Large charges in QFT

Diego Rodriguez-Gomez
U. of Oviedo

Based on joint work w/ G.Arias-Tamargo, A.Bourget & J.Russo

- QFT is hard. It simplifies when there is a small parameter on which one can expand

The star example: the semiclassical limit

- Take

$$Z = \int e^{-\frac{S}{\hbar}} \,, \qquad S = \int d^4x \, \partial\phi^2 + \lambda \, \phi^4 \,. \tag{1}$$

- Do $\phi = \lambda^{-\frac{1}{2}} \, \varphi$ to find

$$Z = \int e^{-\frac{S}{\hbar\lambda}} \,, \qquad S = \int d^4x \, \partial\varphi^2 + \varphi^4 \,. \tag{2}$$

- For very small $\hbar\lambda$ the saddle point approximation becomes exact.
- Moreover, one can systematically compute the corrections: perturbation theory.

- This allows to access a “corner” of QFT. There is however a huge wild to explore. One way is to look for similar expansions on other parameters

Another star example: the large N limit

- Take

$$Z = \int D\vec{\phi}\, e^{-S}\,, \qquad S = \int d^d x\, \partial\vec{\phi}^2 + \lambda\,(\vec{\phi}^{\,2})^2\,. \tag{1}$$

- Do a Hubbard-Stratanovich-like transformation

$$Z = \int D\sigma \int D\vec{\phi}\, e^{-S}\,, \qquad S = \int d^d x\, \partial\vec{\phi}^2 + \sqrt{\lambda}\,\sigma\,\vec{\phi}^2 - \frac{\sigma^2}{4}\,. \tag{2}$$

- This may be re-written as

$$Z = \int D\sigma \int D\phi\, e^{-\frac{S}{N^{-1}}}\,, \qquad S = \int d^d x\, \partial\phi^2 + \sqrt{\lambda\, N}\,\frac{\sigma}{\sqrt{N}}\,\phi^2 - \frac{\sigma^2}{4\,N}\,. \tag{3}$$

- So in the limit $N \to \infty$, $\lambda \to 0$, $g = N\,\lambda =$ fixed we have another semiclassical limit, this time with N^{-1} playing the role of $\hbar$.

- Inspired by this, in search for semiclassical expansions to probe more corners of QFT’s, one may considering “cooking parameters” in which to expand
    - Suppose a family of operators labelled by some integer
    - Such integer provides a new ``scale”, and in many cases its large value limit provides a new ``semiclassical” expansion

Note that this may be a window into strong coupling dynamics!

- There are greatly celebrated examples of this idea
    - Regge theory
    - large spin expansions in CFT’s
    - pp-wave
    - …

- A natural and very universal candidate for such integer is a conserved charge

- Moreover, since fixed points play a pivotal role in QFT, we'll concentrate on CFT's: consider a CFT with a conserved charge and study the large charge sector of such theory

- Lots of recent progress along these lines

A lot of recent activity: Hellerman, Maeda, Watanabe, Alvarez-Gaume, Orlando, Reffert.... Apologies if references are missing!

  - ``bottom-up" effective field theory approach
  - ``top-down" case studies
  - applications to SUSY systems
  - ...

- Today we'll discuss some particular cases: we'll probe large charge sectors of interesting CFT's in a ``top-down" approach

Contents

- Motivation
- Correlation functions in N=2 SUSY theories and the large charge expansion
- Large charge operators in WF fixed points
- Conclusions

Correlation functions in N=2 and large charge

- N=2 theories are interesting playgrounds to tinker with QFT: they have SUSY enough so as to constrain dynamics to accessible limits but not too much so as to "trivialize"

- A lot of activity in recent times on N=2 theories (and relatives in other dimensions)

- In particular, one can exploit SUSY to compute observables exactly

LOCALIZATION

↓

This includes correlators, defect operators and even the partition function itself (meaningful for 4d N=2)

CPO correlators through localization

- The 4d superconformal algebra contains

$$\{\overline{Q}^a_{\dot\alpha}, \overline{S}^b_{\dot\beta}\} = \epsilon_{\dot\alpha\dot\beta}\epsilon^{ab}\Big(\Delta - \frac{R}{2}\Big) + \epsilon^{ab}M_{\dot\alpha\dot\beta} + \epsilon_{\dot\alpha\dot\beta}J^{ab}$$

- Hence an interesting shortening condition is

$$[\overline{Q}^a_{\dot\alpha}, O] = 0 \rightsquigarrow \Delta_O = \frac{R_O}{2},\, j_L = s = 0,\, (\text{and } \mathrm{j_R} = 0)$$

- This (together with being annihilated by S's) defines Chiral Primary Operators (CPO's)

- In lagrangian theories, CPO's are composites of scalars in vector multiplets

For completeness: another natural shortening condition is

$$[Q^1_\alpha, O] = [\overline{Q}^1_{\dot\alpha}, O] = 0$$

This corresponds to operators on the Higgs branch (hypermultiplets, in lagrangian theories)

- CPO's have a non-singular OPE (not to violate the BPS bound). As a consequence, they form a ring: the chiral ring

- Their correlation functions are

$$\langle O_I(0)\overline{O}_{\overline{J}}(x)\rangle = \frac{g_{I\overline{J}}(\tau^i,\overline{\tau}^i)}{|x|^{2\Delta_I}}\,\delta_{\Delta_I,\Delta_{\overline{J}}}$$

