Can $\overline{\text{MS}}$ PDF be negative?

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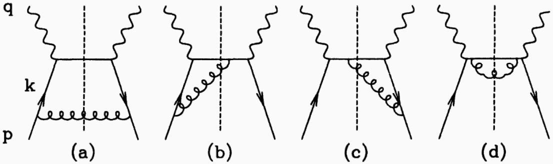
As well known at NLO divergences start to appear, both in **virtual** and real contributions (**soft** and **collinear**).



Collinear divergences

Collinear divergences have *a different origin* that can be related to the asymptotic freedom of strong interactions:

The collinear limit corresponds to a long range (soft) part of the strong interaction, which is not calculable in perturbation theory.

So they must be treated in a different way, defining a suitable **factorization scheme**, i.e. we can hide collinear divergences in a non-perturbative object, the PDF.

$$F_2(x,Q^2) = x\sum_{q,\bar{q}} q_0 \otimes \hat{F}_{2,0}(x,Q^2) = x\sum_{q,\bar{q}} \int_x^1 \frac{\mathrm{d}\xi}{\xi} q_0(\xi)\hat{F}_{2,0}\left(\frac{x}{\xi},Q^2\right)$$
$$= x\sum_{q,\bar{q}} q \otimes \hat{F}_2(x,Q^2)$$

Example: DIS scheme for DIS @ NLO

Effects of the factorization scheme:

1. the effective *subtraction* of the collinear divergence in the partonic cross section
2. the definition of a ”renormalized” PDF
3. the appearance of a new unphysical energy scale: μ_F, the *factorization scale* (which is on the same ground of μ_R).

There is an **arbitrariness** on defining the finite part of the subtraction (as for renormalization).

Counterterm

The formula for a cross section in a generic factorization scheme can be written as an additional counterterm contribution $\mathrm{d}\sigma_a^C$, in this case coming from the PDF redefinition:

$$\sigma_a^{NLO}(p;\mu_F^2) = \int_{m+1} \mathrm{d}\sigma_a^R(p) + \int_m \mathrm{d}\sigma_a^V(p) + \int_m \mathrm{d}\sigma_a^C(p;\mu_F^2)$$

$$\mathrm{d}\sigma_a^C(p;\mu_F^2) = -\frac{\alpha_S}{2\pi}\frac{1}{\Gamma(1-\epsilon)}\sum_b \int_0^1 \mathrm{d}z \left[-\frac{1}{\epsilon}\left(\frac{4\pi\mu^2}{\mu_F^2}\right)^{\epsilon} P^{ab}(z) + K^{ab}(z)\right] \mathrm{d}\sigma_b^B(zp)$$

in dimensional regularization[1].

[1] The counterterm is such that $K^{ab} = 0$ for $\overline{\mathrm{MS}}$ factorization scheme

At LO PDFs are positive by construction, but this is **not true at NLO**, since we **redefined the PDF** with the *subtraction from the factorization scheme*.

Nevertheless it is easy to find a special class of **intrinsically positive** factorization schemes: the PDFs directly defined on **physical observables**, subtracting *all of the finite* contribution.

Like in the well-known DIS scheme we can choose two physical processes for defining the quark and gluon PDF:

quark we can choose a hypothetical $\bar{q}p \to \gamma^* + X$ (pure antiquark beam), or DIS;

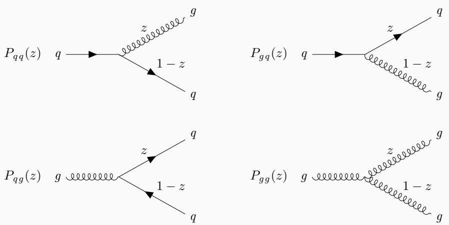
gluon either a hypothetical $gp \to H + X$ (pure gluon beam), or a photon-gluon fusion;

We can choose one of these schemes and call it PHYS, for example:

$$\frac{1}{x}\bar{\sigma}(x, Q^2) = f^{\text{PHYS}}(x, Q^2)$$

$$\bar{\sigma}(x, Q^2) = \begin{pmatrix} \sigma(x, Q^2)[\bar{q}p \to \gamma^* + X] \\ \sigma(x, Q^2)[gp \to H + X] \end{pmatrix}$$

This scheme is positive by construction, since the PDF is proportional to a physical cross-section.



