Compressed Sensing Quantum State Tomography: An Alternate Approach

Revanth Badveli, Vinayak Jagadish, R. Srikanth, and Francesco Petruccione

Department of Computer Science, BITS-Pilani, India

Abstract

The matrix generalizations of Compressed Sensing (CS) were adapted to Quantum State Tomography (QST) previously by Gross et al. [Phys. Rev. Lett. 105, 150401 (2010)], where they consider the tomography of n spin-1/2 systems. For the density matrix of dimension $d = 2^n$ and rank r with $r << 2^n$, it was shown that randomly chosen Pauli measurements of the order $O[dr\log(d)^2]$ are enough to fully reconstruct the density matrix by running a specific convex optimization algorithm. However, these results utilized the low operator-norm of the Pauli operator basis, which are available only in power-of-two dimensional Hilbert spaces. In the present work [Phys. Rev. A 101, 062328 (2020)], we propose an alternate CS-QST protocol for states in Hilbert spaces of non-power-of-two dimensions, which still achieves the bounds on number of measurement settings $O[dr\log(d)^2]$presented in [Phys. Rev. Lett. 105, 150401 (2010)]. In this alternate protocol, we use a unitary operator W to "move" the quantum information from a d dimensional system to a d_1 dimensional ancilla, where d_1 is a power of two. We prove that choosing the optimal value for d_1 and performing the standard CS-QST protocol using simple Pauli measurements on the ancilla will guarentee full recovery from $O[dr\log(d)^2]$ measurements. We show that the unitary operator W can be efficiently implemented using only $poly[log(d)^2]$ single qubit gates at most, which is relatively a small overheard compared to the cost of CS-QST protocol. For states in Hilbert spaces of non-power-of-two dimensions, one may consider performing the standard CS-QST protocol using the $SU(d)$ operators. We point out that the $SU(d)$ operators, owing to their high operator norm, do not provide a significant savings in the number of measurement settings required for successful recovery of all rank-r states. We use numerical simulations to show that the proposed alternate approach outperforms the one using $SU(d)$ operators.

Introduction

The matrix generalization of CS techniques, known as matrix completion [1], are adapted to quantum state tomography (QST) by Gross *et al.* [2] where they consider tomography of n spin-1/2 systems, whose density matrix ρ is of dimension $d = 2^n$ and rank-r.

Theorem

[2] Let ρ $(d \times d)$ be an arbitrary state of rank r. Let $\Omega \subset \{w_a\}_{a=1}^{d^2}$ be a randomly chosen set. Each operator w_a is a k-fold tensor product of the Pauli basis operators $\{\sigma_i\}_{i=0}^{3}$ for matrices on $(\mathbb{C}^2)^{\otimes k}$, where $d^2 = 2^k$. If the number of Pauli expectation values $m = |\Omega| = cdr\log(d)^2$ then the solution σ^* to the following optimization program,

$$\min \ \|\sigma\|_1 \quad \text{subject to} \ \mathrm{Tr}(w_a\sigma) = \mathrm{Tr}(w_a\rho)\ \forall w_a \in \Omega, \tag{1}$$

is unique and equal to ρ with failure probability exponentially small in c.

The main results of Ref. [2] were generalized to any given matrix basis in Ref. [3].

Theorem

[3] Let ρ $(d \times d)$ be a rank-r matrix with coherence ν with respect to the operator basis $\{w_a\}_{a=1}^{d^2}$. Let $\Omega \subset \{w_a\}_{a=1}^{d^2}$ be a randomly chosen set. The solution σ^* to the following optimization program,

$$\min \ \|\sigma\|_1 \quad \text{subject to} \ \mathrm{Tr}(w_a\sigma) = \mathrm{Tr}(w_a\rho)\ \ \forall w_a \in \Omega, \tag{2}$$

is unique and equal to ρ with probability of failure smaller than $e^{-\beta}$ provided that

$$|\Omega| \geq O[dr\nu(\beta+1)\log(d)^2].$$

The number ν is the 'coherence' of the density matrix with respect to the given matrix basis.

