# Topics in Effective Field Theory

PhD-course, Rome 2020 by B. Bellazzini

Lecture - 10$^{th}$

## — Causality Constraints on EFT

We have seen that integrating out heavy modes, either entire massive fields or some energy shells for the fields, one generates infinitely many terms in the EFT lagrangian that are needed to recover the non-locality when the energy is such that some of those modes is put back on-shell. The only two requirements we used to guide us in building such an IR EFT were:

- symmetries
- expansion in Energy

In a sense, the expansion in energy is like the use of a spurionic scale-invariance of which $(E/\Lambda)^{\Delta-D}$ is the associated spurion, so that in fact everything boils down to symmetries, again.

In fact, in this lecture we focus on two new requirements

- causality
- unitarity

of the underlying UV completion of the EFT.

Let's go by examples, and consider the EFT for a single Goldstone Boson (GB) $\pi$ which enjoys a shift symmetry:

(1) $\pi \to \pi + \text{const}$

The associated EFT in the IR is built out of the invariants $\partial_\mu\pi \to \partial_\mu\pi$ and its derivatives

(2) $\mathcal{L}^{(\pi)}_{EFT} = \frac{1}{2}\partial_\mu\pi\,\partial^\mu\pi + \frac{a}{\Lambda^4}(\partial_\mu\pi\,\partial^\mu\pi)^2 + \dots$

Any value of $a$ is consistent with the symmetries, 

(3) $a=0$ , $a>0$ , $a<0$ (a priori)

but, in fact, only $\boxed{a\geq 0}$ is consistent with causality (& unitarity): no UV-completion generates $a<0$. [example: $|\partial\Phi|^2 - V(|\Phi|^2)$ with $V(|\Phi|^2) = \frac{\lambda}{4}(|\Phi|^2 - v^2)^2$ $\Phi \in \mathbb{C}$,
$\Phi = e^{i\frac{\pi}{\sqrt{2}v}}\left(v + \frac{h}{\sqrt{2}}\right) \Rightarrow \partial_\mu\Phi = \frac{e^{i\frac{\pi}{\sqrt{2}v}}}{\sqrt{2}}\left[\partial_\mu h + i\partial_\mu\pi\left(1+\frac{h}{v\sqrt{2}}\right)\right] \Rightarrow |\partial\Phi|^2 = \frac{1}{2}(\partial h)^2 + \frac{1}{2}(\partial\pi)^2\left(1+\frac{h}{v\sqrt{2}}\right)^2$, $V = V(h-\text{only})$
with $m_h^2 = \lambda v^2$, integrating out $h$ $\left(\sum_{\partial\pi}^{\partial\pi} h \left(^{\partial\pi}_{\partial\pi} + \dots\right)\right)$ generates $(\partial\pi)^4$-vertices: $\mathcal{L}^\pi \supset \frac{a}{\Lambda^4}(\partial\pi\partial\pi)^2$ with
$a/\Lambda^4 = \frac{1}{4m_h^2 v^2} = \frac{\lambda}{4}\frac{1}{m_h^4} > 0$ ]

There is a simple way to see this: let's imagine to turn-on a constant $\partial\pi$-background (situation analogous to constant $F_{\mu\nu}$ in Euler-Heisenberg Lagrangian) which breaks spontaneously boosts but respects translations & 3D rotations:

(4) $\langle\partial_\mu\bar{\pi}\rangle = \delta_\mu^0 v^2$

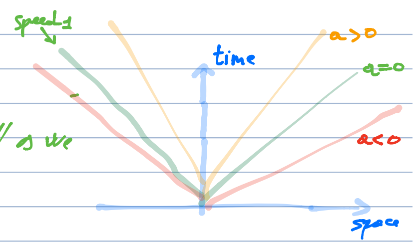
which solves the leading-order e.o.m. of (2): $\partial_\mu\left[\partial_\mu\varphi\left(1 + 4\frac{a}{\Lambda^4}(\partial\varphi)^2\right) + \dots\right] = 0$ & it's thus a legitimate solution (at least if this is over a finite region of spacetime). Now, one can look at perturbations $\varphi$ around this background: $\pi = \bar{\pi} + \varphi$ $\partial_\mu\pi = \partial_\mu\bar{\pi} + \partial_\mu\varphi$

