Compressed Sensing Quantum State Tomography: An Alternate Approach

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Abstract

The matrix generalizations of Compressed Sensing (CS) were adapted to Quantum State Tomography (QST) previously by Gross et al. [Phys. Rev. Lett. 105, 150401 (2010)], where they consider the tomography of $n$ spin-1/2 systems. For the density matrix of dimension $d=2^n$ and rank $r$ with $r < c2^n$, it was shown that randomly chosen Pauli measurements of the order $O(d r \log(d)^2)$ are enough to fully reconstruct the density matrix by running a specific convex optimization algorithm. However, these results utilized the low operator-norm of the Pauli operator basis, which are available only in power-of-two dimensional Hilbert spaces. In the present work [Phys. Rev. A 101, 062328 (2020)], we propose an alternate CS-QST protocol for states in Hilbert spaces of non-power-of-two dimensions, which still achieves the bounds on number of measurement settings $O(dr \log(d)^2)$ presented in [Phys. Rev. Lett. 105, 150401 (2010)]. In this alternate protocol, we use a unitary operator $W$ to “move” the quantum information from a $d$ dimensional system to a $d_1$ dimensional ancilla, where $d_1$ is a power of two. We prove that choosing the optimal value for $d_1$ and performing the standard CS-QST protocol using simple Pauli measurements on the ancilla will guarantee full recovery from $O(dr \log(d)^2)$ measurements. We show that the unitary operator $W$ can be efficiently implemented using only $\text{poly}(\log(d))^2$ single qubit gates at most, which is relatively a small overhead compared to the cost of CS-QST protocol. For states in Hilbert spaces of non-power-of-two dimensions, one may consider performing the standard CS-QST protocol using the $SU(d)$ operators. We point out that the $SU(d)$ operators, owing to their high operator norm, do not provide a significant savings in the number of measurement settings required for successful recovery of all rank-$r$ states. We use numerical simulations to show that the proposed alternate approach outperforms the one using $SU(d)$ operators.

Introduction

The matrix generalization of CS techniques, known as matrix completion [1], are adapted to quantum state tomography (QST) by Gross et al. [2] where they consider tomography of $n$ spin-1/2 systems, whose density matrix $\rho$ is of dimension $d=2^n$ and rank-$r$.

**Theorem**

[2] Let $\rho (d \times d)$ be an arbitrary state of rank $r$. Let $\Omega \subset \{w_a\}^{d^2}_{a=1}$ be a randomly chosen set. Each operator $w_a$ is a $k$-fold tensor product of the Pauli basis operators $\{\sigma_i\}_{i=1}^4$ for matrices on $(\mathbb{C}^2)^k$, where $d^2=2^k$. If the number of Pauli expectation values $m = |\Omega| = c d r \log(d)^2$ then the solution $\sigma^*$ to the following optimization program,

$$\min_{\sigma} ||\sigma||_1 \quad \text{subject to} \quad \text{Tr}(w_a \sigma) = \text{Tr}(w_a \rho) \quad \forall w_a \in \Omega, \quad (1)$$

is unique and equal to $\rho$ with failure probability exponentially small in $c$.

The main results of Ref. [2] were generalized to any given matrix basis in Ref. [3].

**Theorem**

[3] Let $\rho (d \times d)$ be a rank-$r$ matrix with coherence $\nu$ with respect to the operator basis $\{w_a\}^{d^2}_{a=1}$. Let $\Omega \subset \{w_a\}^{d^2}_{a=1}$ be a randomly chosen set. The solution $\sigma^*$ to the following optimization program,

$$\min_{\sigma} ||\sigma||_1 \quad \text{subject to} \quad \text{Tr}(w_a \sigma) = \text{Tr}(w_a \rho) \quad \forall w_a \in \Omega, \quad (2)$$

is unique and equal to $\rho$ with probability of failure smaller than $e^{-\beta}$ provided that $|\Omega| \geq O(d r \log(\beta + 1) \log(d)^2)$.

