Topics in Effective Field Theory

PhD-course, Rome 2020 by B. Bellazzini

lecture - 9^{th}

In this lecture we are going to study the leading corrections to GR and Maxwell in an expansion in derivatives, that is in E/Λ. But before doing so we will see a general fact about redundant operators identified via field redefinitions.

- Field Redefinitions

Physical observables do not depend on the choice of variables we use to describe the system. For example, the S-matrix elements are obtained by the LSZ projection, schematically

$$(1) \quad \mathcal{M}_{\text{connected}}(p_1, \dots) = \lim_{p_1^2 \to m_1^2} \frac{(p_1^2 - m_1^2)}{N} \dots \langle 0 | T \tilde{\phi}(p_1) \dots \rangle$$

where one is free to use any scalar field, composite or not, as long as it is interpolating field with non-vanishing matrix element

$$(2) \quad \langle 0 | \phi(0) | p_1 \rangle = N \neq 0 .$$

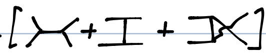
The S-matrix is invariant not only under symmetries but also under non-symmetries of various sort, e.g. $\phi \to \phi' = f(\phi)$ with $f'(0) = 1$ in the scalar example.

Example: Free scalar theory in disguise

Consider the Lagrangian

$$(3) \quad \mathcal{L} = \frac{1}{2}(\partial\phi)^2 K^2(\phi) = \frac{1}{2}(\partial\phi)^2 \left[1 + a\phi + \frac{b\phi^2}{2!} + \dots\right]^2$$

If one calculates S-matrix elements (on-shell) they are all trivial, e.g.

(4) $\mathcal{M}(2\to 2) = a\,[$  $] + (a^2+b)$  $+ (\text{loops}) \propto (s+t+u) = 0$

which becomes manifest if one performs the field redefinition:

(5) $\phi \to f(\phi)\,,\quad \partial_\mu\phi \to (\partial_\mu\phi) f'(\phi)\,,\quad \mathcal{L} \to \frac{1}{2}(\partial\phi)^2 f'^2(\phi) K^2(f(\phi)) \Rightarrow$ choose $f' = \dfrac{1}{K(f)}$

one lands on the free theory.

[Explicitly, say we had $\frac{1}{2}(\partial\phi)^2 e^{-2\phi} = \frac{1}{2}(\partial\phi)^2(1-2\phi+\dots) \Rightarrow$ define $\phi = f(\bar\phi)$ with $f' = e^f$

$\Rightarrow \phi = -\ln(1-\bar\phi) = \bar\phi + \dots$ gives $\partial_\mu\phi = \frac{1}{1-\bar\phi}\partial_\mu\bar\phi = e^{\phi}\partial_\mu\bar\phi \to (\partial\phi)^2 e^{-2\phi} = (\partial\bar\phi)^2$.]

— Field Redefinition & Equations of motion —

There is a very useful connection between the use of E.o.m. and field redefinitions.

Suppose we have an expansion in a parameter ϵ (that could be $\epsilon = E/\Lambda$ in an EFT, or a loop-expansion $\epsilon = \frac{g^2\hbar}{16\pi^2}$, or a spurion expansion etc...)

(6) $S[\Phi] = S^{(0)}[\Phi] + \epsilon\, S^{(1)}[\Phi] + \epsilon^2 S^{(2)}[\Phi] + \dots$

and such that an operator $\mathcal{O}^{(n)}(x)$ that enters a level n^{th} in S, $S^{(n)} \supset \int d^4x\, \mathcal{O}^{(n)}(x)$, would vanish on the lowest order e.o.m. $\frac{\delta S^{(0)}}{\delta\Phi(x)} = 0$:

(7) $\int d^4x\, \mathcal{O}^{(n)}(x) = \int d^4x\, \frac{\delta S^{(0)}[\phi]}{\delta\Phi(x)} f(\Phi(x), \partial_\mu\Phi(x))$

Then, this operator is called redundant (@ level n^{th}) because its effect on physical observables is actually of order higher than ϵ^n. Indeed, one can perform the field redefinition

(8) $\Phi(x) \to \Phi(x) - \epsilon^n f(\Phi(x), \partial\Phi(x))$

such that $S[\phi] = S^{(0)}[\phi] + \epsilon S^{(1)}[\phi] + \dots$ becomes

(9) $S[\phi] \to S[\phi] = S^{(0)}[\phi] + \epsilon S^{(1)}[\phi] + \dots + \epsilon^{n-1} S^{(n-1)}[\phi] + \epsilon^n\Big(S^{(n)}[\phi] \underbrace{- \int d^4x \frac{\delta S^{(0)}[\phi]}{\delta\Phi(x)} f(\Phi(x), \partial\Phi(x))}_{-\int d^4x\, \mathcal{O}^{(n)}(x)}\Big) + \dots$

that is $S^{(m<n)}[\phi] \to S^{(m<n)}[\phi]$ while $S^{(n)}[\Phi] \to S^{(n)}[\phi] - \int d^4x\, \mathcal{O}^{(n)}(x)$ which

is removing the operator $O^{(n)}(x)$ from $S^{(m\le n)}[\Phi]$ without changing the other operators at the same and lower order. (Higher orders get corrected, e.g. $\delta S^{(n+1)} \supset \int \frac{\delta S^{(1)}}{\delta\Phi} f \dots$)

– Remark: often one can complete operators into redundant ones so that effectively one is plugging-in the lowest order e.o.m.