Endowes the Coulomb branch of a very interesting geometry…but that's another story. See Papadodimas; Baggio, Niarchos & Papadodimas

- SUSY implies that

$$\langle O_1(x_1)\cdots O_n(x_n)\overline{O}(\infty)\rangle = \lim_{x_i\rightsquigarrow x}\langle O_1(x_1)\cdots O_n(x_n)\overline{O}(\infty)\rangle =$$

$$C_{\Delta_1\Delta_2}^{\Delta'_1}C_{\Delta'_1\Delta_3}^{\Delta'_2}\cdots C_{\Delta'_{n-2}\Delta_n}^{\Delta'_{n-1}}g_{\Delta'_{n-1}\Delta_y}$$

- We will be interested on 2-point functions of extremal correlators

- The 2-point functions can be mapped to the sphere

$$\langle A(x)\overline{B}(0)\rangle = \frac{C_{AB}}{|x|^{2\Delta_A}}\,\delta_{\Delta_A\Delta_B} \rightsquigarrow \langle |x|^{2\Delta_A} A(x)\overline{B}(0)\rangle = C_{AB}\delta_{\Delta_A\Delta_B}$$

- To extract C, we can take the large x limit

$$\lim_{|x|\to\infty} |x|^{2\Delta_A} A(x) = 4^{\Delta_A} \lim_{|x|\to\infty} \left(1 + \frac{|x|^2}{4}\right)^{\Delta_A} A(x)$$

- Since $ds^2_{\mathbb{R}^4} = \left(1 + \frac{|x|^2}{4}\right)^4 ds^2_{\mathbb{S}^4}$

- …it follows that $4^{\Delta_A}\langle A(N)\overline{B}(S)\rangle_{\mathbb{S}^4} = C_{AB}$

- It turns out that, due to the very special SUSY properties of CPO's, upon deforming the theory by

$$-\frac{1}{32\pi^2}\int d^4x \int d^4\theta\, \mathcal{E}\, \sum_I \tau_I\, \mathcal{O}_I$$

- …the correlation functions are

$$\frac{1}{\tilde{\mathcal{Z}}(\mathbb{S}^4)}\partial_{\tau_I}\partial_{\overline{\tau}_{\overline{J}}}\tilde{\mathcal{Z}}(\mathbb{S}^4)\Big|_{\tau_I=0\,(I\neq YM)} = \langle A(N)\overline{B}(S)\rangle_{\mathbb{S}^4}$$

- There is one subtlety, though: due to the conformal anomaly there can be mixing

$$O_{\Delta}^{\mathbb{R}^4} \to O_{\Delta}^{\mathbb{S}^4} + \frac{\alpha_1}{R^2}\, O_{\Delta-2}^{\mathbb{S}^4} + \frac{\alpha_2}{R^4}\, O_{\Delta-4}^{\mathbb{S}^4} + \cdots$$

- To remove this mixing, one has to run a Gram-Schmidt orthogonalization (to recover the delta in dimensions in the correlator!)

Gerchkovitz, Gomis, Ishtiaque, Komargodski & Pufu, 1602.05971

- For N=4 write the sphere operators as $\mathcal{O}^{(m)}_n = (\mathrm{Tr}\phi^2)^n \prod_{n_2=0} (\mathrm{Tr}\phi^k)^{n_k} = \phi_2^n\, \mathcal{O}^{(m)}$
- Fixing m, the correlators group into separate Toda chains

$$16\,\partial_\tau\partial_{\bar\tau}\log G^{(m)}_{2n} = \frac{G^{(m)}_{2n+2}}{G^{(m)}_{2n}} - \frac{G^{(m)}_{2n}}{G^{(m)}_{2n-2}} - G^{(0)}_2$$

- For N=2 (superconformal) SQCD these chains are not obviously decoupled

  - Recall that the GS is needed because the sphere supplements a scale which allows mixings
  - Using this, it is possible to come up with an ordering for the GS such that one finds families of decoupled Toda chains
  - This typically requires mixing coefficients proportional to positive powers of R: not clear what the flat space limit is
  - For N=4 things arrange such that these terms are absent (and one recovers the decoupled Toda chains)

Bourget, R-G & Russo, 1810.00840

- One can nevertheless simply compute the lowest correlators by brute force.

- The simplest ones are $\mathcal{O}_n = (\mathrm{Tr}\phi^2)^n$. They can be computed from derivates of Z wrt. the coupling (because they correspond to “insertions of the action”!)

- One finds

$$\frac{G_{2n}^{\rm QCD}}{G_{2n}^{\mathcal{N}=4}} = 1 - \frac{9\,n\,(N^2+2n-1)\,\zeta(3)}{4\pi^2\,(\mathrm{Im}\tau)^2} + \frac{5\,n\,(2N^2-1)\,(3N^4+(15n-3)N^2+(20n^2-15n+4))\,\zeta(5)}{4\pi^3\,N\,(N^2+3)\,(\mathrm{Im}\tau)^3} + \cdots .$$

$$G_{2n}^{\mathcal{N}=4,\mathfrak{g}} = \frac{n!\,2^{2n}}{(\mathrm{Im}\tau)^{2n}}\alpha\,(1+\alpha)_{n-1}\,, \qquad \alpha = \frac{1}{2}\dim(\mathfrak{g})\,.$$

- The polynomial in n multiplying each order in the coupling is just the appropriate so as to define the double scaling limit (at FIXED N!)

$$n\to\infty\,, \qquad g\to 0\,, \qquad \lambda\equiv g^2 n = \text{fixed}\,,$$

(Gauge instantons truly supressed!)

