Nonclassical steering and the Gaussian steering triangoloids

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Overview on quantum steering
Overview on quantum steering

- Introduced by Schrödinger in 1935 to discuss EPR argument
- Rigorously defined for generic mixed states only in 2007 (Wiseman et al. [3])
- General definition based on one-party entanglement verification
- Finds applications in Quantum Key Distribution (QKD)
Overview on quantum steering

- Intermediate between entanglement and Bell’s inequality violation, in the hierarchy of **quantum correlations**
Nonclassicality
Nonclassicality according to the Glauber P-function

\[ \hat{\rho} = \int_{\mathbb{C}^n} d^{2n} \alpha \ P[\hat{\rho}](\alpha) \ |\alpha\rangle\langle\alpha| \]

- **Most physical** definition of nonclassicality
- **Nonclassical** whenever the P-function is *not* a valid probability distribution
- Necessary for photon antibunching and sub-Poissonian statistics
Nonclassical steering with TMSTs
Two mode squeezed thermal states

\[ \hat{\rho}_{AB} := \hat{S}^{(2)}(r) [\hat{\nu}_{th}(N_A) \otimes \hat{\nu}_{th}(N_B)] \hat{S}^{(2)}(r)^\dagger \]

\[ \hat{\nu}_{th}(N) = \frac{1}{1 + N} \sum_{n=0}^{\infty} \left( \frac{N}{1 + N} \right)^n |n\rangle \langle n| \]

\[ \hat{S}^{(2)}(r) := e^{r(\hat{a}^\dagger \hat{b}^\dagger - \hat{a} \hat{b})}, \quad r \in \mathbb{R} \]

They can be **separable** or **entangled**: interesting enough for quantum correlations

\[ \hat{\rho}_A = \text{Tr}_B[\hat{\rho}_{AB}] \text{ and } \hat{\rho}_B = \text{Tr}_A[\hat{\rho}_{AB}] \] are **always classical**
Nonclassical steering
Nonclassical steering

\[
\hat{\Pi}_\alpha = \frac{1}{\pi} \hat{D}(\alpha) \hat{\rho}_M \hat{D}^\dagger(\alpha), \quad p_\alpha = \text{Tr}_{AB} \left[ \hat{\rho}_{AB} \left( I_A \otimes \hat{\Pi}_\alpha \right) \right]
\]
Nonclassical steering

\[ \hat{\rho}_A \to \hat{\rho}_A^{(\alpha)} = \text{Tr}_B[\hat{\rho}_{AB}^{(\alpha)}] = \frac{1}{\rho_\alpha} \text{Tr}_B \left[ \hat{\rho}_{AB} \left( I_A \otimes \hat{\Pi}_\alpha \right) \right] \]
Gaussian steering triangoloids

**Figure 1**: Gaussian triangoloid of a TMST, in parameter’s space of conditional CM. $\mu_c$ is the conditional purity, $\mu_{sc}$ is related to conditional squeezing.
Nonclassical steering with TMST states

Figure 2: Triangoloids for TMST states with $N_A = N_B = 0.75$, $r = 1.2$ (left) and with $N_A = N_B = 2.8$, $r = 1.2$ (right)
Nonclassical steering with TMST states

- Necessary and sufficient condition (clearly asymmetric):
  \[ \sinh^2 r > \frac{N_A (1 + 2 N_B)}{1 + N_A + N_B} \]

- Entanglement is necessary but not sufficient

Figure 3: Slice of parameter's space of TMST states
Nonclassical steering with generic two-mode Gaussian state
\( \hat{\rho}_{AB} \) is:

- **Weakly nonclassically steerable (WNS)** if it is possible to generate nonclassical conditional state with some Gaussian measurement
  
  Does **not** imply entanglement
  
  Possible with arbitrarily low (but nonzero) Gaussian Quantum Discord

- **Strongly nonclassically steerable (SNS)** if *any* quadrature measurement on Bob’s mode will prepare a nonclassical conditional state of Alice’s mode
  
  Implies **EPR steerability**, thus also entanglement
For **TMSTs**, weak and strong nonclassical steering are equivalent.

They also coincide with **EPR steering**.

This fact has a powerful implication: **nonclassicality** and **entanglement** are strongly related, at least for TMST states.
New quantum correlations

Figure 4: Quantum correlations for two-mode Gaussian states
**Nonclassical steering and the Gaussian steering triangoloids.**  


H. M. Wiseman, S. J. Jones, and A. C. Doherty.  
Questions?
WNS
necessary and sufficient condition:

\[ a - \frac{c^2}{b} < \frac{1}{2}, \quad c = \max\{|c_1|, |c_2|\}. \]

SNS
necessary and sufficient condition:

\[ a - \frac{c'^2}{b} < \frac{1}{2}, \quad c' = \min\{|c_1|, |c_2|\} \]

EPR steerability
necessary and sufficient condition:

\[ \left( a - \frac{c_1^2}{b} \right) \left( a - \frac{c_2^2}{b} \right) < \frac{1}{4} \]
Backup slides: Gaussian quantum states
CV quantum systems