For example, take a scalar theory with

(10) $$\mathcal{L} = \underbrace{\frac{1}{2}(\partial\Phi)^2 - V(\Phi)}_{\mathcal{L}^{(0)}} + \frac{1}{\Lambda^2}\underbrace{\left\{\Phi^3\Box\Phi + g\,\Phi^6 + \dots\right\}}_{\mathcal{L}^{(1)}} + \frac{1}{\Lambda^4}\underbrace{\{\dots\}}_{\mathcal{L}^{(2)}} + \dots$$

with $V(\Phi) = \frac{\lambda}{4!}\Phi^4$ and $\frac{\delta S^{(0)}}{\delta\Phi(x)} = -\Box\Phi - V'(\Phi) = -\Box\Phi - \frac{\lambda}{3!}\Phi^3$

so that $\frac{\Phi^3}{\Lambda^2}\Box\Phi$ is physically equivalent to $-\frac{1}{\Lambda^2}\Phi^3 V'(\Phi)$ (up to $O(\frac{1}{\Lambda^4})$ terms)

(11) $$\frac{\Phi^3}{\Lambda^2}\Box\Phi = \underbrace{\frac{\Phi^3}{\Lambda^2}\big(\overbrace{\Box\Phi + V'(\Phi)}^{-\frac{\delta S^{(0)}}{\delta\Phi}}\big)}_{} - \frac{\Phi^3 V'(\Phi)}{\Lambda^2} \sim -\frac{\Phi^3 V'(\Phi)}{\Lambda^2} = -\frac{\lambda\Phi^6}{3!\,\Lambda^2}$$

removed by $\Phi \to \Phi + \frac{\Phi^3}{\Lambda^2}$, extra $\frac{1}{\Lambda^4}$ in $\mathcal{L}^{(2)}$ generated

that is, one between $\Phi^3\Box\Phi$ and Φ^6 is redundant and it can be eliminated in favor of the other: $\mathcal{L}^{(1)}$ in (10) is equivalent to

(12) $$\mathcal{L}^{(1)}_{(10)} \sim g'\frac{\Phi^6}{\Lambda^2} \quad \text{with } g' = g - \frac{\lambda}{3!}$$

(the effect of $\Phi^3\Box\Phi$ is equivalent to a redefinition of Φ^6 coupling, to this order in $1/\Lambda^2$)

— Example: – EFT photon coupled to a massless fermion

(13) $$S^{(0)} = \int d^4x\, \mathcal{L}^{(\Delta=4)} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar\psi i\partial\!\!\!/\psi - g\underbrace{\bar\psi\gamma^\mu\psi}_{J^\mu} A_\mu$$

(expansion in Δ of the op. that is in $1/\Lambda^2$)

(14) $$S^{(1)} = \int d^4x\, \mathcal{L}^{(\Delta=6)} = \int d^4x\, \frac{\alpha}{\Lambda^2}(\bar\psi\gamma^\mu\psi)(\bar\psi\gamma_\mu\psi) + \frac{\beta}{\Lambda^2}(\partial_\mu F_{\mu\nu})^2 + \frac{\gamma}{\Lambda^2}(\partial_\rho F_{\mu\nu})^2 + \dots$$

In $S^{(1)}$ only two operators are actually independent, we can e.g. remove β and γ in favor of a new α

This is immediately visible from looking at the leading order e.o.m. $\frac{\delta S^{(0)}}{\delta \Phi(x)} = 0$

$$(15)\quad \partial_\mu F^{\mu\nu} = g J^\nu, \qquad i \not{\partial} \psi = g \not{A} \psi, \qquad -i \bar{\psi} \overleftarrow{\not{\partial}} = g \bar{\psi} \not{A} \qquad \left(\Rightarrow \partial_\mu J^\mu = \partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0\right)$$

which plugged into the $\Delta = 6$ operators in (14) give

$$(16)\quad S^{(1)} \underset{\text{int. by parts}}{=} \int d^4x \left[\frac{\alpha}{\Lambda^2} J^\mu J_\mu + \frac{\beta}{\Lambda^2}(\partial_\mu F_{\mu\nu})^2 + \frac{2\gamma}{\Lambda^2} \Box A_\nu \partial_\mu F_{\mu\nu}\right] \underset{\text{e.o.m. of } S^{(0)}}{\sim} \int d^4x \left[\frac{\alpha}{\Lambda^2} J_\mu J^\mu + \frac{\beta g^2}{\Lambda^2} J^\mu J^\mu + \frac{2\gamma g}{\Lambda^2} \Box A_\nu J_\nu\right]$$

$$\Box A_\nu \to g J_\nu + \partial_\nu \partial_\mu A_\mu, \qquad \partial_\mu J^\mu \big|_{\text{e.o.m.}} = 0$$

$$\sim \boxed{\frac{\alpha + (\beta + 2\gamma) g^2}{\Lambda^2} \int J^\mu J_\mu \, d^4x}$$

All physical effect of β and γ is equivalent to $\boxed{\alpha \to \alpha + g^2(\beta + 2\gamma) = \tilde{\alpha}}$.

