# Topics in Effective Field Theory

## PhD-course, Rome 2020 by B. Bellazzini

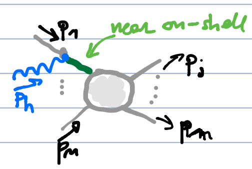
Lecture 8th

---

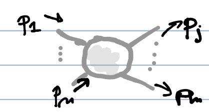
— Bootstrapping GR —

We have seen in L6 that the field $h_{\mu\nu}(x)$ associated to a massless spin-2 particle is Lorentz-covariant only up to gauge transformations. In turn, S-matrix elements $\epsilon^{\sigma}_{\mu\nu}(p)\, \mathcal{M}^{\mu\nu}(p)$ are Lorentz-cov. only if satisfy the Ward identity $\mathcal{M}_{\mu\nu}(p)\, p^\mu = 0$.

This condition was shown in L7 to be consistent with the absorption of a soft-graviton




vs



$\mathcal{M}(X \to Y)$

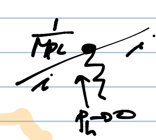
$$(1)\quad \mathcal{M}(h(\underset{\to 0}{p_h})\, X \to Y) = \left[\sum_{\text{incoming}} \frac{g_i}{2p_i\cdot p_h}\, p_i^\mu p_i^\nu\, \epsilon^{\sigma_h}_{\mu\nu}(p_h) - \sum_{\text{outgoing}} \frac{g_j}{2p_j\cdot p_h}\, p_j^\mu p_j^\nu\, \epsilon^{\sigma_h}_{\mu\nu}(p_h)\right] \mathcal{M}(X \to Y)$$

$$\left(\langle p_i \sigma | T_{\mu\nu}(0) | p_i \sigma' \rangle = g_i\, p_i^\mu p_i^\nu\, \delta_{\sigma\sigma'} \qquad \mathcal{L}_I = \int T_{\mu\nu} h_{\mu\nu}\, d^4x \right)$$

$$(2)\quad \boxed{\mathcal{M}_{\mu\nu}\, p_h^\mu = 0 \iff \sum_{\text{incoming}} g_i\, p_i^\mu - \sum_{\text{outgoing}} g_j\, p_j^\mu = 0}$$

only if the gravitational coupling $g_i$ of the soft graviton to $T_{\mu\nu}$ is universal:

$$(3)\quad \boxed{g_i = i\text{-independent} \equiv \frac{1}{M_{Pl}}}$$



(in the following we redefine $T_{\mu\nu} h_{\mu\nu} \to \frac{T_{\mu\nu} h^{\mu\nu}}{M_{Pl}}$)

This is the way quantum-mechanics + Relativity imply the equivalence principle

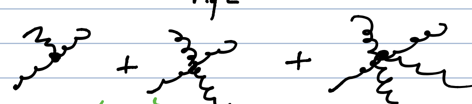
Now, for $g_i = 1/M_{PL}$ for all particles, the (2) means that $T_{\mu\nu}$ must be exactly conserved (as expected, btw, by $h_{\mu\nu}T^{\mu\nu}$ being Lorentz-invariant while the $h_{\mu\nu}$ is covariant up to a gauge)

(4) $\partial_\mu T^{\mu\nu} = 0$

Since the graviton carries itself energy-momentum, it's clear that $h_{\mu\nu}T^{\mu\nu}$ gives rise to self-interactions  $\frac{1}{M_{PL}} \partial_\mu^2 h^3$

In fact, it must give rise to infinitely many interactions since $T^{free}_{\mu\nu}(h,\partial^2)$ is obtained from $\mathcal{L}^{(2)}_{free}(h^2,\partial^2)$, but once the interactions $h_{\mu\nu}T^{free}_{\mu\nu}(\partial^2, h^2)$ are added the actual conserved $T_{\mu\nu}$ is redefined:

(5) $$T_{\mu\nu} = T^{free}_{\mu\nu}(h^2,\partial^2) + \frac{1}{M_{PL}} T^{(1)}_{\mu\nu}(h^3,\partial^2) + \frac{1}{M_{PL}^2} T^{(2)}_{\mu\nu}(h^4,\partial^2) + \dots$$

corresponding to  $+ \dots = h_{\mu\nu}T_{\mu\nu}\frac{1}{M_{PL}}$

initial seed from KT
that's why it's universal

from consistency of $\partial_\mu T^{\mu\nu}=0$ including the int.