(5) $\int d^4x\, \mathcal{L}^{(\varphi)} = \int d^4x\, \frac{1}{2}\partial_\mu\varphi\left[\eta^{\mu\nu}\left(1 + \frac{8a(\partial\bar{\pi})^2}{\Lambda^4}\right) + \frac{16a}{\Lambda^4}\partial^\mu\bar{\pi}\,\partial^\nu\bar{\pi}\right]\partial_\nu\varphi + O\left((\partial\varphi)^3\right)$

and see, as expected, that the effective inverse-metric $g^{\mu\nu}$ is Lorentz-breaking (spontaneously), that is the kinetic term for the $\varphi$ gives rise to a deformed light-cone

(6) $g^{\mu\nu} = \eta^{\mu\nu}\left(1 + \frac{8a(\partial\bar{\pi})^2}{\Lambda^4}\right) + \frac{16a}{\Lambda^4}\underbrace{\partial^\mu\bar{\pi}\,\partial^\nu\bar{\pi}}_{\eta^{\mu0}\eta^{\nu0}v^4} + \dots$ $\rightarrow$ suppressed by extra $v/\Lambda \ll 1$ can be as small as we like

(7) $g_{\mu\nu} = \eta_{\mu\nu}\left(1 - \frac{8a(\partial\bar{\pi})^2}{\Lambda^4}\right) - \frac{16a}{\Lambda^4}\underbrace{\partial_\mu\bar{\pi}\,\partial_\nu\bar{\pi}}_{\delta_\mu^0\delta_\nu^0 v^4} + \dots$




In particular, expanding in plane-waves $\varphi \propto e^{ik_\mu x^\mu}$ we get

(8) $(k\cdot k)\left(1 + \frac{8av^4}{\Lambda^4}\right) + \frac{16av^4}{\Lambda^4}(k^0)^2 = 0 \Rightarrow \boxed{(k^0)^2\left(1 + 16\frac{av^4}{\Lambda^4}\right) = |\vec{k}|^2}$ (working to leading order we can drop the coeff. of $k\cdot k$ $\frac{a}{\Lambda^4}$)

Now, the speed of perturbations is the ratio of $k^0/|\vec{k}|$ (we have rotational invariance)

(9) $\boxed{V_\varphi^2 = 1 - 16\frac{av^4}{\Lambda^4}}$ $\leq 1$ iff $\boxed{a \geq 0}$ (see hep-th/0602178)

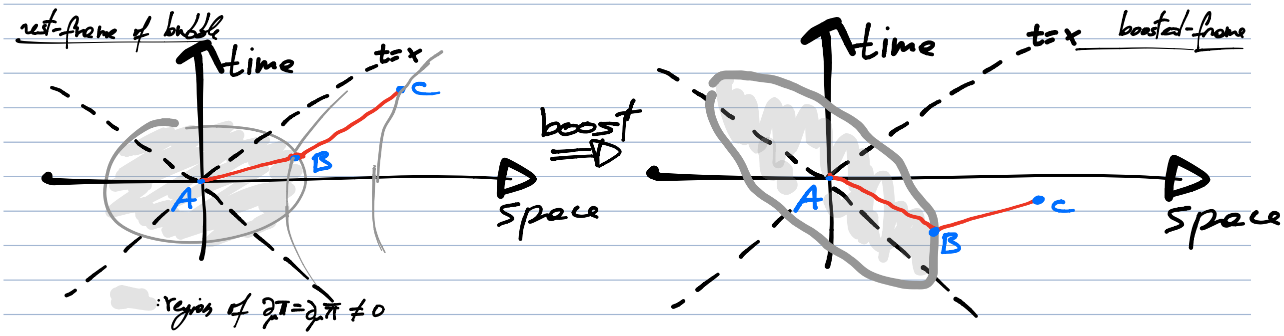
In other words, the index of refraction for the medium generated by $\partial_\mu\bar\pi$ upon which 

the perturbation $\varphi$ propagates is larger than 1 only if $a\geq 0$, despite the fact any sign for $a$ is consistent with the symmetries.

$\Rightarrow$ $\boxed{\text{subluminality of perturbations put a non-trivial constraint on the EFT, indep. of those by symmetries.}}$

One may wonder if violation of $a\geq 0$, i.e. superluminal propagation of perturbation, would cause any inconsistency with causality in our Lorentz-breaking background.