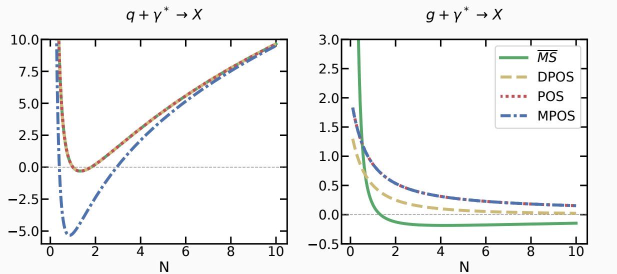
In $\overline{\text{MS}}$ the subtracted cross section can be negative: negative finite parts are factored away from the regularized cross sections, into the PDFs, that can become negative as well.

On the other hand, the residue of the collinear pole is universal—it is given by process-independent splitting functions P_{ab}.

For this reason we can go on with the analysis of some particular process to catch the universal behavior involved in the factorization scheme.

The problem

The original d-dimensional expression is positive by construction, since it is a **cross-section**, but the $\overline{\text{MS}}$ prescription leads to an over-subtraction in the gluon channel.



Gluon channel

We can make it positive again analyzing the source of over-subtraction.

$$C_g^{(1)\,\overline{\text{MS}}}(x) = \lim_{\epsilon\to 0^-}\left[C_g^{(1)}(x,Q^2,\epsilon) - \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}\left(-\frac{1}{\epsilon}+\gamma_E\right)P_{qg}(x)\right]$$

$$C_g^{(1)\,\text{DPOS}}(x) = \lim_{\epsilon\to 0^-}\left[C_g^{(1)}(x,Q^2,\epsilon) - \frac{1}{1-\epsilon}\left(\frac{\mu_D^2}{\pi\mu^2}\right)^{-\epsilon}\left(-\frac{1}{\epsilon}+\gamma_E\right)P_{qg}(x)\right]$$

$$\mu_D^2 \equiv |k_T^{\text{max, DIS}}|^2 = \frac{s}{4} = \frac{Q^2(1-x)}{4x},$$

The changes applied have an intrinsic physical meaning:

- the DPOS scheme is subtracting at correct energy scale (maximal transverse momentum, not virtuality)
- d-dimensional gluon polarizations' average is taken into account

From the definitions of $\overline{\text{MS}}$ and DPOS it is possible to define a *change of scheme* matrix K:

$$K_{qg}^{\text{DPOS}}(x) = C_g^{(1)\,\overline{\text{MS}}}(x) - C_g^{(1)\,\text{DPOS}}(x) = P_{qg}(x)\left[\log\left(\frac{1-x}{x}\right) - 1\right]$$

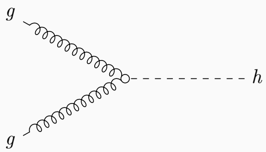
Quark channel

Since in this case it would be an *under-subtraction* for the sake of positivity it is fine to choose $K_{qq}^{\text{DPOS}}(x) = 0$.

Still to do:

- there are still two elements of matrix missing $K_{qg}^{\text{DPOS}}(x)$ and $K_{gg}^{\text{DPOS}}(x)$, so we need a process with an incoming gluon at LO.
- the factorization scheme is linked to collinear structure, that is universal *w.r.t. the hard process*, but is affected by the **incoming kinematics**.

A suitable process for both is $gg \to h + X$ in HEFT:



The only difference is just in the incoming kinematics[2]. This yields that the only new thing is just the **different energy available for transverse momentum**:

$$\mu_h^2 = |k_T^{\text{max, had}}|^2 = \frac{(s - Q^2)^2}{4s} = \frac{Q^2(1-x)^2}{4x}$$

where $x = \frac{M_H^2}{s}$ and $Q^2 = M_H^2$.

The **off-diagonal** matrix element it is similar to the DIS, but with a further contribution coming from the second incoming particle

$$K_{gq}^{\text{POS}}(x) = P_{gq}(x) \left[\ln\left(\frac{(1-x)^2}{x} \right) - 1 \right]$$

and $K_{gg}^{\text{POS}}(x) = 0$

POS for both

We changed to POS applying the same energy correction μ_h^2 in all the channels (instead of μ_D^2).