We are now going to show how to effectively resum this series and recover GR as the only consistent theory, at leading order in derivatives (hence in the IR), once the coupling $\frac{1}{M_{PL}}T_{\mu\nu}h^{\mu\nu}$ is finite (corresponding to the long-range $1/r^2$ force at large distances).

To this end, we observe that the infinite series is generated because the interaction added, $T_{\mu\nu}h^{\mu\nu}\frac{1}{M_{PL}}$, contains derivatives of the field and it thus changes the conserved current.

[We recall that a conserved Noether current is defined by the response of the action to local version of the continuous symmetry: $S[\Phi] = S[\Phi + \varepsilon\delta\Phi]$ $\forall \varepsilon \in \mathbb{R}$

$\Rightarrow S[\phi+\varepsilon(x)\delta\Phi] = S[\phi] - \int \partial_\mu\varepsilon(x) J^\mu(x)$ defines $J^\mu$ = Noether

If one add $\delta S_I$ to a symmetric action $S^0$, and $\delta S_I$ is without derivatives and also invariant under the global transformation, it will not contribute to $J_\mu = J_\mu^0$]

## — $1^{st}$-order formalism: generalities

These observations bring us to the $1^{st}$-order formalism where the free theory is expressed in terms of first derivatives only (at the price of working with auxiliary fields), and most importantly where the interaction "field-current", $h_{\mu\nu} T^{\mu\nu}$, will not involve derivatives so that the theory will automatically be consistent (and reproduce the $2^{nd}$-order theory with second derivatives once the auxiliary fields are integrated out).

We have used it already in L7 to bootstrap YM as the only consistent theory of several self-interacting massless spin-1 particles, because we were facing a similar problem of finding $J_\mu^a$ such that $J_\mu^a A_\mu^a$ does not break the symmetry and leave $J_\mu^a$ conserved.

Let's recall the essential ingredients:

- consider a set of dynamical variables $\vec{q}$ along with a set of auxiliary (i.e. non-dynamical) ones $\vec{P}$, such that the free theory contains only one derivative, schematically:

free theory: $1^{st}$-order

$$(6) \quad \boxed{\mathcal{L}^{1-st}_{free}(\vec{q}, \dot{\vec{q}}, \vec{P}) = \vec{P}\dot{\vec{q}} - \frac{\vec{P}^2}{2}} \quad \xRightarrow{\text{e.o.m. } \vec{P}} \begin{cases} \vec{P} = \dot{\vec{q}} \\ \mathcal{L}^{2nd}_{free}(\vec{q}, \dot{\vec{q}}) = \frac{\dot{\vec{q}}^2}{2} \end{cases}$$

In the YM example: $\vec{q} = \vec{A}_\mu$ and $\vec{P} = \vec{F}_{\mu\nu}$, $\mathcal{L}^{1st}_{free} = -\frac{1}{2}\vec{F}_{\mu\nu}(\partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu) + \frac{1}{4}\vec{F}_{\mu\nu}\vec{F}_{\mu\nu}$

(where the factors of $1/2$ and $1/4$ come from the antisymmetric nature of $F_{\mu\nu}$ while we are summing over all $\mu\nu$-pairs, not just $\mu<\nu$: $\sum_{\mu<\nu} \vec{F}_{\mu\nu}\partial_\mu \vec{A}_\nu = \frac{1}{2}\sum_{\mu,\nu} \vec{F}_{\mu\nu}(\partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu)$, $\sum_{\mu\nu}\frac{1}{4}\vec{F}_{\mu\nu}\vec{F}^{\mu\nu} = \frac{1}{2}\sum_{\mu<\nu}\vec{F}_{\mu\nu}\vec{F}^{\mu\nu}$]

- Identify the conserved current associated with the continuous symmetry of $\mathcal{L}^{1st}_{free}$

(7) $\boxed{\begin{cases} \vec{q} \to R\vec{q} \\ \vec{p} \to R\vec{p} \end{cases}}$ with $R$ some "sort of rotation" that leaves $\mathcal{L}^{1st}$ invariant, i.e.

