Contextuality-by-default for behaviours in compatibility scenarios

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Compatibility-hypergraph approach to contextuality

Compatibility scenarios

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- $\mathcal{X}$ is a finite set (set of measurements)
- $\mathcal{C}$ is a collection of contexts
- O is a finite set (set of outcomes)

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- A **behaviour** p for a scenario $(\mathcal{X}, \mathcal{C}, O)$ is a function which associates, to each context C, a probability distribution p^C over O^C

From behaviours to random variables

"We label all measurements contextually: this means that a property q is represented by different random variables R_q^C depending on the context C." [1]

[1] [1] J. V. Kujala, E. N. Dzhafarov, and J.-A. Larsson, "Necessary and sufficient conditions for an extended noncontextuality in abroad class of quantum mechanical systems,"Phys. Rev. Lett.,vol. 115, p. 150401, Oct 2015

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System

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From behaviours to random variables

- “Scenario $+$ behaviour $\Rightarrow$ system”?

From behaviours to random variables

- "Scenario + behaviour ⇒ system"?
- We do that using marginal distributions

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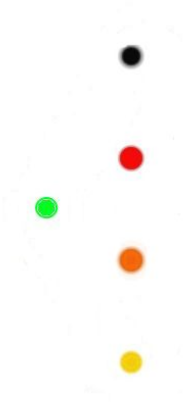

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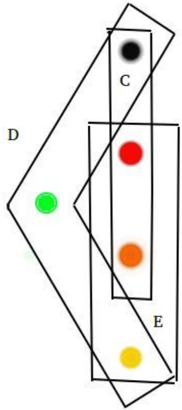

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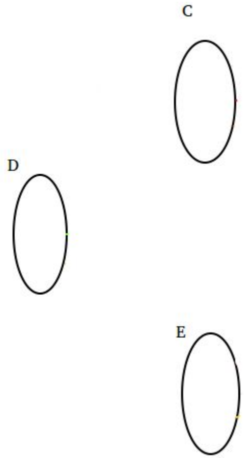

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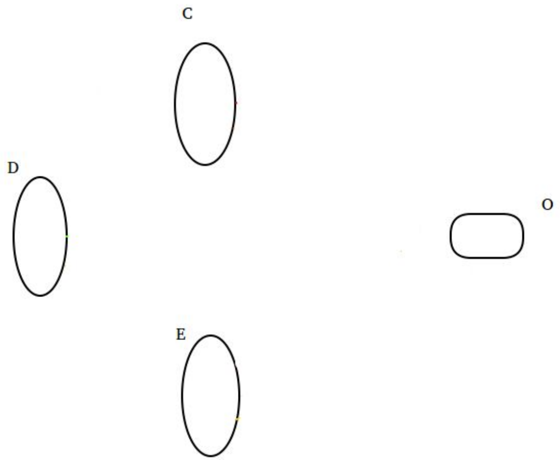

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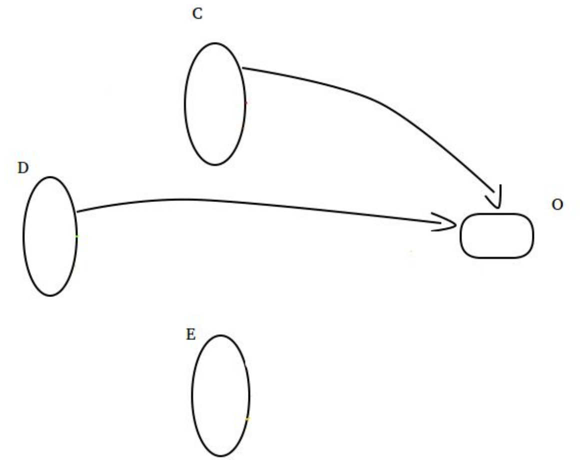

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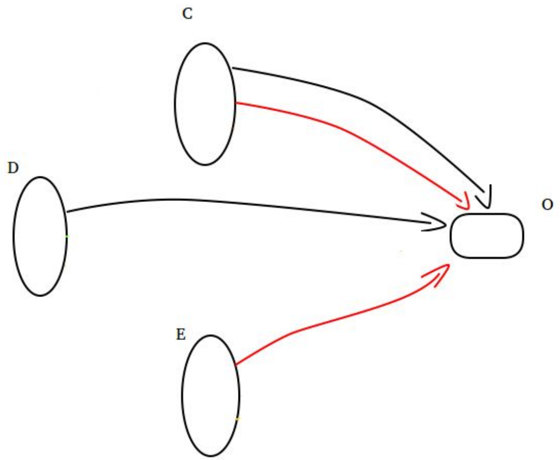