(of course one could have instead chosen to eliminate any pair of parameters, say α and γ)

We can get to the same conclusion via explicit field redefinitions:

$$(17)\quad \delta S^{(0)} = \int d^4x \, \underbrace{(\partial_\mu F^{\mu\nu} - g \bar{\psi}\gamma^\nu\psi)}_{\frac{\delta S^{(0)}}{\delta A^\nu}} \delta A^\nu + \underbrace{\bar{\psi}(-i \overleftarrow{\not{\partial}} - g \not{A})}_{\frac{\delta S^{(0)}}{\delta \psi}} \delta\psi + \delta\bar{\psi} \underbrace{(i\not{\partial} - g\not{A})\psi}_{\frac{\delta S^{(0)}}{\delta \bar{\psi}}}$$

where, transforming the fields up to $O(\frac{1}{\Lambda^2})$ included

$$(18)\quad \begin{cases} A_\nu \to A_\nu + \delta A_\nu = A_\nu + \frac{1}{\Lambda^2}\left(x_1 \partial_\mu F_{\mu\nu} + x_2 \Box A_\nu + x_3 \overbrace{\bar{\psi}\gamma_\nu\psi}^{J_\nu}\right) + \dots \\ \psi \to \psi + \delta\psi = \psi + \frac{i x_4}{\Lambda^2}(\partial_\mu A^\mu)\psi \simeq \exp\left[i \frac{x_4}{\Lambda^2}(\partial \cdot A)\right]\psi + O\left(\frac{1}{\Lambda^4}\right) \\ \bar{\psi} \to \bar{\psi} + \delta\bar{\psi} = \bar{\psi} - \frac{i x_4}{\Lambda^2} \bar{\psi}(\partial_\mu A^\mu) \simeq \bar{\psi} \exp\left[-i \frac{x_4}{\Lambda^2}(\partial \cdot A)\right] + \dots \end{cases}$$

so that $S^{(0)} \to S^{(0)}$, whereas for $S^{(1)} = \int d^4x \, \frac{\alpha}{\Lambda^2}(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi) + \frac{\beta}{\Lambda^2}(\partial_\mu F_{\mu\nu})^2 + \frac{2\gamma}{\Lambda^2} \Box A_\nu \partial_\mu F_{\mu\nu}$

$$(19)\quad S^{(1)} \to S^{(1)} + \frac{1}{\Lambda^2} \int d^4x \, (\partial_\mu F^{\mu\nu} - g J^\nu)(x_1 \partial_\mu F_{\mu\nu} + x_2 \Box A_\nu + x_3 J_\nu) - x_4 \partial_\mu(\partial \cdot A) J^\mu$$

$$= \int d^4x \frac{1}{\Lambda^2} \Big\{ (\partial_\mu F^{\mu\nu})^2 (\beta + x_1) + \Box A_\nu \partial^\mu F^{\mu\nu} (2\gamma + x_2) + J_\nu J^\nu (\alpha - g x_3) + \underbrace{\partial_\mu F^{\mu\nu} J^\nu (x_3 - g x_1) - J^\nu \Box A_\nu \, g x_2 - x_4 \partial_\nu(\partial \cdot A) J^\nu}_{} \Big\}$$

$$\hookrightarrow \Box A^\nu J^\nu (x_3 - g x_1 - g x_2) - \partial^\nu(\partial \cdot A) J^\nu (x_3 - g x_1 + x_4)$$

Picking $x_1 = -\beta$, $x_2 = -2\gamma$, $x_3 = g x_1 + g x_2$, $x_4 = g x_1 - x_3 = -g x_2$

we can set all $\Delta = 6$ operators to zero except one, we choose $J^\mu J^\mu$, that has $\boxed{\alpha \to \alpha - g x_3 = \alpha + g^2(\beta + 2\gamma)}$

in agreement with (16).

Clearly, any other term (such as e.g. $\bar\psi\gamma^\mu\Box\psi A_\mu$) can be also removed in a similar way (e.g. $\psi\to\psi+\frac{\delta}{\Lambda^2}\Box\psi$.)

