The quantum Wasserstein distance of order 1

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Motivations

- Hamming distance ubiquitous in classical probability, information theory, machine learning
- Yet no quantum version for qudits!!
- Bit flip small change wrt Hamming distance, but can generate orthogonal state
- Orthogonal states maximally far for any unitarily invariant distance
- Want: distance such that
  - Recovers Hamming distance for canonical basis states
  - One-qudit channels induce small changes
  - Global quantities (e.g., entropy) continuous

The quantum W_1 distance

- Neighboring states: coincide after discarding one qudit
- Require: neighboring states have distance at most one
- Definition: maximum distance induced by a norm with the above property

Properties

- Recovers Hamming distance for canonical basis vectors
- Symmetries: local unitaries, qudit permutations
- Contractive wrt one-qudit quantum channels
- Additive wrt tensor product

$$\|\rho \otimes \rho' - \sigma \otimes \sigma'\|_{W_1} = \|\rho - \sigma\|_{W_1} + \|\rho' - \sigma'\|_{W_1}$$

- Relation with trace distance

$$\frac{1}{2}\|\rho - \sigma\|_1 \leq \|\rho - \sigma\|_{W_1} \leq \frac{n}{2}\|\rho - \sigma\|_1$$

- Local operations: if Φ acts on *k* qudits,

$$\|\Phi(\rho) - \rho\|_{W_1} \leq 2\,k$$

Continuity of the von Neumann entropy

- Continuity bounds wrt trace distance / fidelity void for orthogonal states, but one-qudit channel can turn state into orthogonal state with entropy change at most 2 ln *d*
- Continuity bound wrt quantum W_1 distance

$$|S(\rho) - S(\sigma)| \le g\left(\|\rho - \sigma\|_{W_1}\right) + \|\rho - \sigma\|_{W_1} \ln\left(d^2\, n\right)$$

$$g(t) = (t+1)\ln(t+1) - t\ln t \le \ln(t+1) + 1$$

$$\|\rho - \sigma\|_{W_1} = o\left(\frac{n}{\ln n}\right) \quad \Longrightarrow \quad |S(\rho) - S(\sigma)| = o(n)$$

Perspectives

- Quantum state estimation
- Robustness of quantum machine learning
- Quantum Generative Adversarial Networks
- Quantum rate distortion theory
- Quantum differential privacy
- Mixing time of quantum Markov semigroups
- Shallow quantum circuits
- Quantum many-body Hamiltonians

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