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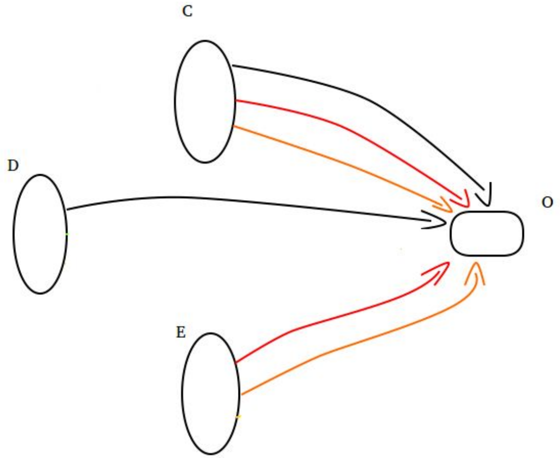

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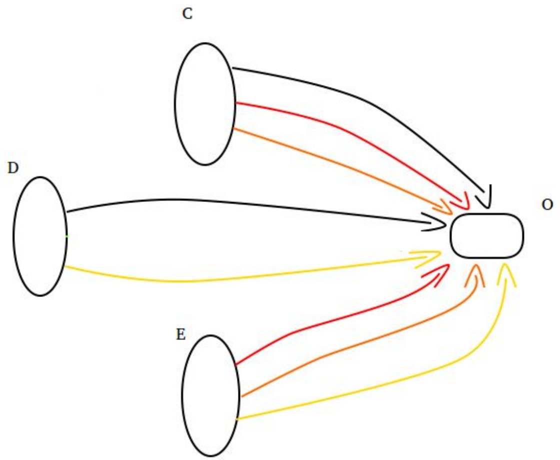

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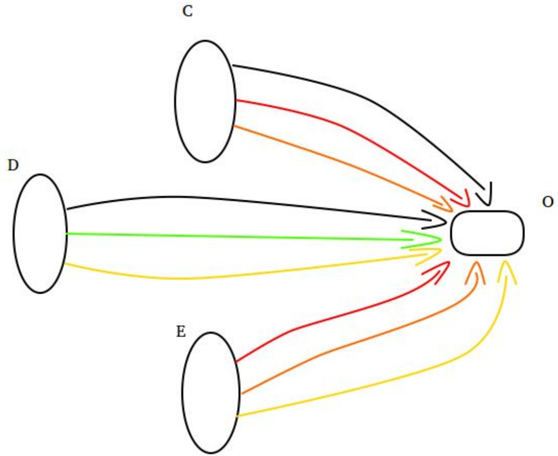

From behaviours to random variables

System

Behaviour

From behaviours to random variables

System | **Behaviour**

consistent connected

From behaviours to random variables

System

consistent connected

Behaviour

non-degenerate

From behaviours to random variables

System

consistent connected

maximally non-contextual description

Behaviour

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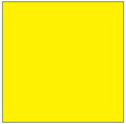
From behaviours to random variables

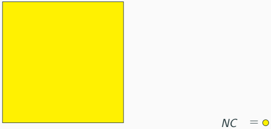
System	Behaviour
consistent connected	non-degenerate
maximally non-contextual description	non-contextual in the extended sense

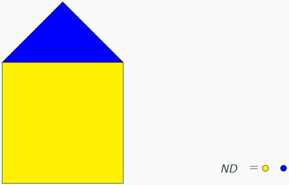
Results and conclusions



$NC =$

Results and conclusions





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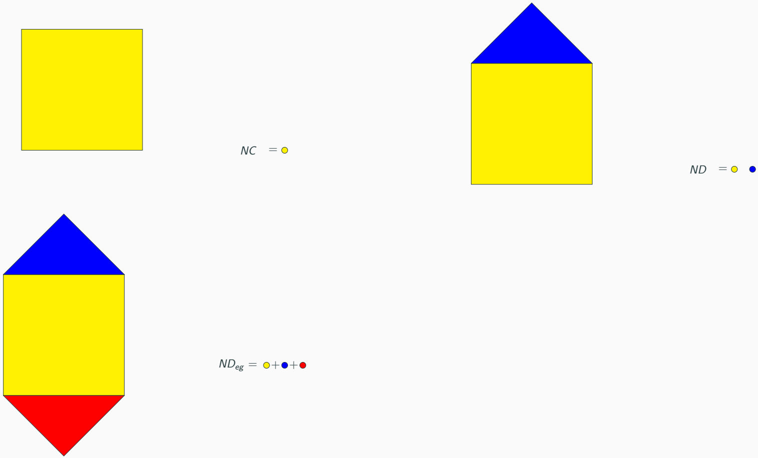