Bourget, R-G & Russo, 1803.00580

- Going beyond this tower by explicit computation is very hard. The next simplest case is SU(3): there is only one more CPO. Explicitly computing the correlators (heavy use of Toda chain!) shows that the limit continues to exist

Beccaria, 1809.06280
Beccaria, 1810.10483

- It turns out that the existence of the limit is rooted in the structure of the correlators: the GS can be recasted as a matrix model

  - Very sketchy: for SU(2) there is only one CPO, whose sphere correlators are derivatives of Z wrt. the coupling. The flat space correlators are rations of subdeterminants of the matrix of derivatives

  - It turns out that each such subdeterminant can be written as a matrix integral: convert the computation of correlators into a matrix model!

$$\det \mathcal{M}_{(n)} = \frac{1}{n!} \int_0^\infty \prod_{j=0}^{n-1} dx_j e^{-4\pi \operatorname{Im} \tau x_j} x_j^{\frac{1}{2}} Z_{\text{1-Loop}} \left(\sqrt{x_j}\right) \prod_{j<k} (x_j - x_k)^2 \ .$$

  - The ’t Hooft limit of this matrix model is well defined: it is our double scaling limit (strictly speaking, the latter is the weak ’t Hooft coupling regime)

Grassi, Komargodski & Tizziano, 1908.10306
Beccaria, Galvagno & Hasan, 2001.06645

↓

(note that in any case, gauge instantons are safely supressed in this regime)

Large charge in WF

- The mere existence of the previous limit is somewhat surprising. One can wonder wether a toy model version exists. The natural candidate is

$$S = \int |\partial\phi|^2 + \frac{g}{4}|\phi|^4$$

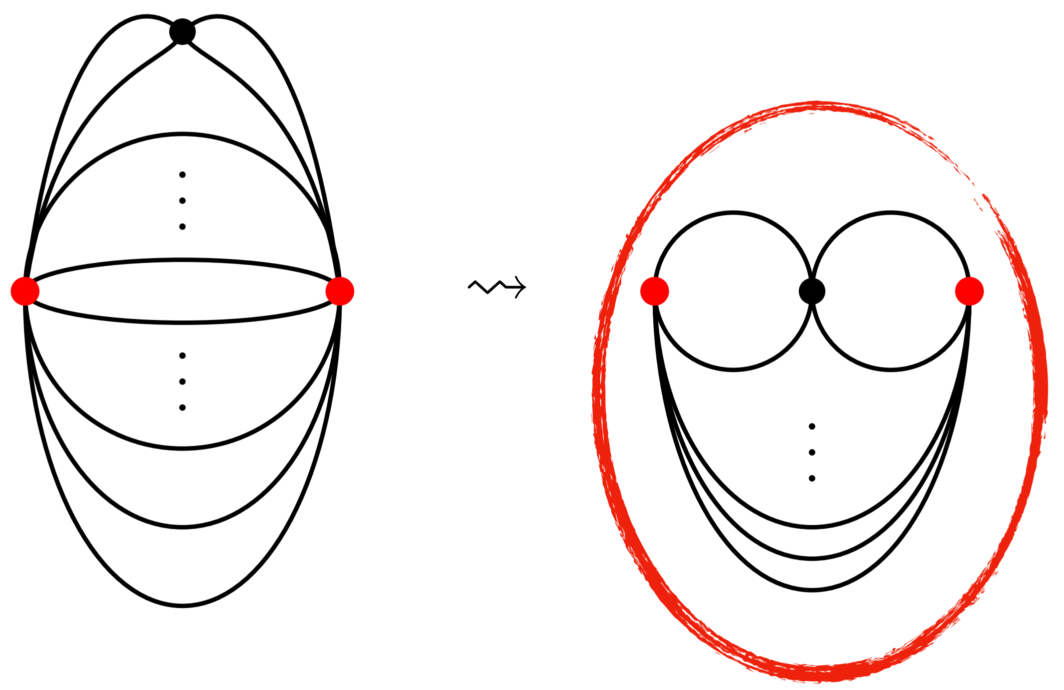
- This theory has a U(1) symmetry $\phi \to e^{i\alpha}\,\phi$

