Superconformal theories from S-fold geometries

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Plan of the talk

- We construct $\mathcal{N}=2$ $\mathcal{S}$-fold geometries (generalizing the $\mathcal{N}=3$ construction). These realize uniformly all rank 1 SCFTs and define higher rank generalizations thereof.
- We develop various tools to study 't Hooft anomalies, the moduli space and RG flows of these theories. We find that the HB of these theories is related to ALE instantons with holonomy at infinity involving outer-automorphisms.
- We discuss the connection of $\mathcal{S}$-fold geometries with the torus compactifications of 6d $\mathcal{N}=(1,0)$ SCFTs.

$\mathcal{N} = 3$ theories in 4d from F-theory

$\mathcal{S}$-folds: Type IIB compactifications involving an S-duality twist:

Consider $U(1)$ $\mathcal{N} = 4$ SYM and gauge a $\mathbb{Z}_\ell \subset U(1)_R \times SU(2)_F$. To preserve $\mathcal{N} = 3$ susy we embed $\mathbb{Z}_\ell$ in $SL(2, \mathbb{Z})$.

We can construct $\mathcal{N} = 3$ SCFTs by probing with r D3 branes a $\mathbb{Z}_\ell$ $\mathcal{S}$-fold geometry: Garcia-Etxebarria, Regalado '15; Aharony, Tachikawa '16.

$$\frac{T^2 \quad \mathbb{C} \quad \mathbb{C} \quad \mathbb{C}}{\omega_\ell \quad \omega_\ell^{-1} \quad \omega_\ell \quad \omega_\ell^{-1}}; \quad \omega_\ell = e^{2\pi i/\ell} \quad \ell = 2, 3, 4, 6.$$

We can introduce discrete flux for H_3 and B_3: Aharony, Tachikawa '16.

Value of ℓ	2	3	4	6
Discrete flux?	yes	yes	yes	no

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$\mathcal{N} = 2$ instanton theories from $D3$ branes in F-theory

We probe with $D3$ branes a 7-brane with constant τ: Mukhi, Dasgupta '96.

G	Δ_7	Weierstrass	Value of τ
$\emptyset$	$\frac{6}{5}$	$y^2 = x^3 + z$	$e^{\pi i/3}$
A_1	$\frac{4}{3}$	$y^2 = x^3 + xz$	$e^{\pi i/2}$
A_2	$\frac{3}{2}$	$y^2 = x^3 + z^2$	$e^{\pi i/3}$
D_4	2	$y^2 = x^3 + x\tau z^2 + z^3$	τ
E_6	3	$y^2 = x^3 + z^4$	$e^{\pi i/3}$
E_7	4	$y^2 = x^3 + xz^3$	$e^{\pi i/2}$
E_8	6	$y^2 = x^3 + z^5$	$e^{\pi i/3}$

In the 4d theory z is the CB operator with dimension Δ_7. The gauge symmetry G on the 7-brane is the global symmetry in 4d.

$\mathcal{N} = 2$ $\mathcal{S}$-folds = $\mathcal{S}$-folds + 7-branes

We wrap the 7-brane on $\mathbb{C}^2/\mathbb{Z}_\ell$ and combine this with a $\mathbb{Z}_\ell$ quotient of the z-plane. To preserve $\mathcal{N} = 2$ supersymmetry this must be accompanied (for $\ell \neq 1$) by the action of $\mathbb{Z}_{\ell\Delta_7} \subset SL(2, \mathbb{Z})$.

We find the following possibilities:

- For $\ell = 1$ $\Delta_7 = 1, 6/5, 4/3, 3/2, 2, 3, 4$ and 6;
- For $\ell = 2$ $\Delta_7 = 1, 3/2, 2$ and 3;
- For $\ell = 3$ $\Delta_7 = 1, 4/3$ and 2;
- For $\ell = 4$ $\Delta_7 = 1$ and $3/2$;
- For $\ell = 5$ $\Delta_7 = 6/5$;
- For $\ell = 6$ $\Delta_7 = 1$.

Each possibility leads to an infinite family of 4d $\mathcal{N} = 2$ SCFTs.