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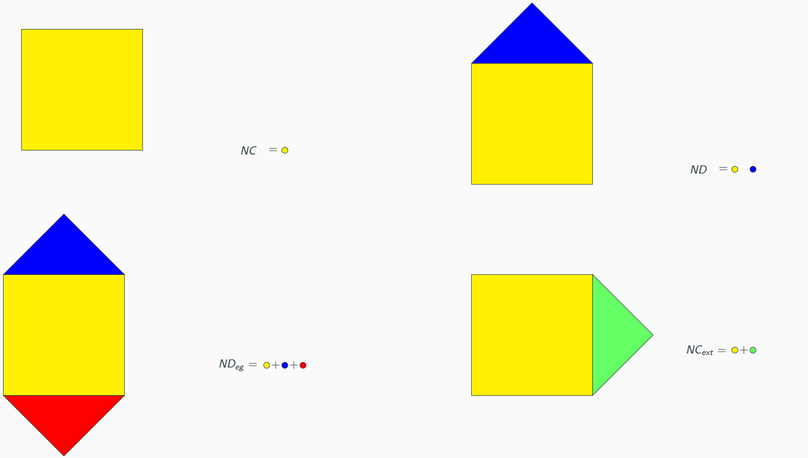
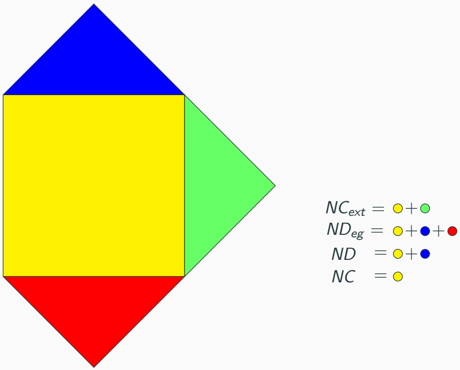


Results and conclusions



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- Non-degeneracy (consistent connectedness) defines a polytope
- We can relax the non-disturbance condition as a physical requirement

Thank you

Appendix

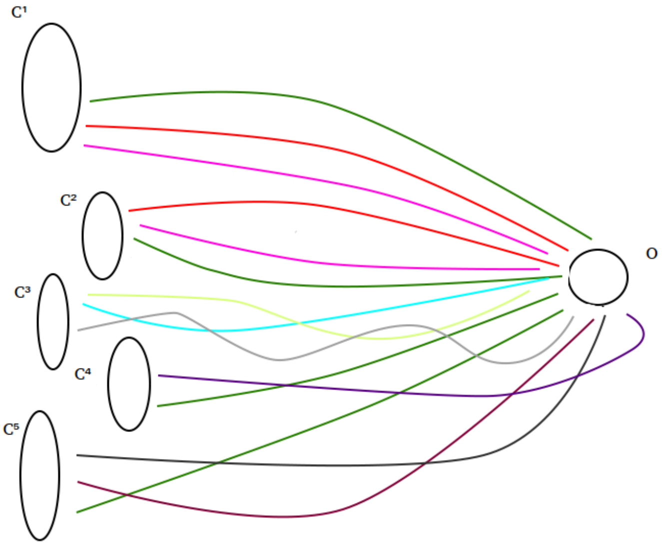
A collection of classical systems

For a context C we can associate a probability space $(\Omega^C, \Sigma^C, \mu^C)$ where

$$x_C : \Omega^C \to O \quad (1)$$

$$p^C(s) = \mu^C\left(\bigcap_{x \in C} x_C^{-1}(s_x)\right) \quad (2)$$

$$p_x^C(o) = \mu^C(x_C^{-1}(o)) \quad (3)$$


C¹
C²
C³
C⁴
C⁵
O

Classical behaviours

p in $(\mathcal{X}, \mathcal{C}, O)$ is classical iff exists:

(a) a measurable space (Ω, Σ)