The number $\nu$ is the 'coherence' of the density matrix with respect to the given matrix basis.

**Definition:** The coherence $\nu$ of a $d \times d$ matrix $\rho$ with respect to an operator basis $\{w_a\}^{d^2}_{a=1}$ is given by $\min(\nu_i, \nu_j)$ if

$$\max \frac{||w_a||}{a} \nu_i \leq \nu \quad (3)$$

and

$$\max \frac{||P_rw_a + w_aP_r - P_rw_aP_r||_2}{d} \leq 2\nu \quad (4)$$

hold. $P_r$ is the projection operator onto the column (or row) space of $\rho$.

The operator norm of any normalized Pauli operator is $\sqrt{m}$, and hence, $\nu_1 = 1$, which makes it incoherent to all low rank matrices. Theorem [2] utilizes the above property of the Pauli operator basis, which are available only in power-of-two dimensional Hilbert spaces.

Problem Statement

- How do we achieve the bounds on number of measurement settings $O(dr \log(d)^2)$ even for qudits in Hilbert spaces of non-power-of-two dimensions?

**SU (d) Operator basis**

Since the Pauli operators can only be defined in $\mathbb{C}^2 \times \mathbb{C}^2$ as a $k$-fold tensor product of $SU(2)$ operators, a natural candidate would be to use the $SU(d)$ operator basis [4]. The operator norm of $SU(d)$ basis elements is greater than or equal to $1/2$, and hence, $\nu_1 > d/2$. In this case, we [5] find that one can obtain non-trivial bounds on the number of $SU(d)$ measurement settings from Theorem 1 only if $d_0$ is small, which is true for a very limited set of states.

An Alternate Approach

- Move the quantum information from a $d$ dimensional system to a $d_1$ dimensional ancilla, where $d_1$ is a power of two, using the following unitary operator,

$$W = \sum_{j} |j_0\rangle \langle j_0| \otimes |j_{d_1}\rangle \langle j_{d_1}| + \sum_{j} |j_0\rangle \langle j_0| \otimes |j_{d_1}\rangle \langle j_{d_1}|,$$

**Example**

$$\begin{pmatrix}
\sigma_{11} & \sigma_{12} & 0 & 0 \\
\sigma_{12} & \sigma_{22} & 0 & 0 \\
0 & 0 & \sigma_{33} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \rightarrow_W \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

- The new state of Ancilla $\rho_A = \sum |i\rangle \langle i| |j_{d_1}\rangle$ has the same rank as $\rho_A$
- We use the following program to reconstruct $\rho_A$:

$$\min_{\sigma} ||\sigma||_1 \quad \text{subject to} \quad \text{Tr}(w_a \sigma) = \text{Tr}(w_a \rho_A) \quad \forall w_a \in \Omega,$$

where $\Omega$ is the set of randomly chosen Pauli operators.

- From Theorem 1, $|\Omega| = c d r \log(d)^2$ Pauli measurements are enough for the output of the program (6) to be unique and equal to $\rho_A$ with failure probability exponentially low in $c$.
- Set $d_2$ as the smallest power of two greater than or equal to $d_1$ ($d_1 < d_2 < 2d_1$).

$$c d r \log(d)^2 < c d r \log(d_2)^2 \quad (7)$$

- The Unitary $W (d_1 \times d_2 \times d_2)$ is 1-sparse matrix and hence can be implemented with accuracy $\epsilon$ using at most $\text{poly} \log(d_1), 1/\epsilon$ gates [6].

Numerical Simulations

**Figure 1** The fidelity $F(\rho, \sigma^*)$ between the estimated $|\sigma^*\rangle$ and the true states $|\rho\rangle$ against the number of measurement settings (m) for $SU(31)$ basis measurements (orange) and Pauli measurements on the ancilla (blue) is shown. Fidelity is calculated over 1000 randomly generated $31 \times 31$ rank-1 density matrices.

References