— *Example: EFT of photon only* —

At energy below the mass of any charged particle, there is no J^μ operator in the EFT (besides those identically conserved e.g. $\partial_\nu(F^{\mu\nu}F^2)=J^\mu$) so that $(\partial F^{\mu\nu})^2$-operators can be field-redefined away completely [namely $\mathcal{L}^0=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\xrightarrow[A_\nu\to A_\nu+\frac{\alpha}{\Lambda^2}\partial^\mu F_{\mu\nu}+\frac{\beta}{\Lambda^2}\Box A_\nu]{}\mathcal{L}^0+\frac{\alpha}{\Lambda^2}(\partial_\mu F^{\mu\nu})^2+\frac{\beta}{\Lambda^2}\partial_\mu F^{\mu\nu}\Box A_\nu\underset{\text{int. by parts}}{\sim}\mathcal{L}^0+\frac{\alpha}{\Lambda^2}(\partial^\mu F_{\mu\nu})^2+\frac{\beta}{2\Lambda^2}(\partial_\rho F_{\mu\nu})^2$]

Going therefore to $\Delta=8$ operators we have (see also L2)

(20) $$\boxed{\mathcal{L}^{\Delta=8}=\frac{\alpha}{\Lambda^4}(F_{\mu\nu}F^{\mu\nu})^2+\frac{\beta}{\Lambda^4}(F_{\mu\nu}F_{\nu\rho})^2}$$ (up to parity breaking op. such as $FF\tilde{F}\tilde{F}$ and recall: $(F_{\mu\nu}\tilde{F}^{\mu\nu})^2=4(F_{\mu\nu}F_{\nu\rho})^2-2(F_{\mu\nu}F^{\mu\nu})^2$)

The scattering amplitudes start at $O(E^4)$, $M(\gamma\gamma\to\gamma\gamma)\sim\frac{E^4}{\Lambda^4}$ to lowest order in E.

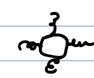
Moreover, the α and β in (20) do not run because of dimension-analysis

(dim-analysis works best in dim-reg)

(21) 

$\left(\frac{\alpha}{\Lambda^4}\right)^n\left(\frac{\beta}{\Lambda^4}\right)^m\Lambda^4\,\partial^{4(n+m-1)}\frac{F^4}{\Lambda^4}$ with $n+m>1$

they renormalize $(\partial F)^4$ & higher-dim. operators (any insertion of even higher-dim operator can't change this because they have even higher $1/\Lambda$ powers than F^4's op.

In the EFT with matter, the α and β run instead because of $\Delta\leq 6$ insertions: 

strong-coupling scale:

As all EFT this theory of photons has a range of validity given by the encounter of the lightest charged state (the electron in real-world QED) of mass Λ. But we can't determine Λ from EFT inputs alone in (20) which depends on α/Λ^4 and β/Λ^4 where α & β depend on the UV details (such as the coupling constants $\alpha\sim\frac{g^4q^4}{16\pi^2}$ in a QED-like UV completion, or $\beta\sim\frac{\Lambda^2}{f^2}$ with f a decay-constant like for an axion-like partial UV-completion ). A larger α,β can mimick a smaller mass Λ of the integrated states.

However, recalling that we are treating EFT of particles as perturbations of the free Gaussian theory, one can give an upper bound on the regime of validity of such an EFT when the interactions would become as important as the kinetic term: $\Lambda < \Lambda_*$ where Λ_* is the strong-coupling scale.

In this regime scattering amplitudes become large, and a rough estimate of Λ_* is given by

(22) $\mathcal{M}(\gamma\gamma \to \gamma\gamma)\big|_{E=\Lambda_*} = 16\pi^2 \gg 1$ (in comparison $\mathcal{M}(\phi\phi\to\phi\phi) = \lambda$ in $\frac{\lambda}{4!}\phi^4$ & $\lambda \simeq 16\pi^2$ is when theory is maximally strongly coupled.)

$$\Rightarrow \frac{\alpha E^4}{\Lambda^4}\bigg|_{E=\Lambda_*} = 16\pi^2 \qquad \Lambda_* \simeq \left(\frac{16\pi^2}{\alpha}\right)^{1/4}\Lambda$$

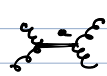
In a QED-like UV-completion $\alpha = \frac{g^4 q^4}{16\pi^2}$ & $\Lambda_* \simeq \left(\frac{4\pi}{g q}\right)\Lambda \gg \Lambda$ in a weakly coupled theory. The Λ_* represents when the theory is no longer near a Gaussian fixed point, that is weakly coupled theory of particles, so that weakly coupled UV-completion enter at $\Lambda \ll \Lambda_*$.