Definition: The coherence ν of a $d \times d$ matrix ρ with respect to an operator basis $\{w_a\}_{a=1}^{d^2}$ is given by $\min(\nu_1, \nu_2)$ if

$$\max_a \|w_a\|^2 \leq \nu_1 \left(\frac{1}{d}\right) \tag{3}$$

and

$$\max_a \|P_U w_a + w_a P_U - P_U w_a P_U\|_2^2 \leq 2\nu_2 \left(\frac{r}{d}\right) \tag{4}$$

hold. P_U is the projection operator onto the column (or row) space of ρ.

The operator norm of any normalized Pauli operator is $\sqrt{1/d}$, and hence, $\nu_1 = 1$, which makes it incoherent to all low rank matrices. Theorem [2] utilizes the above property of the Pauli operator basis, which are available only in power-of-two dimensional Hilbert spaces.

Problem Statement

- How do we acheive the bounds on number of measurement settings $O(dr\log(d)^2)$ even for qudits in Hilbert spaces of non-power-of-two dimensions?

SU (d) Operator basis

Since the Pauli operators can only be defined in $\mathbb{C}^{2^k \times 2^k}$ as a k-fold tensor product of $SU(2)$ operators, a natural candidate would be to use the SU(d) operator basis [4]. The operator norm of SU(d) basis elements is greater than or equal to 1/2, and hence, $\nu_1 > d/2$. In this case, we [5] find that one can obtain non-trivial bounds on the number of SU(d) measurement settings from Theorem 1 only if ν_2 is small, which is true for a very limited set of states.

An Alternate Approach

- Move the quantum information from a d dimensional system to a d_1 dimensional ancilla, where d_1 is a power of two, using the following unitary operator,

$$W = \sum_{i,j}^{d_1} |i_S\rangle\langle j_S| \otimes |j_A\rangle\langle i_A| + \sum_{i}^{d_2-d_1} 1 \otimes |i_A\rangle\langle i_A|, \tag{5}$$

Example:

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \otimes \begin{pmatrix} 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} \rightarrow_W \begin{pmatrix} 1&0&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix} \otimes \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & 0 \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- The new state of Ancilla $\rho'_A = \Sigma_{i,j}^{d_1} \rho_{ij} |i_A\rangle\langle j_A|$ has the same rank as ρ_S
- We use the following program to reconstruct ρ'_A,

$$\min \ \|\sigma\|_1 \quad \text{subject to} \ \ \mathrm{Tr}(w_a\sigma) = \mathrm{Tr}(w_a\rho'_A)\ \forall w_a \in \Omega, \tag{6}$$

  where Ω is the set of randomly chosen Pauli operators.
- From Theorem 1, $|\Omega| = cd_2 r\log(d_2)^2$ Pauli measurements are enough for the output of the program (6) to be unique and equal to ρ'_A with failure probability exponentially low in c.
- Set d_2 as the smallest power of two greater than or equal to d_1 $(d_1 < d_2 < 2d_1)$.

$$cd_2 r\log(d_2)^2 < c' d_1 r\log(d_1)^2$$
$$\Omega = O(d_1 r\log(d_1)^2)$$

- The Unitary W $(d_1 d_2 \times d_1 d_2)$ is 1-sparse matrix and hence can be implemented with accuracy ϵ using at most poly $[\log(d_1), 1/\epsilon]$ gates [6].