(b) a function $\pi : \mathcal{X} \to MF(\Omega, O)$

(c) A probability measure μ in (Ω, Σ)

satisfying

- For any $C \in \mathcal{C}$ and any $s \in O^C$,

$$p^C(s) = \mu(\bigcap_{x \in C} \pi(x)^{-1}(s_x))$$

Classical behaviour

p is classical iff exists a distribution $\overline{p} : O^{\mathcal{X}} \to [0,1]$ satisfying, for each context C

$$\overline{p}_C = p^C$$

Quantum behaviours

p in $(\mathcal{X}, \mathcal{C}, O)$ is a quantum behaviour iff exists

(a) A Hilbert space H

(b) A function $\theta : \mathcal{X} \to \mathcal{B}(H)^{\mathbb{R}}$

(c) A density operator $\rho \in \mathbb{B}(H)$

satisfying

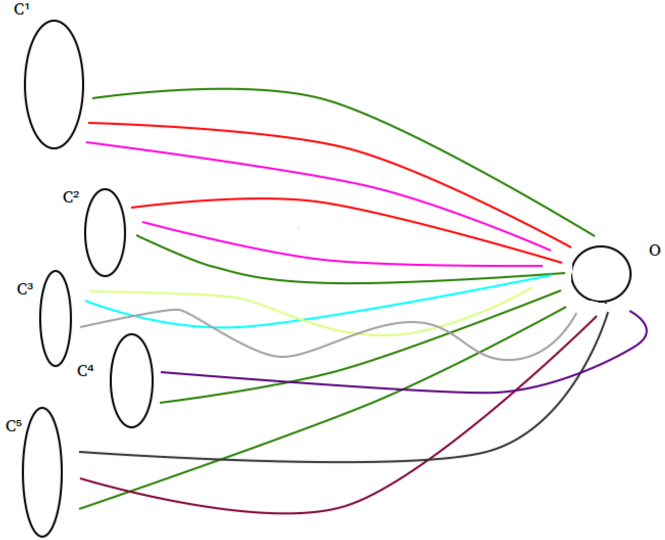
- For any $C \in \mathcal{C}$,

$$[\theta(x), \theta(y)] = 0 \ \forall x, y \in C$$

- For any $C \in \mathcal{C}$ and $s \in O^C$

$$p^C(s) = \mathrm{Tr}(\rho \prod_{x \in C} P_{s_s}^{(x)})$$

Classical behaviours



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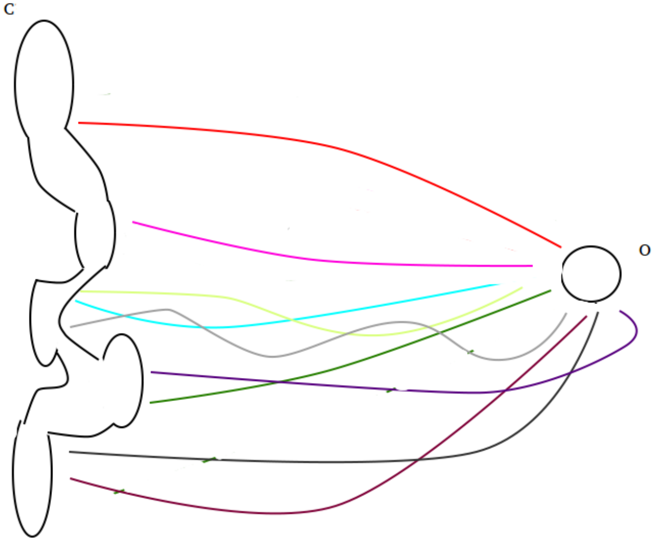
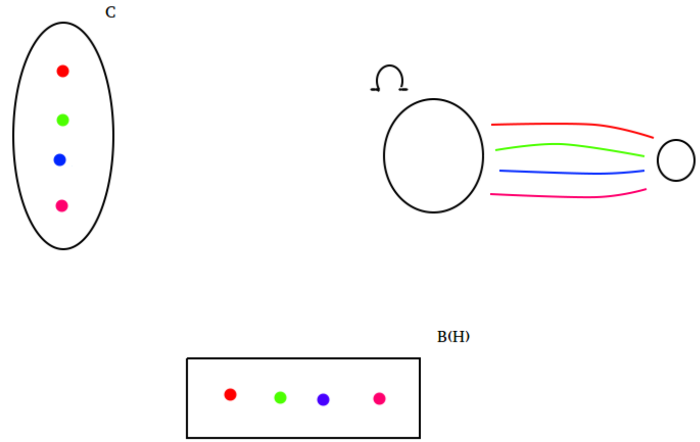


