GENERALIZED COHERENCE VECTOR: DEFINITION AND PROPERTIES

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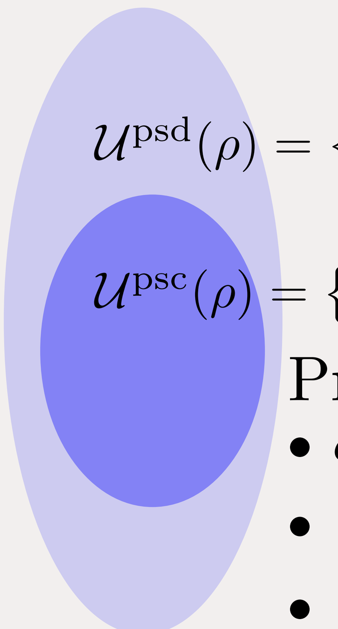
Abstract

A notion of coherence vector of a general quantum state is introduced on the framework of quantum coherence resource theory. This generalized coherence vector completely characterizes the notions of being incoherent, as well as being maximally coherent. Moreover, using this notion and the majorization relation, a necessary condition for the conversion of general quantum states by means of incoherent operations is obtained. Finally, a new family of coherence monotones based on the coherence vector is introduced.

Coherence Vector

see Ref. [7]



$\mathcal{U}^{\rm psd}(\rho) = \{\sum_k q_k \mu^\downarrow(|\psi_k\rangle\langle\psi_k|) : \{q_k, |\psi_k\rangle\} \in \mathcal{D}(\rho)\}$

$\mathcal{U}^{\rm psc}(\rho) = \{\mu^\downarrow(|\psi\rangle\langle\psi|) : |\psi\rangle\langle\psi| \in \mathcal{O}(\rho)\}$

Properties:

- *convex sets s.t.* $\mathcal{U}^{\rm psc}(\rho) \subseteq \mathcal{U}^{\rm psd}(\rho) \subseteq \Delta_d^\downarrow$
- $\bigvee \mathcal{U}^{\rm psc}(\rho) = \bigvee \mathcal{U}^{\rm psd}(\rho)$
- $\bigvee \mathcal{U}^{\rm psc}(\rho) \in \mathcal{U}^{\rm psc}(\rho) \iff \bigvee \mathcal{U}^{\rm psd}(\rho) \in \mathcal{U}^{\rm psd}(\rho)$

Definition.

Coherence vector of ρ

$$\nu(\rho) = \bigvee \mathcal{U}^{\rm psd}(\rho) = \bigvee \mathcal{U}^{\rm psc}(\rho)$$

Properties.

- For pure states: $\nu(|\psi\rangle\langle\psi|) = \mu^\downarrow(|\psi\rangle\langle\psi|)$
- $\nu(\rho) = (1, 0, \dots, 0) \iff \rho$ *is incoherent*
- $\nu(\rho) = \left(\frac{1}{d}, \dots, \frac{1}{d}\right) \iff \rho$ *is maximally coherent*
- $\sum_k p_k \nu(|\psi_k\rangle\langle\psi_k|) \preceq \nu\left(\sum_k p_k |\psi_k\rangle\langle\psi_k|\right)$

References

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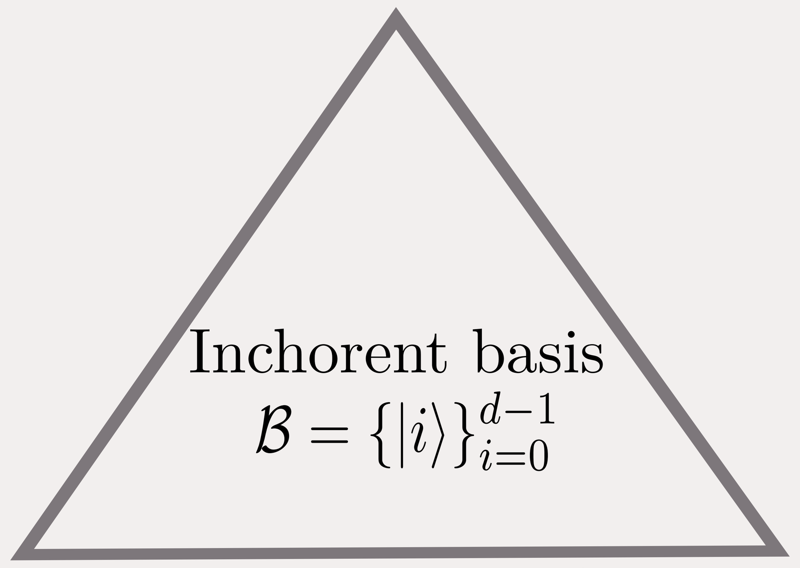
Quantum Resource Theory of Coherence

Incoherent states

$$\mathcal{I} = \{\rho = \sum_i p_i |i\rangle\langle i|\}$$



Coherent states

$\rho \notin \mathcal{I}$

Incoherent operations[1]

$\Lambda(\rho) = \sum_n K_n \rho K_n^\dagger$ s.t.:

$\frac{K_n \rho K_n^\dagger}{{\rm Tr}(K_n \rho K_n^\dagger)} \in \mathcal{I}, \forall \rho \in \mathcal{I} \, \forall n$

$\{K_n\}$ incoherent Kraus operators

Coherent vector for pure states[2,3]:

$\mu(|\psi\rangle\langle\psi|) = \left(|\langle 0|\psi\rangle|^2, \dots, |\langle d-1|\psi\rangle|^2\right)$

$\mu(|\psi\rangle\langle\psi|) \in \Delta_d = \{u \in \mathbb{R}^d : u_i \geq 0 \sum_i u_i = 1\}$

(set of probability vectors)

State transformations[4]:

$|\psi\rangle\langle\psi| \xrightarrow[\rm IO]{} \sigma \iff \exists\{p_n, |\phi_n\rangle\}$ s.t.

1. $\sigma = \sum_n p_n |\phi_n\rangle\langle\phi_n|$
2. $\mu(|\psi\rangle\langle\psi|) \preceq \sum_n p_n \mu^\downarrow(|\phi_n\rangle\langle\phi_n|)$

$\downarrow$ordered components

Majorization lattice[5]:

$u \preceq v \iff \sum_{i=0}^k u_i^\downarrow \leq \sum_{i=0}^k v_i^\downarrow, \forall k \in \{0, \dots, d-1\}$

Preorder (transitive and reflexive) on Δ_d

$\Delta_d^\downarrow$ is a complete lattice (POSET + inf and sup)

Supremum: $\forall \mathcal{U} \subseteq \Delta_d^\downarrow, \exists \bigvee \mathcal{U}$

Coherence measures/monotones[1]:

A *coherence measure* C is a real function such that:

1. Vanishing on incoherent states: $C(\rho) = 0 \; \forall \rho \in \mathcal{I}$
2. Monotonicity under IO: $C(\rho) \geq C(\Lambda(\rho)) \; \forall\, {\rm IO}\,\Lambda$ and $\forall \rho$
3. Strong monotonicity: $C(\rho) \geq \sum_n p_n C(\sigma_n) \; \forall {\rm IO}\Lambda, \forall \rho$, $p_n = {\rm Tr}(K_n \rho K_n^\dagger)$, $\sigma_n = \frac{K_n \rho K_n^\dagger}{p_n}$
4. Normalization: $C(\rho) = 1$ for any ρ maximally coherent
5. Convexity: $C(\sum_k q_k \rho_k) \leq \sum_k q_k C(\rho_k)$

If C safisfies 1-3, it is called *coherence monotone*

Proposition.[2,3]

Any coherence measure for pure states is of the form

$$C(\mu(|\psi\rangle\langle\psi|) = f(\mu(|\psi\rangle\langle\psi|)$$

$f \in \mathcal{F} = \left\{\text{symmetric, concave}, f(1, 0, \dots, 0) = 0, f\left(\frac{1}{d}, \dots, \frac{1}{d}\right) = 1\right\}$

Convex roof measure of coherence[2,3]:

$C_f^{\rm cr}(\rho) = \inf_{\{q_k, |\psi_k\rangle\} \in \mathcal{D}(\rho)} \sum_k q_k f(\mu(|\psi\rangle\langle\psi|))$

$\mathcal{D}(\rho) = \{\{q_k, |\psi_k\rangle\} : \; \rho = \sum_k q_k |\psi_k\rangle\langle\psi_k|\}$ set of all pure states decompositions of ρ

Top monotone of coherence[6]:

$C_f^{\rm top}(\rho) = \inf_{|\psi\rangle\langle\psi| \in \mathcal{O}(\rho)} f(\mu(|\psi\rangle\langle\psi|)$

$\mathcal{O}(\rho) = \left\{|\psi\rangle\langle\psi| : |\psi\rangle\langle\psi| \xrightarrow[\rm IO]{} \rho\right\}$ set of all pure states that can be converted into ρ

Conditions for Incoherent Transformations & Coherence Monotones

see Ref. [7]

Proposition.

$\rho \xrightarrow[\rm IO]{} \sigma \implies \forall \{q_k, |\psi_k\rangle\} \in \mathcal{D}(\rho), \; \exists \{r_l, |\phi_l\rangle\} \in \mathcal{D}(\sigma) :$
$\sum_k q_k \mu^\downarrow(|\psi_k\rangle\langle\psi_k|) \preceq \sum_l r_l \mu^\downarrow(|\phi_l\rangle\langle\phi_l|)$

Corollaries.

- $\rho \xrightarrow[\rm IO]{} \sigma \implies \nu(\rho) \preceq \sum_n p_n \nu(\sigma_n)$
  $p_n = {\rm Tr}(K_n \rho K_n^\dagger)$, $\sigma_n = \frac{K_n \rho K_n^\dagger}{p_n}$
  $\{K_n\}$ incoherent Kraus operators
- $\rho \xrightarrow[\rm IO]{} \sigma \implies \nu(\rho) \preceq \nu(\sigma)$

Definition.

Coherence vector monotone

$C_f^{\rm cv}(\rho) = f(\nu(\rho))$ with $f \in \mathcal{F}$

Proposition.