Numerical Simulations

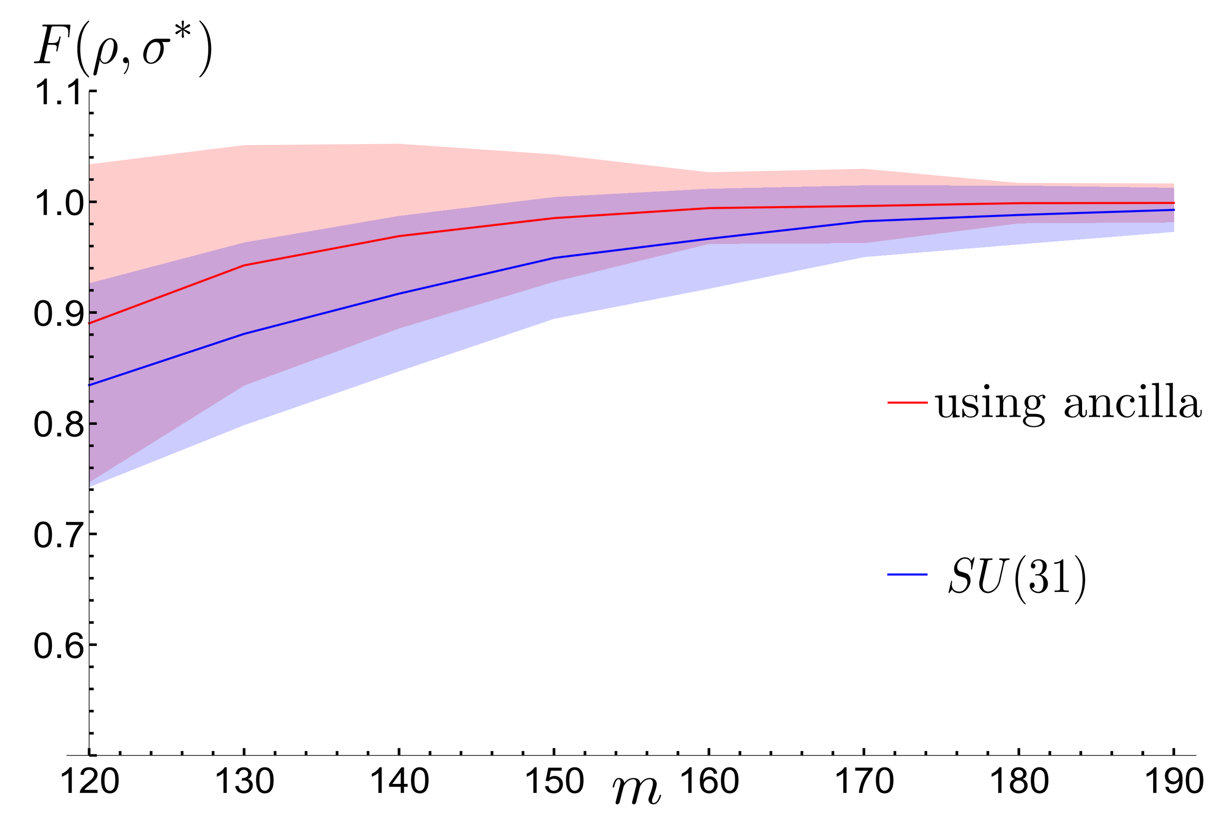

Figure 1: The fidelity $F(\rho, \sigma^*)$ between the estimated (σ^*) and the true states (ρ) against the number of measurement settings (m) for SU(31) basis measurements (orange) and Pauli measurements on the ancilla (blue) is shown. Fidelity is calculated over 1000 randomly generated 31×31 rank-1 density matrices.

References

[1] Emmanuel J. Candès and Benjamin Recht. Exact Matrix Completion via Convex Optimization. *Found. Comput. Math.*, 9(6):717–772, December 2009.

[2] David Gross, Yi-Kai Liu, Steven T. Flammia, Stephen Becker, and Jens Eisert. Quantum state tomography via compressed sensing. *Phys. Rev. Lett.*, 105(15):150401, October 2010.

[3] David Gross. Recovering Low-Rank Matrices From Few Coefficients in Any Basis. *IEEE Trans. Inform. Theory*, 57(3):1548–1566, March 2011.

[4] Walter Greiner and Berndt Müller. *Quantum Mechanics - Symmetries: With 128 Worked Examples and Problems.* Springer, 1994.

[5] Revanth Badveli, Vinayak Jagadish, R. Srikanth, and Francesco Petruccione. Compressed-sensing tomography for qudits in hilbert spaces of non-power-of-two dimensions. *Phys. Rev. A*, 101:062328, Jun 2020.

[6] Stephen P. Jordan and Pawel Wocjan. Efficient quantum circuits for arbitrary sparse unitaries. *Phys. Rev. A*, 80(6):062301, December 2009.