Equivalent to condition (22) is requiring that loops ~ tree-level in the observables:

(23)  ~  gives $\frac{\alpha^2 E^8}{\Lambda^8\, 16\pi^2}\bigg|_{E=\Lambda_*} = \frac{\alpha E^4}{\Lambda^4} \Rightarrow \Lambda_* = \left(\frac{16\pi^2}{\alpha}\right)^{1/4}\Lambda$

As we have stressed, this is generically an overestimate for Λ, as the UV completion can well appear in the weakly coupled regime. In this sense, a better estimate is usually the ratio of two terms in the amplitude with different energy-scaling, they becoming of the same order when the derivative expansion breaks down $\mathcal{M} = \left[\alpha\frac{E^4}{\Lambda^4} + \frac{E^8}{\Lambda^8}\tilde{\alpha} + \dots\right] \Rightarrow \frac{\tilde{\alpha} E^8}{\Lambda^8} \simeq \frac{\alpha E^4}{\Lambda^4} \Rightarrow \boxed{E = \left(\frac{\alpha}{\tilde{\alpha}}\right)^{1/4}\Lambda}$.

If the couplings α & $\tilde{\alpha}$ are not too hierarchical (e.g. a sym. could in principle suppress one in favor of the other) then this is a better estimate of Λ. Think e.g. of when expanding the axion propagator in $\mathcal{L}_{UV} = -\frac{F_{\mu\nu}^2}{4} + \frac{(\partial a)^2}{2} + \frac{a}{f}F\tilde{F}$

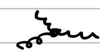
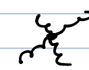
(24)  $= \frac{(F\tilde{F})}{f}\frac{-1}{-\Box - m_a^2}\frac{F\tilde{F}}{f} = \frac{(F\tilde{F})^2}{m_a^2 f^2} + \frac{(F\tilde{F})\Box(F\tilde{F})}{m_a^4 f^2} + \dots$

$\Rightarrow \mathcal{M}(\gamma\gamma\to\gamma\gamma)\big|_{EFT} = \frac{E^4}{m_a^2 f^2}\left(1 + \frac{E^2}{m_a^2} + \dots\right)$ and indeed $E \simeq m_a = \Lambda$ is when the derivative expansion breaks down, despite the fact that $\mathcal{M}(E = m_a = \Lambda) = \frac{m_a^2}{f^2} \ll 1$ can well be in the perturbative regime for a very light m_a.

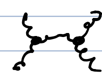
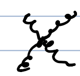
– EFT for Gravitons (only) —— • ——

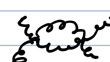
In a theory with the graviton we have seen in L7 and L8 that one has to build a fully diff-invariant Lagrangian (such that $h_{\mu\nu}$ couples universally to an exactly conserved $T_{\mu\nu}$ in order to preserve Lorentz-invariance). In particular, the Kinetic term arises from the Einstein-Hilbert action

$$(25) \qquad S^{(0)} = \int d^4x \sqrt{|g|}\, \frac{M_{PL}^2}{2} R \underset{g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}/M_{PL}}{=} \text{schematically: } \left\{ \partial h \partial h + \partial h \partial h \frac{h}{M_{PL}} + \partial h \partial h \frac{h^2}{M_{PL}^2} + \dots \right\}$$

 $\frac{E^2}{M_{PL}}$  $\frac{E^2}{M_{PL}^2}$

It' clear that the strong-coupling scale Λ_* must be proportional to M_{PL}:

$$(26) \qquad M(hh \to hh)\big|_{E=\Lambda_*} = 16\pi^2 =$$  $+$  $$= \frac{E^2}{M_{PL}^2}\bigg|_{E=\Lambda_*} \Rightarrow \Lambda < \Lambda_* = 4\pi M_{PL}$$

Equivalently: at $E = \Lambda_*$  $\underset{E=\Lambda_*}{\sim}$  $\Rightarrow \Lambda_* = 4\pi M_{PL}$

$$\frac{E^4}{M_{PL}^4}\frac{1}{16\pi^2} \qquad \frac{E^2}{M_{PL}^2}$$

Again, this $\Lambda_* = 4\pi M_{PL}$ is an upper bound on the actual cutoff Λ. Any UV completion entering at $E = \Lambda \ll \Lambda_*$ would be weakly coupled. [this is the case for string theory where $M_S \ll 4\pi M_{PL}$] The M_{PL} is totally analogous to the decay constant f_π for GB's: $(\partial\pi)^2 \frac{\pi^2}{f_\pi^2}$ give $\Lambda_* = 4\pi f_\pi$, but in weakly coupled models the "radial" excitations complete the non-linear sigma-model into the linear one at $E = m_{Radial} \ll 4\pi f_\pi$.

What we have seen, $4\pi M_{PL}$, is the ultimate UV-cutoff, where the theory becomes strongly coupled as E goes up. But there it may also be an IR-cutoff as $E \to 0$.