- Below 4d it has an IR fixed point

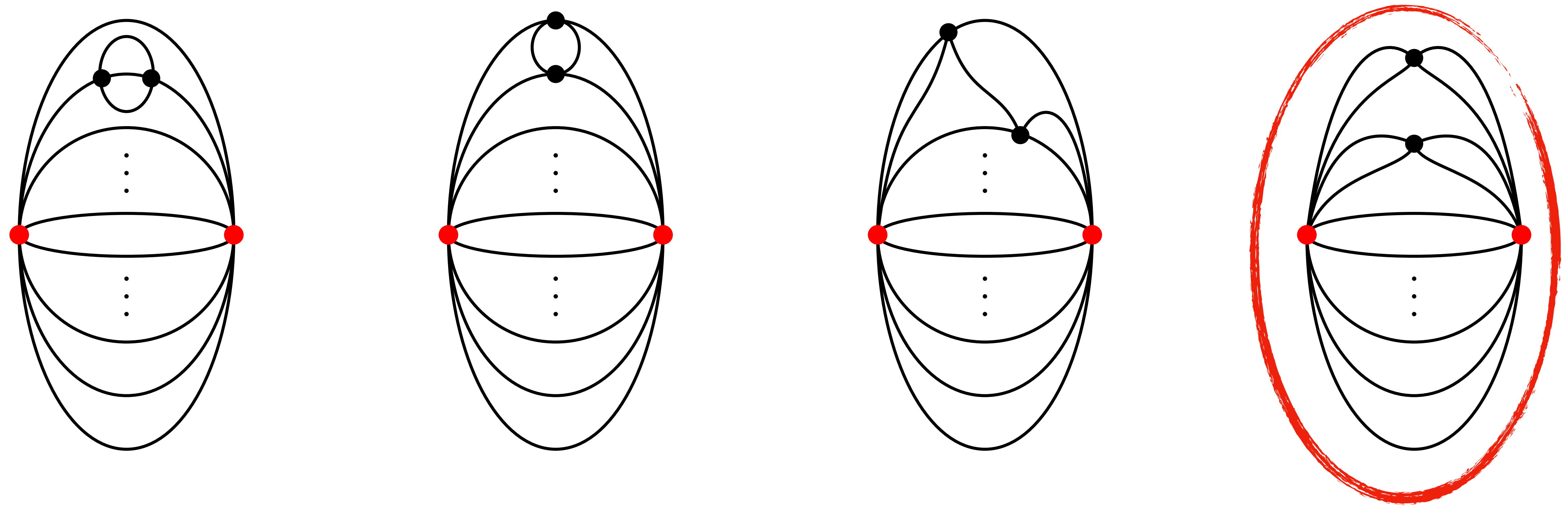
$$d = 4 - \epsilon \quad \rightsquigarrow \quad g_{WF} = \frac{16\pi^2}{5}\,\epsilon$$

- In d=4 the theory is free. The natural (charged) operators are $\mathcal{O}_n = \phi^n$

- Let's compute their 2-point functions in (standard) perturbation theory. To leading order



- To second order



- As n grows, there are many more "kermit diagrams"

- Counting only these, one finds

$$\langle \mathcal{O}_n(x)\,\bar{\mathcal{O}}_n(0)\rangle = n!\,\sum_{m=0}(-ig)^m\,K_m\,\frac{1}{4^m}\,\frac{n!}{(n-2m)!}\,\frac{1}{m!}\,.$$

- In large n $\frac{n!}{(n-2m)!}\sim n^{2m} \rightsquigarrow gn^2=\lambda$, so

$$\langle \mathcal{O}_n(x)\,\bar{\mathcal{O}}_n(0)\rangle = n!\,\sum_{m=0}K_m\left(\frac{-i\,\lambda}{4}\right)^m\frac{1}{m!}\,. \qquad \langle \mathcal{O}_n(x)\,\bar{\mathcal{O}}_n(0)\rangle = \langle \mathcal{O}_n(x)\,\bar{\mathcal{O}}_n(0)\rangle_0\,e^{-i\,\frac{\lambda\,\mathcal{K}}{4}}\,.$$

- …so the result of the 1-loop exponentiates! In the end, it boils down to computing

$$\mathcal{K}=\frac{1}{G(0,\,x)^2}\int d^4z\,G(0,\,z)^2\,G(z,\,x)^2\,.$$

- Using real space renormalization, one finally finds

$$\mathcal{K}=-\frac{i}{8\,\pi^2}\,\log(\Lambda^2x^2)\,. \qquad \langle \mathcal{O}_n(x)\,\bar{\mathcal{O}}_n(0)\rangle = \frac{n!}{(4\,\pi^2)^n\,|x|^{2\,(n+\frac{\lambda}{32\,\pi^2})}}\,.$$

Freedman, Johnson & Latorre, 1992

Arias-Tamargo, R-G & Russo, 1908.11347

- This can be understood from a (deeper) perspective

- The 2-point functions we are interested on can be written through a path integral as

$$\langle \mathcal{O}_n(x_1)\,\bar{\mathcal{O}}(x_2)\rangle = \int \phi(x_1)^n\,\bar{\phi}(x_2)^n\,e^{-S}$$

- Raising the insertions to the exponent

$$\langle \mathcal{O}_n(x_1)\bar{\mathcal{O}}(x_2)\rangle = \int e^{-S_{\text{eff}}}\,, \qquad S_{\text{eff}} = \int |\partial\phi|^2 + \frac{g}{4}|\phi|^4 - n\,\delta(x-x_1)\,\log\varphi - n\,\delta(x-x_2)\,\log\bar{\varphi}$$

- …but the field is a dummy integration variable $(\phi = \sqrt{n}\,\varphi)$

$$\langle \mathcal{O}_n(x_1)\bar{\mathcal{O}}(x_2)\rangle = \int e^{-n\mathcal{S}_{\text{eff}}}\,, \qquad \mathcal{S}_{\text{eff}} = \int |\partial\varphi|^2 + \frac{gn}{4}|\varphi|^4 - \delta(x-x_1)\,\log\varphi - \delta(x-x_2)\,\log\bar{\varphi}$$

→ **emergence of the double-scaling limit** $\quad n\to\infty, \qquad g\to 0, \qquad gn = \kappa = \text{fixed}$

Arias-Tamargo & R-G, Russo, 1908.11347
Badel, Cuomo, Monin & Rattazzi, 1909.01269
Watanabe, 1909.01337

- This suggests that the ``free energy" has a double expansion

$$\log\langle \mathcal{O}_n(x_1)\bar{\mathcal{O}}(x_2)\rangle = n\sum_{k=0} n^{-k}\,\Gamma_k(\kappa)$$