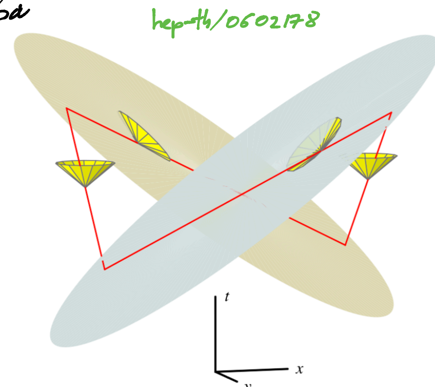
To answer this consider a bubble of bkg $\partial_\mu\bar\pi$ and empty space outside, and let's send a message from A to C passing via the boundary region somewhere at B




What appears to a (highly) boosted observer is that a signal starts at B and propagates to A (to the left) and to C to the right. [this is possible because the coefficient of $\partial_t^2\varphi$ in the $\varphi$-e.o.m. vanishes near the boundary of the bubble, for highly boosted observers]

This does not allow to define causality globally because one can form closed timelike curves: consider two such bubbles of $\partial\bar\pi\neq 0$ bkg flying-by each other with some impact parameter "b" and look at the closed path drawn below

hep-th/0602178



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Any time we can build a non-trivial & sensible background for small perturbations to travel upon, causality gives certain positivity bounds on the EFT such that the light-cone is inside the one Lorentz-preserving one.

Another example is for the EFT of photons only:

(10) $\mathcal{L}^{\gamma} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{a}{\Lambda^4}(F_{\mu\nu}F^{\mu\nu})^2 + \frac{b}{\Lambda^4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2 + \dots$ $\qquad \left(\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\right)$

Consider a non-trivial static bkg $\bar{F}_{\mu\nu}$ (e.g. an electric or magnetic field or both) with perturb. $f_{\mu\nu}$ around it: $A_\mu = \bar{A}_\mu + a_\mu$ , $F_{\mu\nu} = \bar{F}_{\mu\nu} + f_{\mu\nu}$

(11) $\mathcal{L}^{(a)} = -\frac{1}{4}f_{\mu\nu}f^{\mu\nu}\Big(1 - \underbrace{\frac{16a}{\Lambda^4}(\bar{F}_{\rho\sigma}\bar{F}^{\rho\sigma})}\Big) + \frac{8a}{\Lambda^4}(\bar{F}_{\mu\nu}f^{\mu\nu})^2 + \frac{8b}{\Lambda^4}(\tilde{\bar{F}}_{\mu\nu}f^{\mu\nu})^2$

Expanding in plane waves $\qquad$ like before, this Lorentz-preserving part is irrelevant to $O\left(\frac{a}{\Lambda^4}, \frac{b}{\Lambda^4}\right)$

(12) $a_\mu = \epsilon_\mu(k)\, e^{ik_\mu x^\mu}$ $\qquad \epsilon_\mu = \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$ for linear polarizations and $k^\mu = (k^0, 0, 0, k^3)$

the $a_\mu$-e.o.m. (with $\bar{F}$ constant bkg and no currents, $\partial_\mu \bar{F}^{\mu\nu} = \partial_\mu \tilde{\bar{F}}^{\mu\nu} = 0$)

(13) $\partial_\mu f^{\mu\nu} - \frac{32a}{\Lambda^4}\partial_\mu\left[(\bar{F}_{\rho\sigma}f^{\rho\sigma})\bar{F}^{\mu\nu}\right] - \frac{32b}{\Lambda^4}\partial_\mu\left((\tilde{\bar{F}}_{\rho\sigma}f^{\rho\sigma})\tilde{\bar{F}}^{\mu\nu}\right) = 0$

(14) $-(k\cdot k)\vec{\epsilon} - \frac{64a}{\Lambda^4}(k^0)^2\vec{E}\,(\vec{E}\cdot\vec{\epsilon}\,) + (b=0 \text{ for simplicity}) = 0$ $\qquad (\nu = i)$

that contracted with $\vec{\epsilon}$ gives $-(k\cdot k) - \frac{64(\vec{E}\cdot\vec{\epsilon})^2 a}{\Lambda^4}k^{0\,2} = 0$ and therefore $(k^0)^2\left(1 + \frac{64a(\vec{E}\cdot\vec{\epsilon})^2}{\Lambda^4}\right) = \vec{k}^2$