Indeed, there is an even more relevant operator than R in the deep IR which has no derivatives: the cosmological constant contribution: $\sqrt{|g|}\,\Lambda_{cc}$. This is a highly relevant deformation that become important, relative to the K.T. $\sqrt{g} R M_{PL}^2$, when E is so small that:

(27) $M_{PL}^2 E^2 = \Lambda_{cc} \Rightarrow$ for $E < \frac{\sqrt{\Lambda_{cc}}}{M_{PL}}$ that is lengths $L > M_{PL}/\sqrt{\Lambda_{cc}}$

one can't ignore the c.c. In our universe such a scale $L = 1/H_0$ with H_0: Hubble today

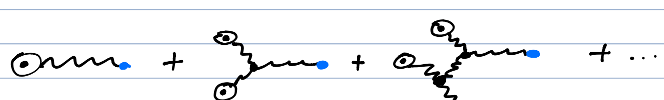
that is $\boxed{M_{PL}^2 H^2 \sim \Lambda_{cc} \sim (\text{few meV})^4}$ $(H_0 \sim 10^{-32}\,eV,\ M_{PL} \sim 10^{18}\,GeV)$. One of the greatest mystery is why a strongly relevant operator (unprotected by a symmetry) has such a small value relative to the vacuum-energy contribution of all other fields

[Specifically: $\Lambda_{cc} = \overbrace{\frac{1}{2}\int \frac{d^3k}{(2\pi)^3} \sqrt{k^2+m^2}\Big|_{|k|<\Lambda_{IR}}}^{\text{IR-calculable}} + \Lambda_{cc}^{UV} \sim \underbrace{\Lambda_{IR}^4}_{(TeV)^4} + \Lambda_{cc}^{UV} \Rightarrow$ as long as Λ_{cc} is calculable in the full UV-completion of GR, like it is in string theory, one would like to understand the cancellation between the calculable IR contribution $\Lambda_{IR}^4 \sim (TeV)^4$ & the a priori unknown Λ_{cc}^{UV}, down to a precision in 1 part in $(meV/TeV)^{-4} = 10^{60}$, suggesting that Λ_{cc}^{UV} & Λ_{IR}^4 have very much to do with each other. (Even in dim-Reg. & even using only interaction corrections to Λ_{cc}, the IR running $\propto m_{top}^4 \frac{g^2}{(16\pi^2)^2} \ln(\Lambda/\Lambda_0)$ remains, sensitive to the highest mass in the theory)]]

– Classical Strong Gravity Effects

The estimate of $E = \Lambda_* = 4\pi M_{PL}$ makes clear that it's where quantum gravity effects are $o(1)$ relative to the classical one. Another interesting scale is instead where classical non-linear effects become important relative to the classical linear ones. One such example is the occurrence of the Horizon around a compact source at $L \sim \frac{M}{M_{PL}^2} \cdot \#$, where $\# = o(1)$.

Let's understand the Schwarzschild radius in the EFT language by coupling a test particle of mass m_{Test} to a heavy source $M_\odot$ via $\frac{h_{\mu\nu} T^{\mu\nu}}{M_{PL}}$ (we also assume $v \ll c$)



(28)
$$\frac{M_\odot m_{TEST}}{M_{PL}^2 r} + \# \frac{M_\odot^2 m_{TEST}}{M_{PL}^4 r^2} + \# \frac{M_\odot^3 m_{TEST}}{M_{PL}^6 r^3} + \ldots = \frac{m_{TEST} M_\odot}{M_{PL}^2 r}\left[1 + \#\left(\frac{M_\odot}{M_{PL}^2 r}\right) + \#\left(\frac{M_\odot}{M_{PL}^2 r}\right)^2 + \ldots\right]$$

where we are taking just leading $o(m_{TEST})$ while all-order in the heavy source $M_\odot$, and neglecting loops (which would generate higher-dim operators, see below). Then we see that large classical effects arise when $r = L \sim M_\odot/M_{PL}^2$, as confirmed by the fully non-linear exact Schwarzschild solution where one sees the formation of a Horizon.

Notice that higher-derivative operators such as e.g. Riemann^3 can't change this conclusion about the existence of a Horizon as long as they are suppressed by a larger scale Λ than $1/r_{sch}$

(29) $\Lambda \gg \frac{1}{r_{sch}} \sim \frac{M_{PL}^2}{M_\odot} = M_{PL} \cdot \left(\frac{M_{PL}}{M_\odot}\right) \quad (\ll M_{PL} \quad \text{for } M_\odot \gg M_{PL})$

Indeed, from $\mathcal{L} = M_{PL}^2 \left[R + \frac{\text{Riemann}^2}{\Lambda^2} + \frac{\text{Riemann}^3}{\Lambda^4} + \dots \right]$ we see that $\text{Riemann} \sim \partial^2 h$ will necessarily pick the relevant scale by dimensional analysis $\boxed{\frac{\text{Riemann}}{\Lambda^2} \sim \frac{1}{r_{sc}^2 \Lambda^2} = \frac{M_{PL}^4}{M_\odot^2 \Lambda^2}}$.

For small Black Holes with $M_\odot \to M_{PL}$ these corrections can't be trusted even with $\Lambda = M_{PL}$. On the other hand, large black Holes with $M_\odot \gg M_{PL}$ are under control as long as (29) holds true.