$C_f^{\rm cv}$ satisfies conditions 1-4 $\forall f \in \mathcal{F}$

Proposition.

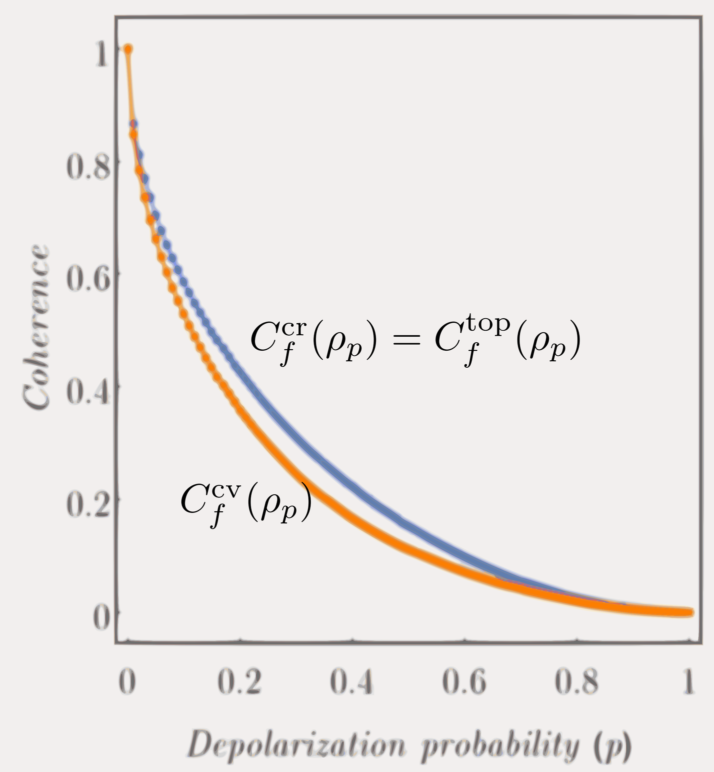
The following statements are equivalent

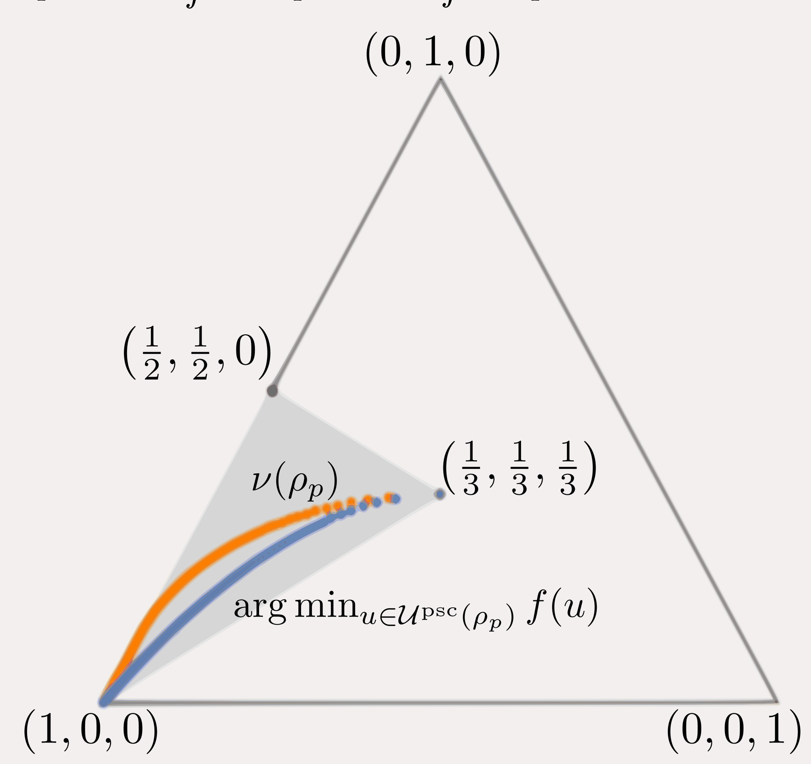
- $\exists$ *optimal pure state decomposition of* ρ
- $C_f^{\rm cv}(\rho) = C_f^{\rm top}(\rho)$ for all $f \in \mathcal{F}$
- $C_f^{\rm cv}(\rho) = C_f^{\rm top}(\rho)$ for $f \in \mathcal{F}$, f strictly Schur-concave

Example: maximally coherent qutrit under depolarizing channel

$\rho_p = \Lambda_p(|\psi\rangle\langle\psi|) = p\frac{I}{3} + (1-p)|\psi\rangle\langle\psi|$, $|\psi\rangle = \frac{|0\rangle + |1\rangle + |2\rangle}{\sqrt{3}}$

For $f(u) = 1 - u_1 + u_3 : C_f^{\rm cr}(\rho_p) = C_f^{\rm top}(\rho_p) > C_f^{\rm cv}(\rho_p)$, $p \in (0,1)$





Proposition.

$\exists$ *optimal pure state decomposition of* ρ
$\implies C_f^{\rm cv}(\rho) \geq C_f^{\rm cr}(\rho)$

Relationships among the families of coherence quantifiers