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Mass deformation of $\mathcal{N}=2$ $\mathcal{S}$-fold theories

By deforming the 7-brane we implement mass deformations for the 4d theories on the probe D3 branes:

$\ell=2$	$\ell=3$	$\ell=4$
$(E_6,\mathbb{Z}_2)$	$(D_4,\mathbb{Z}_3)$	$(A_2,\mathbb{Z}_4)$
$\downarrow$	$\downarrow$	$\downarrow$
$(D_4,\mathbb{Z}_2)$	$(A_1,\mathbb{Z}_3)$	$\mathbb{Z}_4\ \mathcal{S}-fold$
$\downarrow$	$\downarrow$	
$(A_2,\mathbb{Z}_2)$	$\mathbb{Z}_3\ \mathcal{S}-fold$	
$\downarrow$		
$\mathbb{Z}_2\ \mathcal{S}-fold$		

There are two families of $\mathcal{N}=2$ rank-1 SCFTs exhibiting this pattern of mass deformations. Argyres, Martone '16.

They arise on a $D3$ probing $\mathcal{S}$-folds with and without discrete flux.

The holonomy at infinity

We should prescribe the holonomy for G at infinity in $\mathbb{C}^2/\mathbb{Z}_\ell$.
In the case at hand this is an order ℓ automorphism, with order ℓ' as an outer-automorphism. These are classified by Kac's theorem:

ℓ	ℓ'	$G^{(\ell')}$	Dynkin diagram	$H_{\mathcal{T}}$	$H_{\mathcal{S}}$
2	2	$E_6^{(2)}$	$\overset{1}{\underset{\alpha_0}{\circ}} - \overset{2}{\underset{\alpha_1}{\circ}} - \overset{3}{\underset{\alpha_2}{\circ}} \Leftarrow \overset{2}{\underset{\alpha_3}{\circ}} - \overset{1}{\underset{\alpha_4}{\circ}}$	$(F_4)_{\alpha_0}$	$Sp(4)_{\alpha_4}$
2	2	$D_4^{(2)}$	$\overset{1}{\underset{\alpha_0}{\circ}} \Leftarrow \overset{1}{\underset{\alpha_1}{\circ}} - \overset{1}{\underset{\alpha_2}{\circ}} \Rightarrow \overset{1}{\underset{\alpha_3}{\circ}}$	$SO(7)_{\alpha_0}$	$(Sp(2)SU(2))_{\alpha_2}$
2	1	$A_2^{(1)}$	$\overset{1}{\underset{\alpha_0}{\circ}} - \overset{1}{\underset{\alpha_1}{\circ}} - \overset{1}{\underset{\alpha_2}{\circ}} -$	$SU(3)_{\alpha_0\alpha_0}$	$(Sp(1)U(1))_{\alpha_0\alpha_1}$
3	3	$D_4^{(3)}$	$\overset{1}{\underset{\alpha_0}{\circ}} - \overset{2}{\underset{\alpha_1}{\circ}} \Lleftarrow \overset{1}{\underset{\alpha_2}{\circ}}$	$(G_2)_{\alpha_0}$	$SU(3)_{\alpha_2}$
3	1	$A_1^{(1)}$	$\overset{1}{\underset{\alpha_0}{\circ}} - \overset{1}{\underset{\alpha_1}{\circ}} -$	$SU(2)_{\alpha_0\alpha_0\alpha_0}$	$U(1)_{\alpha_0\alpha_0\alpha_1}$
4	2	$A_2^{(2)}$	$\overset{2}{\underset{\alpha_0}{\circ}} \Lleftarrow \overset{1}{\underset{\alpha_1}{\circ}}$	$SU(2)_{\alpha_0}$	$SU(2)_{\alpha_1\alpha_1}$

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Properties of the $\mathcal{T}^{(r)}_{G,\ell}$ SCFTs

Dimension of CB operators: $\ell\Delta_7, 2\ell\Delta_7, \dots, (r-1)\ell\Delta_7, r\Delta_7$.

ℓ	G	Flavor Symmetry	a	c
2	E_6	$(F_4)_{6r} \times SU(2)_{6r^2-5r}$	$\frac{6r^2+r}{4}$	$\frac{6r^2+3r}{4}$
2	D_4	$SO(7)_{4r} \times SU(2)_{4r^2-3r}$	r^2	$\frac{4r^2+r}{4}$
2	A_2	$SU(3)_{3r} \times SU(2)_{3r^2-2r}$	$\frac{6r^2-r}{8}$	$\frac{3r^2}{4}$
3	D_4	$(G_2)_{4r} \times U(1)$	$\frac{3r^2-r}{2}$	$\frac{6r^2-r}{4}$
3	A_1	$SU(2)_{8r/3} \times U(1)$	$\frac{2r^2-r}{2}$	$\frac{12r^2-5r}{12}$
4	A_2	$SU(2)_{3r} \times U(1)$	$\frac{12r^2-7r}{8}$	$\frac{6r^2-3r}{4}$
5	$\emptyset$	$U(1)$	$\frac{15r^2-11r}{10}$	$\frac{30r^2-21r}{20}$

For generic r the global symmetry is $H_{\mathcal{T}}$ times the isometry of the background, but enhances for $r \leq 2$. For $r=1$ these are the G 1-instanton theories.

