# MOND–like Fractional Laplacian Theory

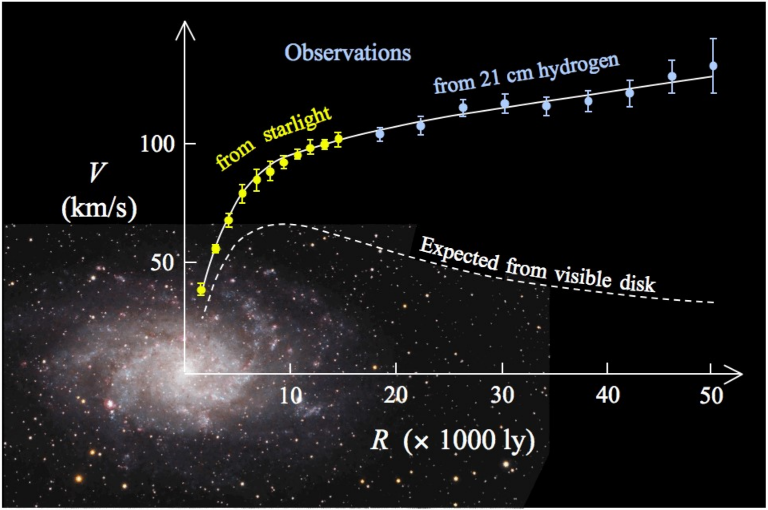
Andrea Giusti

Bishop's University

106th National Congress of the Italian Physical Society
14–18 September 2020

**A. Giusti**, Phys. Rev. D **101**, 124029 (2020) [arXiv:2002.07133 [gr-qc]]
**A. Giusti**, R. Garrappa, G. Vachon, arXiv:2009.04335 [gr-qc]

# Rotation curves of disk galaxies




Source: Wikipedia, based on E. Corbelli and P. Salucci, Mon. Not. Roy. Astron. Soc. **311**, 441-447 (2000).

Comparison of the observed and expected rotation curves of the typical spiral galaxy M33

# Two options

**Dark Matter**

$$\boxed{m(r) \sim r \quad \text{at large}\, r}$$

- No interaction with EM radiation;
- $p_{\rm DM} \simeq 0$.
- Primordial BHs, BSM particles, ...

**Modified Gravity**

$$\boxed{\boldsymbol{F} = m\, \mu(|\boldsymbol{a}|/a_0)\, \boldsymbol{a}}$$

- Modified Newtonian dynamics (MOND);
- Effective theory.
- $f(R)$, TeVeS, BEC of gravitons, ...

**No evidence** for Physics BSM from LHC

$$\Downarrow$$

Let's focus on Modified Gravity
at large $r$ form the Galactic center

# Modified Newtonian dynamics (MOND)

Critical acceleration scale $a_0$

$$a(r) \sim \frac{G_\mathrm{N}\, M}{r^2} \quad \text{for} \quad a \gg a_0\,, \qquad \frac{a^2(r)}{a_0} \sim \frac{G_\mathrm{N}\, M}{r^2} \quad \text{for} \quad a \ll a_0\,,$$

with $M \sim$ core mass (asymptotically far from the Galactic center).

This implies

$$v^2(r) \sim \frac{G_\mathrm{N}\, M}{r} \qquad \text{for} \quad a \gg a_0\,,$$

$$\boxed{v^4(r) \sim G_\mathrm{N}\, M\, a_0 \qquad \text{for} \quad a \ll a_0\,.} \quad \textbf{(Tully–Fisher)}$$

**MOND $\rightsquigarrow$ Bekenstein's modified Poisson equation**

$$\mathrm{div}\left[\mu\left(\frac{|\boldsymbol{\nabla}\Phi|}{a_0}\right)\, \boldsymbol{\nabla}\Phi\right] = 4\,\pi\, G_\mathrm{N}\, \rho(\boldsymbol{x})\,, \qquad \mu(x) \sim \begin{cases} 1\,, & \text{for } x \gg 1\,, \\ x\,, & \text{for } x \ll 1\,. \end{cases}$$

with $\mu(x)$ an "interpolating function".

# Fractional Newtonian Gravity

Fourier transform: $\widehat{f}(\boldsymbol{k}) \equiv \mathcal{F}\left[f(\boldsymbol{x})\,;\,\boldsymbol{k}\right] := \int_{\mathbb{R}^3} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\, f(\boldsymbol{x})\,\mathrm{d}^3x$

The Fourier transform of the Laplacian

$$\mathcal{F}\left[(-\triangle)f(\boldsymbol{x})\,;\,\boldsymbol{k}\right] = |\boldsymbol{k}|^2\,\widehat{f}(\boldsymbol{k})\,,$$

meaning that $(-\triangle)$ is a positive-definite operator.

**Fractional generalization**: $(-\triangle)^s$ such that

$$\boxed{\mathcal{F}\left[(-\triangle)^s f(\boldsymbol{x})\,;\,\boldsymbol{k}\right] = |\boldsymbol{k}|^{2s}\,\widehat{f}(\boldsymbol{k})}$$

for $s > 0$.

**Fractional Poisson equation for gravity**, with $1 \leq s \lesssim 3/2$

$$\boxed{(-\triangle)^s\Phi(\boldsymbol{x}) = -4\,\pi\,G_{\mathrm{N}}\,\ell^{2-2s}\,\rho(\boldsymbol{x})} \quad [\ell] = \text{length}$$

# Kuzmin disk

Mass density

$$\rho(R,z) = \frac{R_0\,M}{2\pi\,(R^2+R_0^2)^{3/2}}\,\delta(z)\,,$$

in cylindrical coordinates $(R,\phi,z)$, $[R_0] = \text{length}$.