Quantum behaviours



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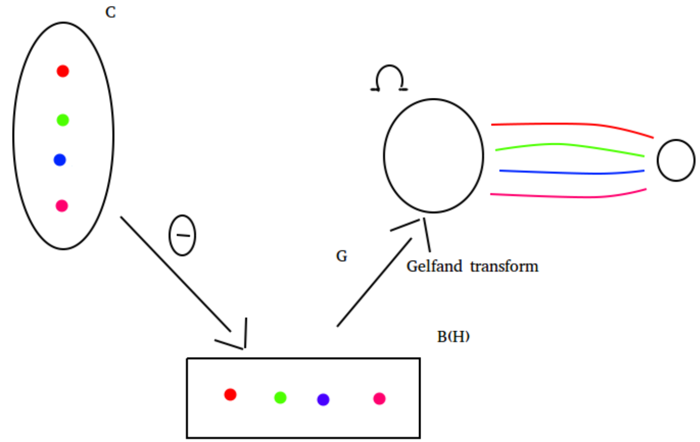

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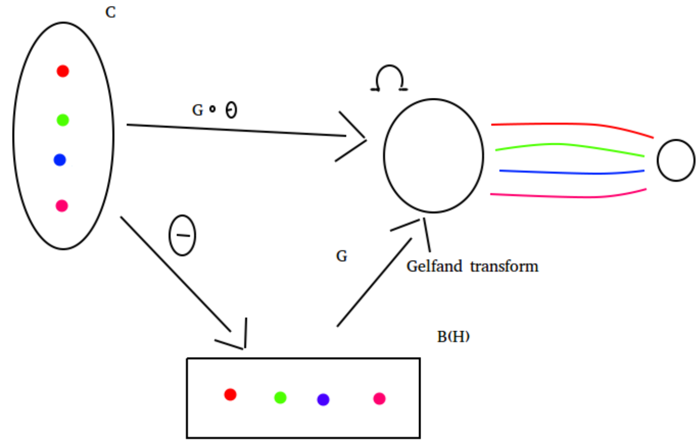

Quantum behaviours

- The joint spectrum of $A_1, ..., A_n \in \mathcal{B}(H)^{\mathbb{R}}$ is

$$\Omega^C \equiv \sigma(\underline{A}) \equiv \{(\lambda_1, ..., \lambda_n) \in \sigma(A_1) \times ... \times \sigma(A_n); \prod_{i=1}^{n} P_{\lambda_i}^{A_i} \neq 0\}.$$

- $\widehat{A}_i : \sigma(\underline{A}) \to \sigma(A_i)$ is the projection

$$\sigma(\underline{A}) \ni (\lambda_1, ..., \lambda_n) \mapsto \lambda_i \in \sigma(A_i).$$

- Consequently

$$(\lambda_1, ..., \lambda_n) = \bigcap_{i=1}^{n} \widehat{A}_i^{-1}(\lambda_i)$$

Quantum behaviours

- A state ρ defines a probability measure $\mu^{\rho}_{\underline{A}}$ in $\sigma(\underline{A})$ by means of the Born rule:

$$\mu^{\rho}_{\underline{A}}(\bigcap_{i=1}^{n} \widehat{A_i}^{-1}(\lambda_i)) = \mu^{\rho}_{\underline{A}}((\lambda_1, ..., \lambda_n)) \doteq \mathrm{Tr}(\rho \prod_{i=1}^{n} P^{A_i}_{\lambda_i})$$

- We also have

$$\mu^{\rho}_{\underline{A}}(\widehat{A_i}^{-1}(\lambda)) = \mathrm{Tr}(\rho P^{A_i}_{\lambda})$$

Quantum behaviours

$(\sigma(\underline{A}), \mathcal{P}(\sigma(\underline{A})), \mu^{\rho}_{\underline{A}})$ "satisfies"

$$x_C : \Omega^C \to O \tag{4}$$

$$p^C(s) = \mu^C(\bigcap_{x \in C} x_C^{-1}(s_x)) \tag{5}$$

$$p_x^C(o) = \mu^C(x_C^{-1}(o)) \tag{6}$$

From behaviours to random variables

- For any $x \in \mathcal{X}$ and $C \in \mathcal{C}_x$ we define $p_x^C : O \to [0,1]$ by

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