– Higher-Derivative operators —

Like in the case of photons, we can look at the next order in derivatives: ∂^4

One could write in principle 3 operators to be added to $S^{(0)} = \int d^4x \sqrt{|g|}\, R \frac{M_{PL}^2}{2}$:

(30) $$\boxed{S^{(1)} = \int d^4x \sqrt{|g|}\, M_{PL}^2 \left\{ \alpha \frac{R^2}{\Lambda^2} + \beta \frac{R_{\mu\nu} R^{\mu\nu}}{\Lambda^2} + \frac{\gamma}{\Lambda^2} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right\}}$$

We can thus use the field redefinitions in the expansion $1/\Lambda^2$ to show that both α (Ricci^2) and β (Ricci-tensor^2) are redundant operators whereas γ does not contribute in $D=4$ because it can be recast in terms of the topological Gauss-Bonnet term

(31) $$\boxed{GB = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2}$$

that can't affect any perturbative calculation $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ since $\delta S_{GB} = \int \sqrt{|g|}\, \nabla_\mu (\,)^\mu$ = Boundary

So, up to redefining α & β (and restricting to $D=4$ spacetime dimensions) we can focus on

(32) $$S^{(1)} = \int d^4x \sqrt{|g|}\, \frac{M_{PL}^2}{2} \left\{ \alpha \frac{R^2}{\Lambda^2} + \beta \frac{R_{\mu\nu} R^{\mu\nu}}{\Lambda^2} \right\}$$

Performing the field redefinition $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$ with

(33) $$\boxed{\delta g_{\mu\nu} = \frac{1}{\Lambda^2}\left\{ x_1 R_{\mu\nu} + x_2 g_{\mu\nu} R \right\} + o\left(\frac{1}{\Lambda^4}\right)}$$

the $S^{(1)}$ action gives rise to a redefinition of α and β in (32),

(34) $$S_0 \to S_0 + \int d^4x \sqrt{|g|}\, \frac{M_{Pl}^2}{2} \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right) \delta g_{\mu\nu} = S_0 + \int d^4x \sqrt{|g|}\, \frac{M_{Pl}^2}{2}\left[\frac{x_1}{\Lambda^2}\left(R_{\mu\nu}^2 - \frac{R^2}{2}\right) + \frac{x_2}{\Lambda^2}\left(R^2 - \frac{4}{2}R^2\right)\right]$$

that is

(35) $$\alpha \to \alpha - \frac{x_1}{2} - x_2 \,, \qquad \beta \to \beta + x_1$$

Choosing $x_1 = -\beta$ and $x_2 = \frac{\beta}{2} + \alpha$, we can move any physical effect to $o\left(\frac{1}{\Lambda^4}\right)$-operators.

[incidentally, this is why 1-loop pure gravity calculations are actually UV-finite,

$\delta\mathcal{L}^{1\text{-loop}} = \frac{R^2}{120 M_{Pl}^2} + \frac{7 R_{\mu\nu}R^{\mu\nu}}{20 M_{Pl}^2}$ as obtained by 't Hooft & Veltman in '74. Indeed one could

move any loop effect to ∂^4's operators that are generated at two loops only].

Genuinely leading corrections to Einstein-Hilbert (in pure gravity without matter) start at $o(\partial^4)$:

(36) $$\boxed{S^{(2)} = \frac{M_{Pl}^2}{2}\int \sqrt{|g|}\, d^4x \left\{ \frac{R_{\mu\nu\rho\sigma} R^{\rho\sigma\alpha\beta} R_{\alpha\beta}{}^{\mu\nu}}{\Lambda^4} + \dots \right\}}$$

(in $D=4$. In $D>4$ add the Gauss-Bonnet term)

— EFT of Gravitons & Photons: our universe (in the deep IR) —

In the case of matter, photons in this case, we need to have a fully diff-invariant theory whose leading 2-∂'s order is obtained by simply covariantizing the KT:

(37) $$S^{(0)} = \int d^4x \sqrt{|g|}\left[\frac{M_{Pl}^2}{2} R - F_{\mu\nu}F_{\rho\sigma}\, g^{\mu\rho} g^{\nu\sigma}\right]$$ Einstein-Maxwell theory

Using just this $S^{(0)}$ there are only gravitational interactions of strength E/M_{Pl} so that the strong coupling scale Λ_* would look the same for photons as for gravitons, $M(\gamma\gamma\to\gamma\gamma) =$  $\sim \frac{E^2}{M_{Pl}^2}$ which is $o(16\pi^2)$ for $E = \Lambda_* = 4\pi M_{Pl}$. But it's clear that generically the UV cutoff will be much lower than that. Let's look e.g. at the leading higher-dimensional op. Like before, performing a field redefinition $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$ with an expansion in ∂/Λ (but not field/Λ)