Fourier domain

$$\widehat{\rho}(\boldsymbol{k}) = M\,e^{-\kappa R_0}\,, \qquad \kappa := \sqrt{k_x^2+k_y^2}\,.$$

Field equation in momentum space:

$$\begin{aligned}\widehat{\Phi}(\boldsymbol{k}) \equiv \widehat{\Phi}(\kappa,k_z) &= -4\,\pi\,G_{\rm N}\,\ell^{2-2s}\,\frac{\widehat{\rho}(\boldsymbol{k})}{|\boldsymbol{k}|^{2s}} \\ &= -4\,\pi\,G_{\rm N}\,M\,\ell^{2-2s}\,\frac{e^{-\kappa R_0}}{(\kappa^2+k_z^2)^s}\end{aligned}$$

$z = 0$ (working on the Galactic plane)

$$\Phi_s(R,0) = \begin{cases} -\dfrac{G_{\rm N}\, M\, \ell^{2-2s}}{\sqrt{\pi}\, R_0^{3-2s}}\, \dfrac{\Gamma(s-1/2)\,\Gamma(3-2s)}{\Gamma(s)} \\ \qquad \times\ {}_2F_1\left(\dfrac{3}{2}-s,\, 2-s,\, 1\,;\, -\dfrac{R^2}{R_0^2}\right)\,, \quad \text{for } 1 \le s < 3/2\,, \\ \\ \dfrac{2}{\pi}\dfrac{G_{\rm N}\, M}{\ell}\, \log\left[1+\sqrt{1+\left(\dfrac{R}{R_0}\right)^2}\,\right]\,, \ \text{for } s = 3/2\,. \end{cases}$$

$z \neq 0$

$$\Phi_s(R,z) = -\frac{2^{\frac{3}{2}-s}\, G_{\rm N}\, M\, \ell^{2-2s} |z|^{s-\frac{1}{2}}}{\sqrt{\pi}\,\Gamma(s)} \times \int_0^\infty {\rm d}\kappa\, \kappa^{\frac{3}{2}-s}\, e^{-\kappa R_0}\, J_0(\kappa R)\, K_{s-\frac{1}{2}}(\kappa|z|)$$

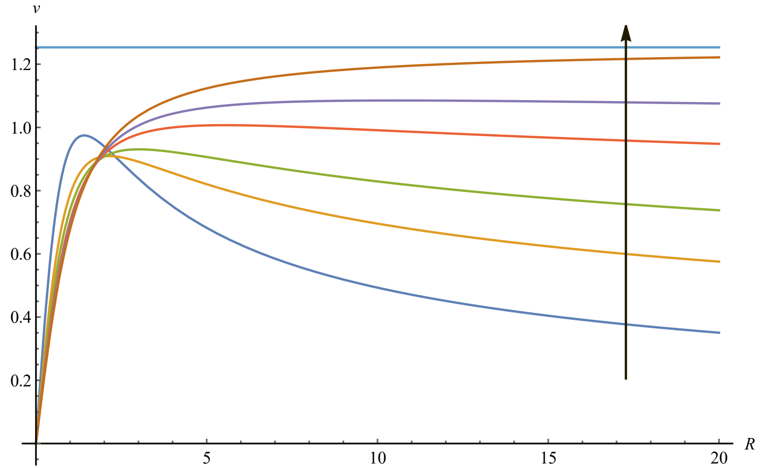
# MOND–like behavior

Orbital speeds of visible stars

$$v_s^2(R) = R\,|\boldsymbol{\nabla}\Phi_s(R,0)| = R\,\frac{\partial\Phi_s(R,0)}{\partial R}$$

$$v_s^2(R) = \begin{cases} \left(\dfrac{\ell}{R_0}\right)^{2-2s} \dfrac{G_\mathrm{N} M R^2}{\sqrt{\pi}\,R_0^3}\,\dfrac{(2-s)\Gamma(4-2s)\Gamma(s-1/2)}{\Gamma(s)} \\ \qquad \times\ {}_2F_1\left(\dfrac{5}{2}-s,\, 3-s,\, 2\,;\, -\dfrac{R^2}{R_0^2}\right)\,, \quad \text{for } 1 \leq s < 3/2\,, \\ \\ \dfrac{2\,G_\mathrm{N}\,M}{\pi\,\ell}\left(1 - \dfrac{R_0}{\sqrt{R^2+R_0^2}}\right)\,, \ \text{for } s = 3/2\,. \end{cases}$$

- $s \to 1$: Newtonian gravity;
- $s \to (3/2)^-$: Tully–Fisher relation for $\ell = \frac{2}{\pi}\sqrt{\frac{G_\mathrm{N}\,M}{a_0}}$ .




Galaxy rotation curves

$\rightsquigarrow$ Scale-dependent (variable-order) fractional theory, $s \to s(|\boldsymbol{x}|/\ell)$.

# Discussion and Conclusions

- We can think of DM phenomenology as the result of an emergent fractional theory of gravity;
- Fractional theory $\leftrightsquigarrow$ non-local field theories;
- Typical scale for non-local effects $\ell = \frac{2}{\pi}\sqrt{\frac{G_{\mathrm{N}}\, M}{a_0}}$;
- Scale-dependence $\leftrightsquigarrow$ variable-order FPDE $\rightsquigarrow$ $s \to s(|\boldsymbol{x}|/\ell)$.

*Thank You!*