Properties of the $\mathcal{S}^{(r)}_{G,\ell}$ SCFTs

Dimension of CB operators: $\ell\Delta_7, 2\ell\Delta_7, \dots, (r-1)\ell\Delta_7, r\ell\Delta_7$.

ℓ	G	Flavor Symmetry	a	c
2	E_6	$Sp(4)_{6r+1} \times SU(2)_{6r^2+r}$	$\frac{36r^2+42r+4}{24}$	$\frac{36r^2+54r+8}{24}$
2	D_4	$Sp(2)_{4r+1} \times SU(2)_{8r} \times SU(2)_{4r^2+r}$	$\frac{24r^2+24r+2}{24}$	$\frac{24r^2+30r+4}{24}$
2	A_2	$SU(2)_{3r+1} \times U(1) \times SU(2)_{3r^2+r}$	$\frac{18r^2+15r+1}{24}$	$\frac{18r^2+18r+2}{24}$
3	D_4	$SU(3)_{12r+2} \times U(1)$	$\frac{36r^2+36r+3}{24}$	$\frac{36r^2+42r+6}{24}$
3	A_1	$U(1) \times U(1)$	$\frac{24r^2+20r+1}{24}$	$\frac{24r^2+22r+2}{24}$
4	A_2	$SU(2)_{12r+2} \times U(1)$	$\frac{36r^2+33r+2}{24}$	$\frac{36r^2+36r+4}{24}$

For generic r the global symmetry is H_S times the isometry of the background, but enhances for $r=1$.

Holography and central charges

The central charges can be computed holographically from the F-theory description. Aharony, Tachikawa '07; Apruzzi, SG, Schafer-Nameki '20

The holographic dual is given by $AdS_5 \times M_5$ ($M_5 = \widetilde{S}^5/\mathbb{Z}_\ell$) where $\widetilde{S}^5$ is $\{|x|^2+|y|^2+|z|^2=1\} \subset \mathbb{C}^3$ with $\theta_z \in [0, 2\pi/\Delta_7]$.

$$2a - c = A(r+\epsilon)^2 + B(r+\epsilon); \quad c - a = C(r+\epsilon),$$

where $\epsilon = \pm\frac{\ell-1}{2\ell}$ is the $D3$ charge of the $\mathcal{N}=2$ $\mathcal{S}$-fold.

- A is proportional to $1/Vol(M_5)$, and we find $A = \ell\Delta_7/4$.
- B and C are proportional to $Vol(M_3)/Vol(M_5)$. The result is $B = C = (\Delta_7 - 1)/4$.

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$\mathcal{S}$-fold theories, Higgs branch and higgsings

For $\ell = 2, 3, 4$ we find the following sequence of RG flows:

$$\cdots \to \mathcal{S}^{(r)}_{G,\ell} \to \mathcal{T}^{(r)}_{G,\ell} \to \mathcal{S}^{(r-1)}_{G,\ell} \to \mathcal{T}^{(r-1)}_{G,\ell} \to \dots$$

From this we conclude that changing the holonomy at infinity (or switching-off the discrete flux) implements an higgsing.

It turns out that the flow $\mathcal{T}^{(r)}_{G,\ell} \to \mathcal{S}^{(r-1)}_{G,\ell}$ is always a minimal nilpotent higgsing for $H_{\mathcal{T}}$.
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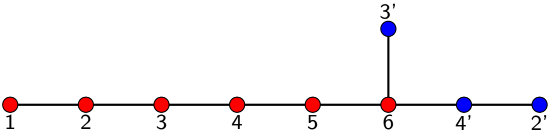
$\mathcal{S}$-fold theories in a nutshell

We can represent graphically $\mathcal{N}=2$ $\mathcal{S}$-fold SCFTs using an auxiliary affine E_8 Dynkin diagram:



$\mathcal{T}$ theories with $\ell = 1, 2, 3, 4, 5, 6.$
$\mathcal{S}$ theories with $\ell = 2, 3, 4.$

This peculiar pattern arises because $\mathcal{S}$-fold SCFTs (with $\ell\Delta_7 = 6$) arise as twisted torus compactifications of 6d $\mathcal{N}=(1,0)$ theories.