(38) $$\delta g_{\mu\nu} = \frac{1}{\Lambda^2}\left[x_1 R_{\mu\nu} + x_2 R g_{\mu\nu} + x_3 \frac{F_{\mu\rho}F^{\rho}{}_{\nu}}{M_{Pl}^2} + x_4 \frac{g_{\mu\nu} F_{\rho\sigma}F^{\rho\sigma}}{M_{Pl}^2}\right]$$

the M_{Pl} just to match dimensions it could be absorbed into $x_{3,4}$. Just didn't use $1/\Lambda^2$ that counts derivatives.

we can redefine away several operators:

(39) $$\delta S^{(0)} = \int \sqrt{|g|}\, \frac{M_{pl}^2}{2} \underbrace{\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R - \frac{1}{M_{pl}^2} T^{\mu\nu}\right)}_{\text{Einst. eq.}} \delta g_{\mu\nu} \qquad \text{where } T^{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\delta (F_{\rho\sigma} F^{\rho\sigma} \sqrt{|g|})}{\delta g_{\mu\nu}} = -F^{\mu\lambda} F^{\nu}{}_{\lambda} + \frac{1}{4} g^{\mu\nu} F^2$$

$$= \int d^4x \sqrt{|g|}\, \frac{M_{pl}^2}{2\Lambda^2} \left\{ x_1 \left(R_{\mu\nu}^2 - \frac{R^2}{2} - \frac{T^{\mu\nu} R_{\mu\nu}}{M_{pl}^2}\right) + x_2 \left(-R^2 - \frac{T}{M_{pl}^2}\right) + x_3 \left(R_{\mu\nu} \frac{F^{\mu}{}_{\rho} F^{\rho}{}_{\nu}}{M_{pl}^2} - \frac{R F^2}{2 M_{pl}^2} - \frac{F_{\mu\rho} F^{\rho}{}_{\nu} T^{\mu\nu}}{M_{pl}^4}\right) + x_4 (\dots) \right\}$$

We can always choose the basis where the leading higher-dimensional corrections to Einstein-Maxwell theory are

(40) $$\boxed{\mathcal{L}^{(1)} = \frac{\alpha}{\Lambda^4} (F_{\mu\nu} F^{\mu\nu})^2 + \frac{\beta}{\Lambda^4} (F_{\mu\nu} F^{\nu\rho})^2 + \frac{\gamma}{\Lambda^2} R_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}}$$

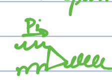
(assuming parity) for simplicity

sometimes people use other basis, e.g. Weyl-tensor $W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - (\text{traces})_{\mu\nu\rho\sigma}$ in (40)

(41) $$\mathcal{L}^{(1)} = \frac{\tilde{\alpha}}{\Lambda^4} (F_{\mu\nu} F^{\mu\nu})^2 + \frac{\tilde{\beta}}{\Lambda^4} (F_{\mu\nu} F^{\nu\rho})^2 + \frac{\gamma}{\Lambda^2} W_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$$

or rather retain instead R^2, $R_{\mu\nu}^2$, $R_{\mu\nu} T^{\mu\nu}$, ...

[contrary to pure photon-theory, the α and β couplings in (40) now run due to gravitational interactions from ∂^2's in $\mathcal{L}^{(0)}$ (loops that generate R^2 & $R_{\mu\nu}^2$ can be turned into F^4-like corrections, or more prosaically  $\sim \frac{F^4}{M_{pl}^4}$. In particular, they run additively, i.e. α & β are generated by $\mathcal{L}^0$ at Λ even if they were zero at a matching scale Λ_0.

The status of γ in (40) is instead different: the $\mathcal{L}^{(0)}$ and α, β in $\mathcal{L}^{(1)}$ are invariant under the duality transformation $F_{\mu\nu} \leftrightarrow \tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$, whereas $\gamma \to -\gamma$. So this symmetry is unbroken if $\gamma = 0$ and loop can't generate it, as they require insertions of the spurion γ itself, $\beta_\gamma \propto \gamma$. Notice that in QED integrating out the electron γ is generated by  which scales as $$\frac{e^2}{M_{pl}} \int \frac{d^4k}{(2\pi)^4} \frac{\not{k}}{(\not{k} - m_e + \not{p}_i)^3} \supset \frac{e^2}{M_{pl}} \underbrace{p_i^\gamma p_j^\gamma p^h p^h}_{\substack{\text{(pull-out momenta} \\ \text{to build the gauge-invariant op.)}}} \int \frac{d^4k}{(2\pi)^4} \frac{\not{k}}{(\not{k} - m_e)^7} \sim \frac{e^2}{16\pi^2} p^\gamma p^\gamma p^h p^h \frac{1}{M_{pl} m_e^2} \Rightarrow \frac{FFW e^2}{16\pi^2 m_e^2}$$

$$\Rightarrow \frac{\gamma_{QED}}{\Lambda_{QED}^2} \sim \frac{e^2}{16\pi} \frac{1}{m_e^2}.$$]