$\vec{q}\cdot\vec{p} \to \vec{q}\vec{p}$ and $\vec{p}\cdot\vec{p} \to \vec{p}\vec{p}$

At the infinitesimal level, $R(\varepsilon) = 1 + i\,\varepsilon^i T^i + \dots$ with $T^i$ a set of generator $\vec{T}$ and $R = R(\vec{\varepsilon}) = 1 + i\,\vec{\varepsilon}\vec{T} + \dots$, so that $i(T^i)^{jk} = f^{ijk}$. For actual $SO(3)$ rotations acting on 3-vectors $f^{ijk} = \varepsilon^{ijk}$ — the fully antisym. 3-tensor with $i,j,k = 1,2,3$, and one can write $R(\varepsilon)\vec{q} = \vec{q} + \vec{\varepsilon}\wedge\vec{q} + \dots$ We will adopt this notation also for other $f^{ijk}$, namely $(\vec{\varepsilon}\wedge\vec{q})^i \equiv f^{ijk}\varepsilon^j q^k$. By the usual definition of Noether current we get:

(8) $\boxed{\vec{J} = \vec{p}\wedge\vec{q}}$ (in YM $\vec{J}_\mu = \vec{F}_{\mu\nu}\wedge\vec{A}_\nu$)

which is conserved on-shell (or $\vec{p} = \dot{\vec{q}}/2$, $\dot{\vec{p}} = 0$), it contains no derivative, and transform like $\vec{q}$ & $\vec{p}$ under the "rotation" $R(\varepsilon)$

- Couple $\vec{q}$ to $\vec{J}$ preserving the global symmetry

(9) $\mathcal{L}^{1st}_3(\vec{q},\dot{\vec{q}},\vec{p}) = \mathcal{L}^{1st}_{free} + g\,\vec{J}\cdot\vec{q} = \vec{p}\dot{\vec{q}} - \frac{\vec{p}^2}{2} + g(\vec{p}\wedge\vec{q})\vec{q}$

$1^{st}$ order interacting with still conserved curr.

$\Rightarrow$ $\vec{J} = \vec{p}\wedge\vec{q}$ is still exactly conserved even in the interacting theory since the interaction does not include derivatives and it can't thus change the Noether current that arise from the N.T.

( in YM: $\vec{J}_\mu = \vec{F}_{\mu\nu}\wedge\vec{A}_\nu$ $\quad \partial_\mu\vec{J}^\mu = \partial_\mu\vec{F}_{\mu\nu}\wedge\vec{A}_\nu + \vec{F}_{\mu\nu}\wedge\partial_\mu\vec{A}_\nu \big|_{on-shell} = 0$ )

- solve for the auxiliary fields:

(10) $\dot{\vec{q}} - \vec{p} - g\,\vec{q}\wedge\vec{q} = 0 \quad \Rightarrow \mathcal{L}^{2nd}_{int}(\vec{q},\dot{\vec{q}}) = \mathcal{L}^{1st}_{int}(\vec{q},\dot{\vec{q}},\vec{p}(\vec{q},\dot{\vec{q}})) = \vec{p}(\dot{\vec{q}} - \frac{\vec{p}}{2} - g\,\vec{q}\wedge\vec{q})$

$= \vec{p}(\vec{p} - \vec{p}/2) = \frac{1}{2}\vec{p}^2 = \frac{1}{2}(\dot{\vec{q}} + g\,\vec{q}\wedge\vec{q})^2$

$\Rightarrow$ (11) $\boxed{\mathcal{L}^{2nd}_{int}(\vec{q},\dot{\vec{q}}) = \frac{1}{2}(\dot{\vec{q}} + g\,\vec{q}\wedge\vec{q})^2}$ $\longleftarrow$ consistent self-interacting theory