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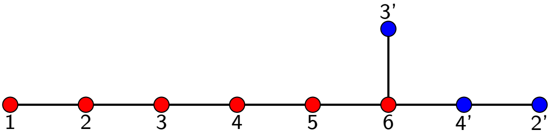

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An alternative construction from M-theory

Two realizations of rank-r E_8 MN theories: Minahan, Nemeschansky '96.

- $\mathcal{S}$-fold theories with $\ell = 1$ and $\Delta_7 = 6$. These correspond to r D3 branes probing a 7-brane of type E_8.
- Rank-r E-string on a torus: r M5 branes probing the M9 wall in M-theory.

In both cases a $\mathbb{R}^4$ transverse to the probes $(SU(2)_R \times SU(2)_F)$.

To realize $\mathcal{S}$-folds with $\ell > 1$, we orbifold the $\mathbb{R}^4$ in both descriptions. This leads to orbi-instanton theories on the M-theory side!

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6d Orbi-instanton theories

We consider 6d $\mathcal{N} = (1,0)$ obtained by wrapping the M9 brane on a ADE singularity $\mathbb{C}^2/\Gamma$. Del Zotto, Heckman, Tomasiello, Vafa '14.

The 6d SCFT is specified by the choice of holonomy at infinity for the E_8 gauge field (in one-to-one correspondence with homomorphisms $\rho : \Gamma \to E_8$).
Its global symmetry is $G \times \Gamma$, where $G \subset E_8$ commutes with $\rho(\Gamma)$.

When $\Gamma = \mathbb{Z}_\ell$ the homomorphisms are specified by selecting nodes of the affine E_8 Dynkin diagram such that the sum of labels is ℓ.

The 6d theories relevant for constructing $\mathcal{S}$-fold theories are specified by selecting **a single node.**

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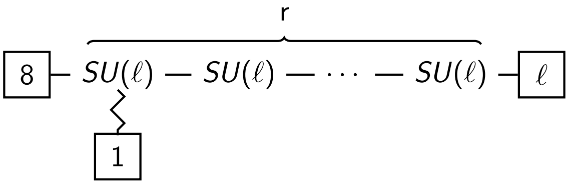
The $\mathcal{S}^{(r)}_{G,\ell}$ theories from six dimensions

When the holonomy involves the nodes $2'$, $3'$ or $4'$ (therefore for $\ell = 2, 3, 4$) we find Mekareeya, Ohmori, Tachikawa, Zafrir '17; Cabrera, Hanany, Sperling '19.



We can construct $\mathcal{S}^{(r)}_{G,\ell}$ theories (with $\ell\Delta_7 = 6$) by compactifying these 6d theories on T^2 with almost commuting $\mathbb{Z}_\ell$ holonomies (case $r = 1$ studied by Ohmori, Tachikawa and Zafrir).

Almost commuting holonomies on T^2

Let's consider the case $\ell = 2$: $\boxed{8} - SU(2) - \cdots - SU(2) - \boxed{2}$

The 8 flavors transform as $(\mathbf{8}, \mathbf{2})$ under $USp(8) \times SU(2) \subset SO(16)$.

We embed in the $SU(2)$ factor the almost commuting holonomies

$$P = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}; \quad Q = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

To have all fields uncharged under $\mathbb{Z}_2$ we embed the holonomy in all the gauge groups, breaking the gauge symmetry completely.

We end up in 4d with a rank r theory with $USp(8) \times SU(2)$ global symmetry. This is the $\mathcal{S}^{(r)}_{E_6,2}$ $\mathcal{S}$-fold theory.

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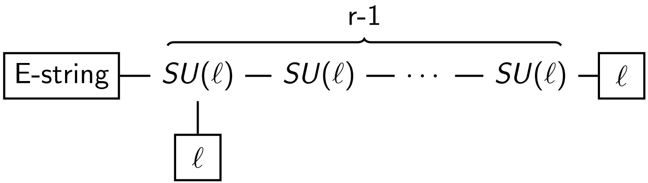
The $\mathcal{T}^{(r)}_{G,\ell}$ theories from six dimensions

Similarly, when the E_8 holonomy involves selecting the nodes $1, 2, 3, 4, 5$ or 6 (for any $\ell \leq 6$) we get the theories



where a $SU(\ell)$ subgroup of E_8 (from the E-string) is gauged.

