matter, different from the Kerr solution. To understand this, let us try to "anchor" to the Kerr background a test field with spin s, which can be a scalar field ($s = 0$), a vector field ($s = 1$) or a gravitational perturbation ($s = 2$); this can also be extended, with minor modifications, to fermion fields $s = 1/2$. For simplicity, let's consider a Schwarzschild background; the discussion is analogue for Kerr.

If we expand the field in spherical harmonics Y^{lm} and consider the linearized field equation of the field around the Schwarzschild background, we find that the component $\Psi^s_{lm}(r)$ of the field satisfies an equation which reads:

$$\left(\left(1 - \frac{2M}{r}\right)\Psi^{s\,\prime}_{lm}\right)' - \left(\frac{l(l+1)}{r^2} + \frac{2M(1-s^2)}{r^3}\right)\Psi^s_{lm} = 0$$

where the prime denotes derivative with respect to r. If we multiply this equation by the complex conjugate $(\Psi^s_{lm})^*$ and integrate from the horizon to infinity we get, by integrating by parts (assuming the field to fall off at infinity)

$$-\int_{2M}^{+\infty} dr \left[\left(1 - \frac{2M}{r}\right)|\Psi^{s\,\prime}_{lm}|^2 + \frac{l(l+1)}{r^3}\left(r + \frac{2M(1-s^2)}{l(l+1)}\right)|\Psi^s_{lm}|^2\right] = 0\,.$$

Elementary algebraic manipulation shows that the integrand is definite positive, leading to a violation of the equation, with few exceptions: $\Phi = const,\ s = 1,\ l = 0$, corresponding to the BH solution with electric charge (and thus a monopolar electric field), as predicted by the BH theorems; and $s = 2,\ l = 0, 1$, which is a metric perturbation which can be reabsorbed by redefining M and J: it corresponds to a change in the mass and the angular momentum of the BH. So, it is impossible (besides a monopolar electric field) to anchor matter fields to a stationary BH; if we add them, they will be eventually "swallowed" by the BH. One can not have, then, a BH with other charges (besides the electric charge): one can not have scalar charges, baryon charges, lepton charges.

All this, the lack of other charges besides M and J, but also the fact that all the infinite multipole moments are functions of M and J, as we have seen, has been summarized by the famous sentence of Wheeler (also attributed to Bekenstein):

a BH has no hair.

For this reason, these results are generally referred to as **no-hair theorem**. This is a remarkable property of GR, and is natural to try to test it. A violation of the no-hair theorem would be extremely important: it could be the **"smoking gun" of new physics**, like modifications of GR, or a violation of the BH hypothesis, i.e. that the objects we think are BHs are actually ECO BH mimickers.

Still, there are other consideration we have to do about the no-hair theorem.

- The first is that there is some (limited) room to **circumvent the theorem** with some particular kind of matter fields. One is vector fields different from the electromagnetic field, such as Yang-Mills fields, or some candidate dark matter fields; another is a complex scalar field which oscillates, so it is not stationary, but such that in its stress-energy tensor the time dependence $\sim e^{i\omega t}$ cancels, and thus the spacetime is stationary; then, under some conditions on the frequency ω, the time dependence of the scalar field allows to violate the no-hair theorem (see the review article: Herdeiro and Radu, arXiv:1603.02687). However, while we have a precise understanding of the formation mechanism of BHs, there is presently no indication of a realistic collapse scenario leading to a significant amount of these hair.

- The second remark is that, of course, there is the hypothesis of **stationarity**. BHs are with very good approximation stationary for mostly their entire life, but we must be aware that the no-hair theorem do not apply to the dynamical processes involving BHs. For instance, in the last stages before the coalescence of a BBH, the two BHs are deformed and for sure they are very different from the Kerr solution. Still, in several cases it can be shown that if matter can not be "anchored" to a stationary BH, its effects on dynamical processes involving BHs is also negligible. We will come back to this when discussing GR deviations.

A natural test of the no-hair theorem is that of measuring the multipole moments, looking at the motion of bodies in the BH spacetime, checking for the simple Kerr relation showed above. Another natural test is that of checking another very characteristic feature of Kerr BHs: the frequencies and damping times of the proper oscillation modes of the BH, the so-called **quasi-normal modes** (QNM) which we shall discuss in the next lecture.