Badel, Cuomo, Monin & Rattazi, 1909.01337

- 1/n plays the role of Planck's constant. The leading term in the sum is the classical trajectory

- However, in terms of "the true Planck constant" this classical approximation is fully quantum (resums an infinite number of corrections)

- Note that the field gets a VEV of order $\sqrt{n}$
  - It is natural to guess that something similar happens in SQCD.
  - But there such VEV puts is in the Coulomb branch. The masses are

$$m_{\text{monopole}} \sim \frac{\phi}{g_{YM}} \sim \frac{\sqrt{n}}{g_{YM}} \sim \frac{n}{\sqrt{\lambda}} \qquad m_{\text{W/hyper}} \sim \phi\, g_{YM} \sim \sqrt{n}\, g_{YM} \sim \sqrt{\lambda}$$

  - So in the gauge theory we expect W/hypers to dominate the limit (specially at weak 't Hooft coupling)

- Large n defines a new “semiclassical expansion” (for a sector of the theory). The saddle point equations are

$$\partial^2\varphi + \frac{\kappa}{2}\varphi\,|\varphi|^2 = -\delta(x-x_2)\,\frac{1}{\bar\varphi} \qquad \partial^2\bar\varphi + \frac{\kappa}{2}\bar\varphi\,|\varphi|^2 = -\delta(x-x_1)\,\frac{1}{\varphi}$$

- At weak ``’t Hooft coupling” we can solve these perturbatively

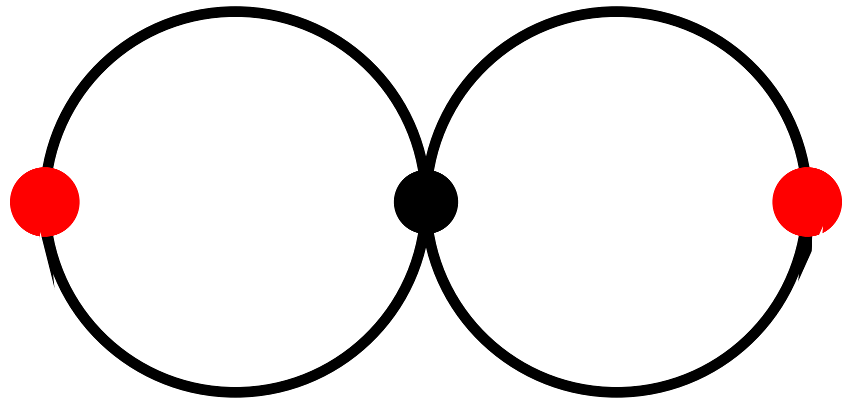
$$\begin{aligned}\partial^2\varphi &= -\delta(x-x_2)\,\frac{1}{\bar\varphi}\\ \partial^2\bar\varphi &= -\delta(x-x_1)\,\frac{1}{\varphi}\end{aligned} \quad\longrightarrow\quad \begin{aligned}\varphi &= \frac{G(x-x_2)}{\sqrt{G(x_1-x_2)}}\\ \bar\varphi &= \frac{G(x-x_1)}{\sqrt{G(x_1-x_2)}}\end{aligned}$$

Hellerman & Maeda, 1710.07336

- Plugging this back into the action we find

$$\langle \mathcal{O}_n(x_1)\bar{\mathcal{O}}(x_2)\rangle = G(x_1-x_2)^n\,e^{-\frac{\kappa\,n}{4}\,\frac{1}{G(x_1-x_2)^2}\int d^dx\,G(x-x_1)^2\,G(x-x_2)^2}$$

- In the exponent we recognize the 1-loop diagram which we had before



- The result is then

$$\langle \mathcal{O}_n(x_1)\bar{\mathcal{O}}(x_2)\rangle = \frac{1}{|x_1-x_2|^{2\Delta}}\,, \qquad \Delta = n\,(1+\frac{\kappa}{32\pi^2}+\mathcal{O}(\kappa^2))$$

- We may write

$$\Delta = n + \frac{gn^2}{32\pi^2} + \sum c_i\, g^i\, n^{i+1} = n + \frac{\lambda}{32\pi^2} + \frac{1}{n}\sum c_i\,\lambda^i \to n + \frac{\lambda}{32\pi^2}$$

The strict large n, extreme weak coupling limit selects the first correction, which is what the diagramatic computation captures

- We can also compute higher point functions of large charge operators, at least perturbatively in the "'t Hooft coupling". Consider

$$\langle \phi(x_1)^{n_1} \cdots \phi(x_r)^{n_r} \bar{\phi}(y_1)^{m_1} \cdots \bar{\phi}(y_s)^{m_s} \rangle , \qquad \sum n_i = \sum m_i$$

- The path integral representation is

$$\langle \phi(x_1)^{n_1} \cdots \phi(x_r)^{n_r} \bar{\phi}(y_1)^{m_1} \cdots \bar{\phi}(y_s)^{m_s} \rangle = \int e^{-n\mathcal{S}_{\text{eff}}}$$

$$\mathcal{S}_{\text{eff}} = \int |\partial\varphi|^2 + \frac{\kappa}{4}|\varphi|^4 - \sum^{r} \delta(x - x_i)\, \log\varphi - \sum^{s} \delta(x - x_i)\, \log\bar{\varphi}$$