Via torus compactification we find the $\mathcal{T}^{(r)}_{G,\ell}$ theories with $\ell\Delta_7 = 6$.

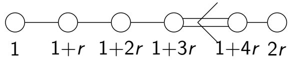
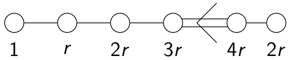
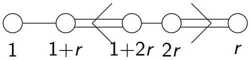
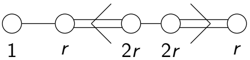
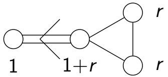
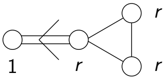
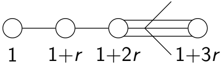
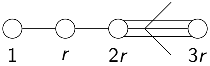
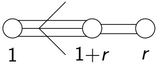
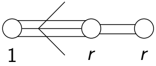
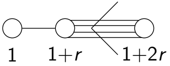
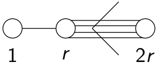
Magnetic quivers from 6d

For $\mathcal{S}$-fold SCFTs we can derive magnetic quivers, whose Coulomb Branch is the Higgs Branch of the underlying 4d theory.

To derive the quivers we start from the parent 6d theory and interpret the T^2 reduction as a 'two-step process', reducing first to a 5d SCFT which is then compactified on a circle to 4d with a twist in a $\mathbb{Z}_\ell$ discrete symmetry. Ohmori, Tachikawa, Zafrir '18.

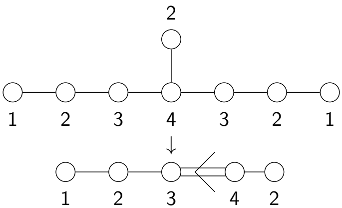
Once we know the magnetic quiver for the 5d SCFT (from the quiver of the 6d theory or from the corresponding brane web), the effect of the discrete twist is accounted for by **folding** the quiver of the 5d theory. Bourget, Hanany, Miketa '20.

Magnetic quivers for $\mathcal{S}$-fold theories

(G,ℓ)	$\mathcal{S}^{(r)}_{G,\ell}$	$\mathcal{T}^{(r)}_{G,\ell}$
$(E_6,2)$		
$(D_4,2)$		
$(A_2,2)$		
$(D_4,3)$		
$(A_1,3)$		
$(A_2,4)$		

Folding quivers

The magnetic quiver for the E_7 theory has a $\mathbb{Z}_2$ symmetry:



Folding means restricting to monopoles whose magnetic charges are $\mathbb{Z}_2$ invariant.

- Ungauging on the right (long side) we find $\mathcal{O}^{E_6}_{min}$;
- Ungauging on the left (short side) we find $\mathcal{O}^{F_4}_{Nmin} = \mathcal{O}^{E_6}_{min}/\mathbb{Z}_2$.

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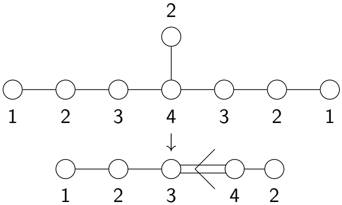

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$\mathcal{S}$-fold theories and moduli spaces of instantons

The $\mathcal{S}$-fold magnetic quivers describe two moduli spaces, depending on the ungauging scheme considered: Hanany, Zajac '20.

If we ungauge on a long node we find a space $\mathcal{C}_L$ (the Higgs Branch of the corresponding $\mathcal{S}$-fold theory), whereas if we ungauge on the short side we find $\mathcal{C}_S = \mathcal{C}_L/\mathbb{Z}_\ell$.

$\mathcal{C}_S$ can be interpreted as the Higgs Branch of $\mathbb{Z}_\ell$-gauged $\mathcal{S}$-fold theories and provides examples of moduli spaces of G instantons on $\mathbb{C}^2/\mathbb{Z}_\ell$ with holonomy involving outer-automorphisms.

Concluding remarks

- We constructed a family of CY4 singularities which interpolate between 7-branes in flat space and $\mathcal{N}=3$ $\mathcal{S}$-folds. When probing these with $D3$ branes we can realize all rank 1 SCFTs and define higher rank generalizations thereof.
- The HB of these theories is related to instantons on ALE spaces with holonomy involving outer-automorphisms and it would be interesting to study these spaces in detail.
- Many other interesting properties of $\mathcal{N}=2$ $\mathcal{S}$-fold models such as CB stratifications, mass deformations, chiral algebras...

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