(15) $\boxed{\begin{cases} v^2 = 1 & \text{if } \vec{\epsilon} \perp \vec{E} \\ v^2 = \frac{k_0^2}{|\vec{k}|^2} = \left(1 - \frac{64a|\vec{E}|^2}{\Lambda^4}\right) & \text{if } \vec{\epsilon} \parallel \vec{E} \end{cases}}$

(16) $\Rightarrow$ $a \geq 0$

Analogously, if we had taken also $b \neq 0$, we would have found a bkg and $\vec{\epsilon}$ configuration such that $b \geq 0$, too. No causal theory can give rise neither to $a < 0$ nor $b < 0$.

Notice that, even though $|v - 1| \ll 1$, any deviation can be seen within the regime of validity of the EFT if one can build a sufficiently large bubble of background (with very small gradient of it).

[this is e.g. the case for the homogeneous bkg, but it may not be the case for bkg from localized sources like Black holes ... ]

There is a nice connection between these positivities & properties of scattering amplitudes.

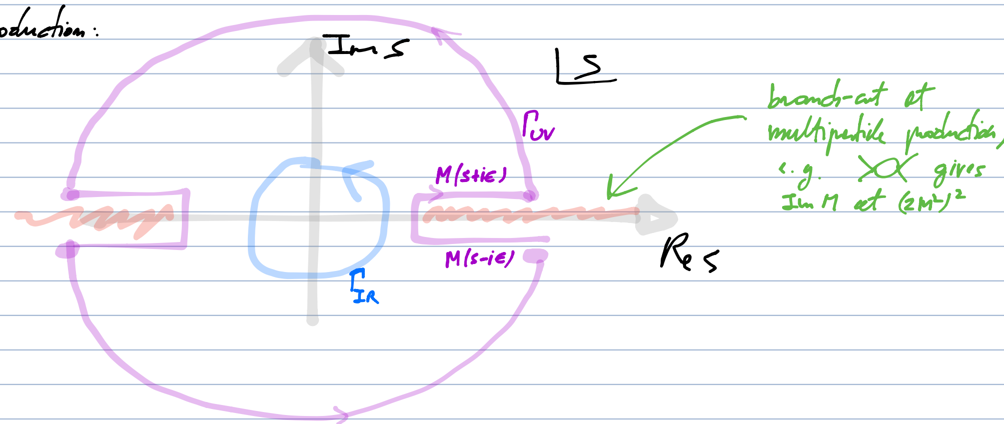
Indeed, it turns out that the refraction index is given by the forward elastic 2-to-2 amplitude:

(17) $n-1 \propto \# \frac{M^{\text{elastic}}_{\text{forward}}}{s^2}(s,t=0)$ with $\# > 0$

So that $n = \frac{1}{v_{\text{pert.}}} > 1$ must be related to $\frac{1}{s^2} M^{\text{elastic}}_{\text{forward}}(s,t=0) > 0$.

This is indeed true and it follows from (micro)causality (which implies analyticity of the $M(s,t=0)$) and unitarity.

Skipping all (in fact very interesting) steps we just claim that $M(s,t=0)$ (elastic forward) is analytic in the complex s-plane except at branch-cuts located on the real axis, associated with multiparticle production:




We can thus consider a contour integral along a path $\Gamma_{IR}$ which by Cauchy theorem gives the same on $\Gamma_{UV}$

(18) $\frac{1}{2\pi i}\oint_{\Gamma_{IR}} \frac{M(s,t=0)}{s^3}\,ds = \frac{1}{2\pi i}\oint_{\Gamma_{UV}} \frac{M(s,t=0)}{s^3}\,ds$ ← UV-IR connection by analyticity i.e. via microcausality.