- **Continuous-variable (CV) quantum system of $n$ modes**
- Hilbert space: $L^2(\mathbb{R}^n)$

$$[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk},$$

$$\hat{x}_j = \frac{\hat{a}_j + \hat{a}_j^\dagger}{\sqrt{2}}, \quad \hat{p}_j = \frac{\hat{a}_j - \hat{a}_j^\dagger}{i\sqrt{2}}, \quad \hat{R} = \begin{pmatrix}
\hat{x}_1 \\
\hat{p}_1 \\
\vdots \\
\hat{x}_n \\
\hat{p}_n
\end{pmatrix}$$
CV quantum systems

- $\hat{\rho} \geq 0, \quad \text{Tr}[\hat{\rho}] = 1$

- Characteristic and \textbf{Wigner} functions:

\[
\chi[\hat{\rho}](\Lambda) = \text{Tr} \left[ \hat{\rho} \exp (-i\Lambda^T \hat{R}) \right]
\]

\[
\mathcal{W}[\hat{\rho}](X) = \int_{\mathbb{R}^{2n}} \frac{d^{2n}\Lambda}{(2\pi^2)^n} \, \chi[\hat{\rho}](\Lambda) e^{i\Lambda^T X}
\]
Gaussian quantum states

- \( \hat{\rho} \) is **Gaussian** when:

\[
W[\hat{\rho}](X) = \frac{1}{\pi^n \sqrt{\det[\sigma]}} \exp \left[ -\frac{1}{2} (X - \langle \hat{R} \rangle)^T \sigma^{-1} (X - \langle \hat{R} \rangle) \right]
\]

- **First-moments vector**:

\[
\langle \hat{R} \rangle = \text{Tr}[\hat{\rho} \hat{R}]
\]

- **covariance matrix (CM)**:

\[
[\sigma]_{jk} = \frac{1}{2} \langle \hat{R}_j \hat{R}_k + \hat{R}_k \hat{R}_j \rangle - \langle \hat{R}_k \rangle \langle \hat{R}_k \rangle
\]
Gaussian quantum states

- Similarly, a POVM \( \{\hat{\Pi}_\alpha\}_{\alpha \in \mathbb{C}^n} \):

  \[
  \forall \alpha \in \mathbb{C}^n : \quad \hat{\Pi}_\alpha \geq 0, \quad \int_{\mathbb{C}^n} d^{2n} \alpha \; \hat{\Pi}_\alpha = \mathbb{I}
  \]

  is Gaussian if all \( \hat{\Pi}_\alpha \) have Gaussian Wigner functions

- Uncertainty Relations:

  \[
  \sigma + i\frac{\Omega}{2} \geq 0
  \]
Backup slides: Nonclassicality
P-nonclassicality

- It can be quantified by **nonclassical depth**:

  \[ \Sigma[\hat{\rho}] = \frac{1 - s_{\text{max}}}{2} \]

- It has a **resource** character

- Number states \((n > 0)\), Schrödinger cat’s states and squeezed vacuum states are all **nonclassical**

- Coherent states are only classical pure states. Thermal states are also classical
Backup slide: covariance matrices
Conditional covariance matrix

- CM of initial state $\hat{\rho}_{AB}$:

$$\sigma = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}$$

- CM of POVM:

$$\sigma_M = \frac{1}{2\mu\mu_s} \begin{pmatrix} 1 + \sqrt{1 - \mu_s^2 \cos \phi} & -\sqrt{1 - \mu_s^2 \sin \phi} \\ -\sqrt{1 - \mu_s^2 \sin \phi} & 1 - \sqrt{1 - \mu_s^2 \cos \phi} \end{pmatrix}$$

- CM of $\hat{\rho}_A^{(\alpha)}$ (conditional CM) does not depend on outcome $\alpha$:

$$\sigma^c_A = A - C^T (B + \sigma_M)^{-1} C$$
States in canonical form

\[ \sigma = \begin{pmatrix}
    a & 0 & c_1 & 0 \\
    0 & a & 0 & c_2 \\
    c_1 & 0 & b & 0 \\
    0 & c_2 & 0 & b
\end{pmatrix} \]

- Simplest generalization of TMSTs. \( \hat{\rho}_A \) and \( \hat{\rho}_B \) are still always classical.
Backup slides: Noisy propagation of twin-beam states
Two-mode squeezed vacuum states:

\[ |r \rangle \rangle = e^{r(\hat{a}^\dagger \hat{b}^\dagger - \hat{a} \hat{b})} |0\rangle_A \otimes |0\rangle_B \]

- Mode A interacts with a Markovian, purely thermal environment with average number of photons \( N_{th} \) and damping rate \( \Gamma \)

- The TWB state evolves into a generic TMST state
Figure 5: Triangoloid for TWB state with $r = 1.2$. 
Noisy evolution of TWB triangoloid

Figure 6: Snapshots of triangoloid evolution from initial TWB ($r = 1.2$)
Maximum propagation time for nonclassical steering:

\[ t_{ns} = \frac{1}{\Gamma} \log \left[ 1 + \frac{N_s}{N_{th}(1 + 2N_s)} \right] , \quad (N_s = \sinh^2 r) \]

Disentangling time:

\[ t_{ent} = \frac{1}{\Gamma} \log \left( 1 + \frac{1}{N_{th}} \right) \]