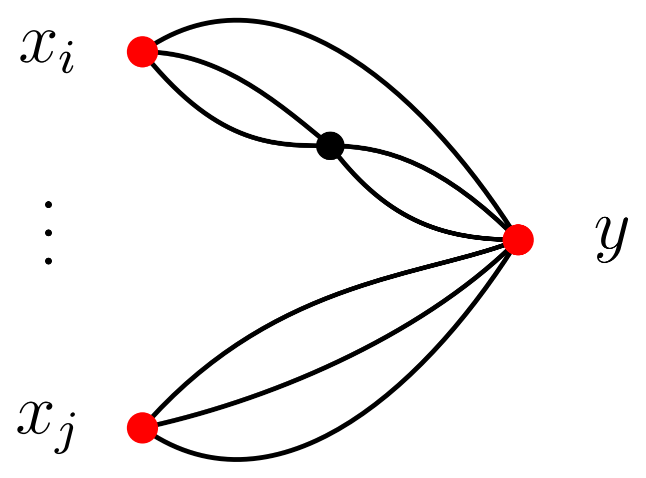
- so the analogous double scaling limit exists

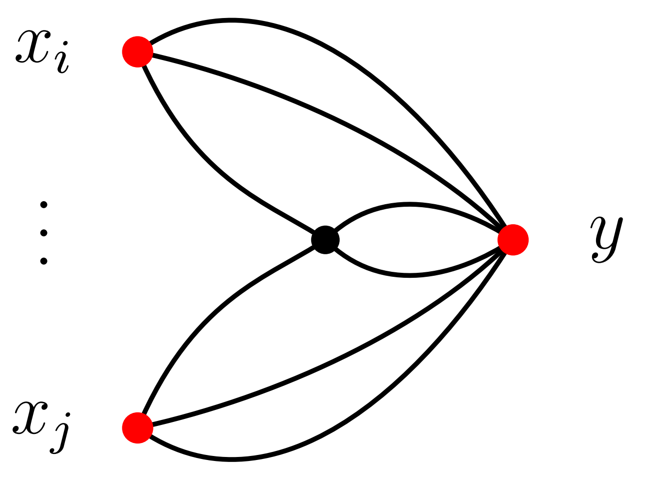
$$n_i = a_i n\ , \qquad m_j = b_j n\ , \qquad g \to 0, \ \ n \to \infty\ , \quad gn^2 = \text{fixed}\ ,$$

Arias-Tamargo, R-G & Russo, 1912.01623

- For “extremal correlators”

$$\langle \phi(x_1)^{n_1} \cdots \phi(x_r)^{n_r} \, \bar{\phi}(y)^m \rangle = \frac{m!}{(4\pi^2)^m \prod_{i=1}^{r} |x_i - y|^{2(n_i + \frac{\lambda \, a_i b}{32\pi^2})} \prod_{i<j}^{r} |x_i - x_j|^{-\frac{\lambda \, a_i \, a_j}{16 \, \pi^2}}} \, .$$





- For more general correlators things are more complicated. Concentrate on 4-point function (only extremal and one non-extremal)

$$\langle \phi(x_1)^n \, \phi(x_2)^n \, \bar{\phi}(x_3)^n \, \bar{\phi}(x_4)^n \rangle = \frac{(n!)^2}{(4\pi^2)^{2n}} \frac{(r_{14}r_{23} + r_{13}r_{24})^{2n} \, (r_{12}r_{34})^{\frac{\lambda}{16\pi^2}}}{(r_{14}r_{23}r_{13}r_{24})^{2\Delta}} \; e^{-S'_{\rm int}}$$

$$S'_{\rm int} \equiv \frac{\lambda}{16\pi^2} \frac{1}{(r_{14}r_{23} + r_{13}r_{24})^2} \left(H \, r_{14}^2 \, r_{23}^2 - r_{13}^2 r_{24}^2 \log \frac{r_{13}r_{24}}{r_{12}r_{34}} - r_{14}^2 r_{23}^2 \log \frac{r_{14}r_{23}}{r_{12}r_{34}} \right)$$

$$r_{ij} \equiv |x_i - x_j|$$

$$H = \frac{1}{1-x-y} \Big(\log x(1-y) \, \log \frac{y}{1-x} - 2\,\mathrm{Li}_2(x) + 2\,\mathrm{Li}_2(1-y) \Big)$$

$$x = \frac{\rho \, u^2}{1 + \rho \, u^2} \, , \quad y = \frac{\rho v^2}{1 + \rho \, v^2} \, , \quad \rho = \frac{2}{1 - u^2 - v^2 - \lambda} \, , \quad \lambda = \sqrt{(1 - u^2 - v^2)^2 - 4 \, u^2 \, v^2}$$

$$u \equiv \frac{r_{12}r_{34}}{r_{13}r_{24}} \, , \qquad v \equiv \frac{r_{14} \, r_{23}}{r_{13} \, r_{24}}$$

- Recovers the extremal case in the suitable coincidence limit
- The free part can be cross-checked against “conformal block computations” in Dolan-Osborn

- So far we have been interested on the weak “’t Hooft” coupling limit. In fact we have concentrated on the leading correction (leading 1/n: resums infinitely many loops in the standard expansion!).

- To go beyond there, just solve in perturbation theory the saddle point equations…unfortunately this is very hard!