But the left-hand side is calculable within the IR EFT

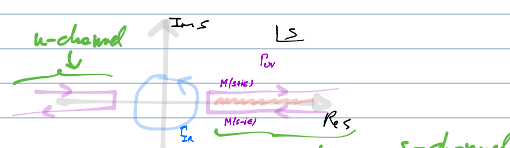
(19) $M(s,t=0) = a_{IR}\, s^2 + b_{IR}\, s^4 + \dots$ ⇒ by Residue Theorem $a_{IR} = \frac{1}{2\pi i}\oint_{\Gamma_{UV}} \frac{M(s,t=0)\,ds}{s^3}$

which links an IR-calculable observable, $a_{IR}$, to an unknown UV-integral

Now, pushing the big circle at infinity & requiring that

(20) $\lim_{|s|\to\infty} \frac{M(s,t=0)}{s^2} \to 0$ (forw. amp. grows less than $s^2$; this is actually a proven fact for gapped theories known as Froissart-Martin bound)

one is left with just the UV integral across the discontinuities




(21) $a_{IR} = \frac{1}{\pi}\int_{s_{IR}}^{\infty} \frac{M(s+i\epsilon,t=0) - M(s-i\epsilon,t=0)}{2i\; s^3} + \text{u-channel}$

Moreover, by crossing symmetry $M(s,t=0)=M(-s,t=0)$ (at $m=0$ for simplicity)

the u-channel gives the same contribution as the s-channel

(22) $a_{IR} = \frac{2}{\pi}\int_{s_{IR}}^{\infty} ds\, \frac{\mathrm{Im}\, M(s,t=0)}{s^3}$ $\quad (M^*(s,t=0)=M(s^*,t=0))$

So, one gets that the IR parameter of the EFT $a_{IR}$ is given by the integral all the way to $s=\infty$, where one can't use the EFT, of $\mathrm{Im}\, M(s,t=0)$. Despite one knows almost nothing about such integral, one thing is known: the optical theorem $\mathrm{Im}\, M(s,t=0)>0$, which implies

(23) $\boxed{a_{IR}>0}$ $\qquad M(s,t=0) = a_{IR}\, s^2 + b_{IR}\, s^4 + \dots$

This result precisely matches the bounds we derived earlier with the request $v\leq 1$ on bkg:

(24) $\mathcal{L} = \frac{1}{2}\partial\pi\,\partial\bar{\pi} + \frac{a}{\Lambda^4}(\partial\pi\,\partial\pi)^2 + \dots \Rightarrow M_{\pi\pi\to\pi\pi}(s,t) = 8\frac{a}{\Lambda^4}(s^2+t^2+u^2) \xrightarrow[t\to 0]{} \frac{16 a\, s^2}{\Lambda^4} \underset{(23)}{\Rightarrow} a>0$

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu}F^{\mu\nu}) + \frac{a}{\Lambda^4}(F_{\mu\nu}F^{\mu\nu})^2 + \frac{b}{\Lambda^4}(\tilde{F}_{\mu\nu}F^{\mu\nu})^2 \Rightarrow \begin{cases} M_{\uparrow\uparrow\to\uparrow\uparrow} \propto a\, s^2 \\ M_{\uparrow\downarrow\to\uparrow\downarrow} \propto b\, s^2 \end{cases} \Rightarrow \begin{matrix} a>0 \\ b>0 \end{matrix}$$

Notice that $\mathrm{Im}\, M(s,t=0)>0$ and it vanishes only for the free theory where $\mathrm{Im}\, M(s,t=0) = s\,\sigma^{TOT} = 0$ the total cross-section $12\to$ anything $\sigma^{TOT}=0$. This means that, actually, $a_{IR}>0$ strictly (hep-th/1605.06111), which is marginally stronger than $a_{IR}\geq 0$.

This is interesting because it means that no symmetry consistent with unitarity of the UV-completion can forbid the $s^2$-term (it would otherwise set to zero $a_{IR}$).

For example, a theory for a chiral fermion $\chi$ with $\chi\to\chi+\epsilon$ a fermionic shift-sym would give rise to $\mathcal{L}^{(\chi)}_{EFT} = \chi^\dagger i\bar{\sigma}\partial\chi + \alpha\, \partial\chi^\dagger\partial\chi^\dagger\partial\chi\,\partial\chi$ & other contr. of $\partial\chi$, and $\partial\chi^\dagger$, the two invariant building blocks. But this can't admit a causal unitary UV-completion since $M(\chi\chi\to\chi\chi)(t=0) = 0$ whereas $a_{IR}>0$.

[We have been a little sloppy and glossed over various points, related e.g. to IR singularities, the presence of mass & spin, etc; a more careful discussion can be found in 1605.06111 & 1310.07535, but a must read is the classic hep-th/0602178]