Since the POS scheme results to make the DIS cross-sections even more positive (non minimal subtraction) it is a good candidate for a positivity scheme.

[2] Process specific ingredients are not involved in PDFs factorization.

Is $\overline{\text{MS}}$ negative? (non-singlet)

First of all we can consider the case of a **non-singlet** PDF: $q^{\text{NS}}(x, Q^2) = q_i(x, Q^2) - q_j(x, Q^2)$ [3].

Master formula for this part of the argument is:

$$\left[q^{\text{NS}}\right]^{\text{DIS}}(x, Q^2) = \left[1 + \frac{\alpha_s}{2\pi}\Delta_q^{(1)\,\overline{\text{MS}}} + \frac{\alpha_s}{2\pi}\bar{C}_q^{(1)}(x)^{\overline{\text{MS}}} \otimes\right]\left[q^{\text{NS}}\right]^{\overline{\text{MS}}}(Q^2)$$

Positivity with a single PDF $\iff C_q(x)^{\overline{\text{MS}}} > 0$

Necessity

Since DIS PDFs are known to be positive, being physical, but assuming a generic positive $\overline{\text{MS}}$ PDF without any other restriction yields the coefficient function to be positive[4].

Sufficiency

Applying a **perturbative inversion** to the master formula, since physical (here DIS) PDF is positive and perturbatively the coefficient cannot become negative[5].

[3] This is *academic* because we are going to **assume it positive in a physical scheme**, but being a *difference* it may also not exist such a positive non-singlet.

[6] Like in a variational argument one assumes the coefficient of the variation to be zero, since the variation can be whatever.

[5] There is a single *caveat*: in the region $x \to 1$ it's not perturbative, so in this case the inverted formula to LL$(1 - x)$.

Instead singlet and gluon are coupled

$$\frac{1}{x}\sigma(x,Q^2) = \hat{\Sigma}_0 \otimes \left[1 + \frac{\alpha_s}{2\pi} C^{(1)} \otimes\right] f(Q^2)\,.$$

where:

$$\sigma(x,Q^2) = \begin{pmatrix} \sigma^q(x,Q^2) \\ \sigma^g(x,Q^2) \end{pmatrix} \qquad \hat{\Sigma}_0(x,Q^2) = \begin{pmatrix} \hat{\sigma}_0^q q(x,Q^2) & 0 \\ 0 & \sigma_0^g g(x,Q^2) \end{pmatrix} \qquad f(x,Q^2) = \begin{pmatrix} q(x,Q^2) \\ g(x,Q^2) \end{pmatrix}$$

so σ represents a couple of processes, and the coefficient function is a two-by-two matrix.

The only new ingredients are **off-diagonal terms**, but POS has been defined taking explicitly care of them. So the argument it's pretty the same, but to apply it one should consider a pair of processes.

To a positive $\overline{\text{MS}}$

With POS > 0 he prove of $\overline{\text{MS}} > 0$ is straightforward, since the explicit transformation between the two schemes is known:

$$\left[\mathbb{I} + \frac{\alpha_s}{2\pi} C^{(1)\,\overline{\text{MS}}}\right] = \left[\mathbb{I} + \frac{\alpha_s}{2\pi} C^{(1)\,\text{POS}}\right] \left[\mathbb{I} + \otimes \frac{\alpha_s}{2\pi} K^{\text{POS}}\right]$$

and so:

$$f^{\text{POS}}(Q^2) = \left[\mathbb{I} + \frac{\alpha_s}{2\pi} K^{\text{POS}} \otimes\right] f^{\overline{\text{MS}}}(Q^2)\,,$$

which **inverse** can be obtained **perturbatively** (with the same caveat for the $x \to 1$ **limit**).

Is it enough?

No it's **not enough**, **nor needed**.

Sufficient

PDF positivity it's still not enough to guarantee observables positivity, since the coefficient functions are not bounded to be positive in x space in 4 dimensions.

Necessary

It's also not necessary, because negative PDF folded with suitable coefficient functions can still lead to positive observables.

So why?

It is a useful physical constraint on the PDFs, so it will add **more theoretical knowledge** in the fit.
In practice it cuts out a region of the hypothesis space that should not be explore by the fitting algorithm, so

The main result will be reducing the variance of the fitted PDFs.

Thanks for your attention