- We can exploit the conformal invariance and map the “partition function” to the energy on the cylinder (at fixed charge!)

$$\langle \mathcal{O}(x)\,\mathcal{O}(0)\rangle = \frac{C}{|x|^{2\Delta}}$$

$$ds^2 = d\vec{x}^2 = dr^2 + r^2\,d\Omega^2 = e^{2\,\tau}\,(d\tau^2 + d\Omega^2)\,, \qquad r = e^{\tau}$$

$$\rightsquigarrow \quad \langle \mathcal{O}_c(-\infty)\,\mathcal{O}_c(\tau)\rangle = \lim_{\tau_i\to\infty} C\,e^{-\Delta\,(\tau-\tau_i)} \quad \rightsquigarrow \quad \langle \mathcal{O}_c(-\infty)\,\mathcal{O}_c(\tau)\rangle = \lim_{\tau_i\to\infty} C\,e^{-\Delta\int_{\tau_i}^{\tau} d\tau}$$

- Since we want spatially homogeneous configurations

$$S = \int \sum_i \partial\phi_i\partial\bar{\phi}_i - V \qquad L = \sum_i \left(\frac{1}{2}\,\Omega\,\dot{\rho}_i^2 + \frac{1}{2}\,\Omega\,\rho_i^2\,\dot{\theta}_i^2\right) - \Omega\,V\,, \qquad Q = \Omega\sum_i q_i\,\rho_i^2\,\dot{\theta}_i$$

- Since we want the charge to be fixed, we may consider instead

$$L' = L - \sum q_i\, \dot\theta_i = \sum_i \left(\frac{1}{2}\,\Omega\, \dot\rho_i^2 + \frac{1}{2}\,\Omega\, \rho_i^2\, \dot\theta_i^2\right) - \Omega\, V - \frac{\bar Q}{\vec q^{\,2}} \sum q_i\, \dot\theta_i \qquad \frac{d}{dt}\left(\Omega\, \rho_i^2\, \dot\theta_i - \frac{\bar Q}{\vec q^{\,2}}\, q_i\right) = 0$$

- Solving that eq. and plugging this back we find

$$\rho_i^2\, \dot\theta_i = \frac{\bar Q}{\Omega\, \vec q^{\,2}}\, q_i \quad \rightsquigarrow \quad L' = \sum_i \frac{1}{2}\,\Omega\, \dot\rho_i^2 - \Omega\, V_{\text{eff}}\,, \qquad V_{\text{eff}} = V + \sum_i \left(\frac{\bar Q}{\vec q^{\,2}}\right)^2 \frac{q_i^2}{2\,\Omega^2\, \rho_i^2} \qquad V = \frac{1}{2}\, m^2\, \vec\rho^{\,2} + U$$

- Extracting an overall factor of the charge, we can establish the double scaling expansion

- Since we also want time-independent configurations, we just have to minimize the effective potential. These are algebraic equations!

$$\Delta = \begin{cases} \kappa << 1: & n\left(1 + \frac{\kappa}{32\pi^2} - \frac{\kappa^2}{512\pi^4} + \mathcal{O}(\kappa^3)\right) \\ \kappa >> 1: & n\left(\frac{3\kappa^{\frac{1}{3}}}{8\pi^{\frac{2}{3}}} + \frac{\pi^{\frac{2}{3}}}{\kappa^{\frac{1}{3}}} + \mathcal{O}(\kappa^{-1})\right) \end{cases}$$

Badel, Cuomo, Monin & Rattazzi, 1909.01269
Watanabe, 1909.01337

- For strong “’t Hooft coupling” we can write $\Delta \sim n^{\frac{4}{3}}$
- We can recognize here $\Delta \sim n^{\frac{d}{d-1}}$ (computing quantum corrections indeed completes the expansion)
- This makes contact with the (very generic=powerful!) expected large charge behavior
    - Mapping to the cylinder, the large charge sector appears as a SSB ground state
    - The radius of the sphere and the charge provide two well-separated scales: regardless on the coupling one can write an EFT (strongly constrained by conformal invariance)
    - The scaling of the dimensions immediately follows, along with other universal properties
    - The extra twist: really it is an expansion in $g^{\#}n$. Depending on the n-scaling of g one can have different regimes.

(In SQCD g is exactly marginal. In WF g is tied with d-4, so we interpolate across dimensions)

Great body of work building up on this, starting with Alvarez-Gaume, Loukas, Orlando & Reffert; Hellerman, Kobayashi, Maeda & Watanabe and many others (apologies!)

- WF in d=3 is very interesting…in particular the O(N) model bears a connetion to holography. One may wonder wether a similar story holds true for the O(N) model around d=5

- The critical dimension is 4: not obvious that the O(N) model is a CFT (UV: it is “flown from”). It was proposed that it has a UV completion in terms of a cubic theory with N+1 fields and O(N) symmetry.

- Such UV completion has (PERTURBATIVE) IR fixed point. (Some) correlators, scaling dimensions etc. were shown to agree with the quartic model

Fei, Giombi & Klebanov, 1404.1094
Fei, Giombi, Klebanov & Tarnopolsky, 1411.1099

- However there are instanton corrections which render scaling dimensions etc. imaginary: not really a CFT

Aizenmann, 1981
Percacci & Vacca, 1405.6622
Eichhorn, Janssen & Scherer, 1604.03561
Kamikado& Kanazawa, 1604.04830
Giombi, Huang, Pufu & Tarnopolsky, 1910.02462

-
    - De-stabilizing instantons are proportional to $e^{-S_{\rm inst}} \sim e^{-\frac{1}{\rm coupling}} \sim e^{-\frac{n}{\lambda}}$
    - In the double-scaling limit these are supressed (just as in SYM)

Large charge sector: instability-free sector in a complex CFT!

