Contextuality-by-default for behaviours in compatibility scenarios

Alisson Tezzin, Rafael Wagner, Barbara Amaral
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Compatibility-hypergraph approach to contextuality
A compatibility scenario is a triple \( S \equiv (X, C, O) \) where

- \( X \) is a finite set (set of measurements)
- \( C \) is a collection of contexts
- \( O \) is a finite set (set of outcomes)
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Behaviours

• $\mathcal{O}$ denotes the set of all functions $\mathcal{C} \rightarrow \mathcal{O}$.

• A behaviour $p$ for a scenario $(\mathcal{X}, \mathcal{C}, \mathcal{O})$ is a function which associates, to each context $\mathcal{C}$, a probability distribution $p_C$ over $\mathcal{O}_C$. 
Behaviours

- $O^C$ denotes the set of all functions $C \rightarrow O$
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- A **behaviour** $p$ for a scenario $(X, C, O)$ is a function which associates, to each context $C$, a probability distribution $p^C$ over $O^C$
From behaviours to random variables
“We label all measurements contextually: this means that a property \( q \) is represented by different random variables \( R^C_q \) depending on the context \( C \).”\(^1\)

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\[\text{System}\]

From behaviours to random variables

- “Scenario + behaviour $\Rightarrow$ system”?
From behaviours to random variables

- “Scenario + behaviour ⇒ system”?
- We do that using marginal distributions
From behaviours to random variables
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<td>non-degenerate</td>
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<tr>
<td>description</td>
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From behaviours to random variables

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- connected

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Results and conclusions

$\text{NC} = \cdot$
Results and conclusions

$NC = \bullet$

$ND = \bullet$
Results and conclusions

\[ NC = \bullet \]

\[ ND = \bullet \]

\[ ND_{eg} = \bullet + \bullet + \bullet \]
Results and conclusions

$NC = \bullet$

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Results and conclusions

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Results and conclusions

- The idea behind contextuality-by-default is implicit in the compatibility-hypergraph approach to contextuality
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- Non-degeneracy (consistent connectedness) defines a polytope
Results and conclusions

- The idea behind contextuality-by-default is implicit in the compatibility-hypergraph approach to contextuality.
- Non-degeneracy (consistent connectedness) defines a polytope.
- We can relax the non-disturbance condition as a physical requirement.
Thank you
Appendix
A collection of classical systems

For a context $C$ we can associate a probability space $(\Omega^C, \Sigma^C, \mu^C)$ where

$$x_C : \Omega^C \rightarrow O$$  \hspace{1cm} (1)  

$$p^C(s) = \mu^C(\bigcap_{x \in C} x^{-1}_C(s_x))$$ \hspace{1cm} (2)  

$$p^C_x(o) = \mu^C(x^{-1}_C(o))$$ \hspace{1cm} (3)
Classical behaviours

$p$ in $(\mathcal{X}, \mathcal{C}, O)$ is classical iff exists:

(a) a measurable space $(\Omega, \Sigma)$
(b) a function $\pi : \mathcal{X} \to MF(\Omega, O)$
(c) A probability measure $\mu$ in $(\Omega, \Sigma)$

satisfying

- For any $C \in \mathcal{C}$ and any $s \in O^C$,

$$p^C(s) = \mu(\bigcap_{x \in C} \pi(x)^{-1}(s_x))$$
$p$ is classical iff exists a distribution $\bar{p} : O^X \to [0, 1]$ satisfying, for each context $C$

$$\bar{p}_C = p^C$$
Quantum behaviours

$p$ in $(\mathcal{X}, \mathcal{C}, \mathcal{O})$ is a quantum behaviour iff exists

(a) A Hilbert space $H$
(b) A function $\theta : \mathcal{X} \to \mathcal{B}(H)^\mathbb{R}$
(c) A density operator $\rho \in \mathcal{B}(H)$

satisfying

- For any $C \in \mathcal{C}$,
  
  $$[\theta(x), \theta(y)] = 0 \ \forall x, y \in C$$

- For any $C \in \mathcal{C}$ and $s \in \mathcal{O}^C$
  
  $$p_C^C(s) = \text{Tr}(\rho \prod_{x \in C} P_{ss}^{(x)})$$
Classical behaviours
Classical behaviours
Quantum behaviours
Quantum behaviours

[C] G \cdot \Theta

Gelfand transform

\[ B(H) \]
Quantum behaviours

- The joint spectrum of $A_1, ..., A_n \in \mathcal{B}(H)^\mathbb{R}$ is

$$\Omega^C \equiv \sigma(A) \equiv \{(\lambda_1, ..., \lambda_n) \in \sigma(A_1) \times ... \times \sigma(A_n); \prod_{i=1}^{n} P_{\lambda_i}^{A_i} \neq 0\}.$$

- $\hat{A}_i : \sigma(A) \to \sigma(A_i)$ is the projection

$$\sigma(A) \ni (\lambda_1, ..., \lambda_n) \mapsto \lambda_i \in \sigma(A_i).$$

- Consequently

$$(\lambda_1, ..., \lambda_n) = \bigcap_{i=1}^{n} \hat{A}_i^{-1}(\lambda_i)$$
Quantum behaviours

• A state $\rho$ defines a probability measure $\mu_A^\rho$ in $\sigma(A)$ by means of the Born rule:

$$\mu_A^\rho(\bigcap_{i=1}^n \hat{A}_i^{-1}(\lambda_i)) = \mu_A^\rho((\lambda_1, \ldots, \lambda_n)) = \text{Tr}(\rho \prod_{i=1}^n P_{\lambda_i}^{A_i})$$

• We also have

$$\mu_A^\rho(\hat{A}_i^{-1}(\lambda)) = \text{Tr}(\rho P_{\lambda}^{A_i})$$
Quantum behaviours

\((\sigma(A), \mathcal{P}(\sigma(A)), \mu^A)\) "satisfies"

\[ x_C : \Omega^C \rightarrow O \quad (4) \]

\[ p^C(s) = \mu^C(\bigcap_{x \in C} x_C^{-1}(s_x)) \quad (5) \]

\[ p^C_x(o) = \mu^C(x_C^{-1}(o)) \quad (6) \]
• For any $x \in \mathcal{X}$ and $C \in C_x$ we define $p^C_x : O \rightarrow [0, 1]$ by

$$p^C_x(o) \doteq p^C_{\{x\}}(o) = \sum_{s \in O^C \atop s_x = o} p^C(s)$$
• For any $x \in \mathcal{X}$ and $C \in C_x$ we define $p_x^C : O \rightarrow [0, 1]$ by

$$p_x^C(o) \doteq p_{\{x\}}(o) = \sum_{s \in O^C \atop s_x = o} p^C(s)$$

• $p_x^C, C \in C_x$
From behaviours to random variables

- For any \( x \in \mathcal{X} \) and \( C \in C_x \) we define \( p^C_x : O \rightarrow [0, 1] \) by
  
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- \( p^C_x, C \in C_x \)
- \( x_C, C \in C_x \)
For any \( x \in \mathcal{X} \) and \( C \in \mathcal{C}_x \) we define \( p^C_x : O \rightarrow [0, 1] \) by

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- \( p^C_x, C \in \mathcal{C}_x \)
- \( x_C, C \in \mathcal{C}_x \)
- A behaviour defines a system
From behaviours to random variables

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- $p^C_x$, $C \in \mathcal{C}_x$
- $x_C$, $C \in \mathcal{C}_x$
- A behaviour defines a system