Arias-Tamargo, R-G & Russo, 2003.13772

- Consider a O(2N) theory with bosonic fields in the vector representation. Generic operators made out of products of fields will be in the

See also Antipin, Bersini, Sannino, Wang & Zhang; 2003.13121, 2006.10078

$$\mathrm{Sym}^n\Big([1,\,0,\cdots,\,0]_{D_N}\Big)=\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}[n-2i,\,0,\cdots,\,0]_{D_N}$$

- O(2N) has a U(1)xSU(N) subgroup under which $[n,\,0,\cdots,\,0]_{D_N}\to\sum_{k=0}^{n}[n-k,\,0\cdots0,\,k]_{(\mathbf{n-2\,k})}$

- Writting $\phi^I=\frac{\varphi^I+i\,\varphi^{I+N}}{\sqrt{2}}$, the operator $\mathcal{O}_n=(\phi^1)^n$ can only be an entry of $[n,\,0,\cdots,\,0]_{(\mathbf{n})}$. So computing their correlators we'll capture the correlators of $[n,\,0,\cdots,\,0]_{(\mathbf{n})}$

- The cubic theory (that with IR perturbative fixed point) is

$$S=\int d^dx\left(|\partial\vec{\phi}|^2+\frac{1}{2}\big(\partial\eta\big)^2+g_1\,\eta\,|\vec{\phi}|^2+\frac{g_2}{6}\,\eta^3\right)$$

- Just as before

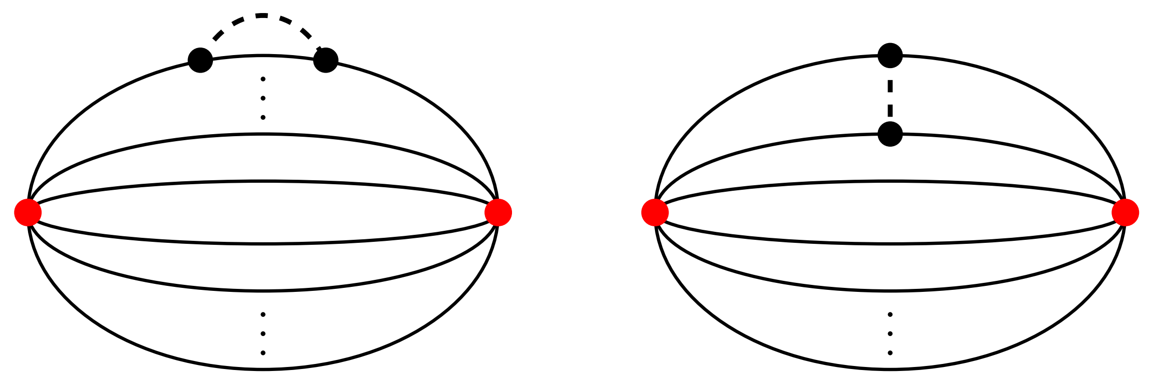
$$\phi^I=\sqrt{n}\,\Phi^I\,,\qquad\eta=\sqrt{n}\,\rho\qquad g_1=\frac{h_1}{\sqrt{n}}\,,\qquad g_2=\frac{h_2}{\sqrt{n}}$$

- Now we find

$$\langle \mathcal{O}_n(x_1)\,\overline{\mathcal{O}}_n(x_2)\rangle \sim \frac{1}{|x_1-x_2|^{2\,(\Delta_{\rm cl}+\gamma_{[n,\,0,\cdots,\,0]_{D_N}})}} \qquad \gamma_{[n,\,0,\cdots,\,0]_{D_N}} = -\frac{3\,\epsilon\,n^2}{N}$$

- In turn, the cuartic theory, upon introducing a HS field and looking to the IR fixed point, becomes

$$S=\int d^dx\left(|\partial\vec{\phi}|^2+\frac{1}{\sqrt{N}}\,\sigma|\vec{\phi}|^2\right) \qquad \langle\sigma(x)\,\sigma(0)\rangle=\frac{C_d}{2\,(x^2)^2} \qquad C_d=\frac{2^{2+d}\,\Gamma(\frac{d-1}{2})\,\sin(\frac{\pi\,d}{2})}{\pi^{\frac{3}{2}}\,\Gamma(\frac{d}{2}-2)}$$

- Computing the same correlator amounts to evaluating 

- …which recovers the same result as in the cubic theory!

- One can generalize the result: consider “meson” operators $\mathcal{M}^n=(\phi^1\,\bar{\phi}_2)^n$ in the $[2\,n,\,0,\cdots,\,0]_{D_N}$
These can be embedded into a gauge theory with an IR fixed point

Conclusions

- Identifying new expansions allowing to explore other corners of QFT is very interesting
- The large charge sector is one such example: allows to prove QFT's in new regimes
- Typically the relevant combination is "gn":
  - If g, n are independent, 1/n expansion (regardless on g: a window into strong coupling!)
  - Interesting regimes arise if g scales with 1/n. Two examples are
    - SQCD theories: the dimensions are protected by SUSY (from the point of view of what we discussed: moduli space makes V=0). But the coefficient depends non-trivially in gn
    - WF theories: interesting regimes arise as gn is varied. Since g=4-d, this interpolates in dimensions

Sharon & Watanabe, 2008.01106

- Potential interesting implications for the physics of WF (including d=5?)

Many thanks!!!