[In the YM case $\mathcal{L}^{2nd}_{int}(A_\mu,\partial_\mu A_\nu) = -\frac{1}{2}\sum_{\mu<\nu}\vec{F}_{\mu\nu}\vec{F}_{\mu\nu} = -\frac{1}{4}\vec{F}_{\mu\nu}\vec{F}_{\mu\nu} = -\frac{1}{4}(\partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu + g\vec{A}_\mu\wedge\vec{A}_\nu)^2$

or step by step: $+\vec{F}_{\mu\nu} - (\partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu) + g\,\vec{A}_\mu\wedge\vec{A}_\nu \underset{on-shell}{=} 0 \Rightarrow \frac{1}{2}\vec{F}_{\mu\nu}\left(\frac{\vec{F}_{\mu\nu}}{2} - \partial_\mu A_\nu + g\,\vec{A}_\mu\wedge\vec{A}_\nu\right) \underset{on\ shell}{=} \frac{1}{4}\vec{F}_{\mu\nu}\vec{F}_{\mu\nu}$]

## 1st order formalism: the case of gravity — o —

Let's adopt this strategy to the theory of a graviton that we want to couple to $T_{\mu\nu}$ conserved exactly.

The 1st order $(p\dot q - \frac{p^2}{2})$-type of Lagrangian for the free $m=0$ spin-2 is

$$(12) \qquad \mathcal{L}^{1\text{-st}}_{\text{free}}(\partial h, \Gamma^{\alpha}_{\mu\nu}) = -\partial_\alpha h^{\mu\nu}\left(\Gamma^{\alpha}_{\mu\nu} - \delta^{\alpha}_{\nu}\Gamma^{\beta}_{\mu\beta}\right) + \eta^{\mu\nu}\left(\Gamma^{\beta}_{\mu\nu}\Gamma^{\alpha}_{\beta\alpha} - \Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\beta\nu}\right)$$

or, as it is more often written after integrating by parts:

$$(13) \qquad \mathcal{L}^{1\text{-st}}_{\text{free}} = h^{\mu\nu}\left(\partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\alpha}\right) + \eta^{\mu\nu}\left(\Gamma^{\beta}_{\mu\nu}\Gamma^{\alpha}_{\beta\alpha} - \Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\beta\nu}\right)$$

where $\Gamma^{\alpha}_{\mu\nu}$ is the auxiliary field, taken symmetric w.r.t. the lower indices

$$(14) \qquad \Gamma^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\nu\mu}$$

(at the end we will recognize tha $\Gamma^{\alpha}_{\mu\nu}$ is the Christoffel connection)

This quadratic action is appropriate for the free massless spin-2 because it gives the correct e.o.m.

$$(15) \qquad -\partial_\alpha h^{\mu\nu} + \frac{1}{2}\left(\partial_\beta h^{\mu\beta}\delta^{\nu}_{\alpha} + \partial_\beta h^{\nu\beta}\delta^{\mu}_{\alpha}\right) + \Gamma^{\beta}_{\alpha\beta}\eta^{\mu\nu} + \frac{1}{2}\left(\eta^{\beta\sigma}\delta^{\nu}_{\alpha}\Gamma^{\mu}_{\beta\sigma} + \mu \leftrightarrow \nu\right) - \eta^{\mu\beta}\Gamma^{\nu}_{\alpha\beta} - \eta^{\beta\nu}\Gamma^{\mu}_{\beta\alpha} = 0$$

(we used $\Gamma^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\nu\mu}$ & $h^{\mu\nu} = h^{\nu\mu}$)

$$(16) \qquad 0 = \partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \frac{1}{2}\left(\partial_\nu \Gamma^{\alpha}_{\mu\alpha} + \partial_\mu \Gamma^{\alpha}_{\nu\alpha}\right) \qquad \left(\frac{\delta S^{1st}_{free}}{\delta h^{\mu\nu}} = 0\right)$$

as it is visible after some manipulations to put put them in the standard Fierz-Pauli form for $m=0$ & $J=2$.

Explicitly:

- contracting $(\nu$-$\alpha)$-indices in (15) we get $\partial_\beta h^{\mu\beta} = -\Gamma^{\mu}_{\beta\sigma}\eta^{\beta\sigma}$
- contracting $(\mu$-$\nu)$-indices in (15) we get $\partial_\alpha h = (D-2)\Gamma^{\beta}_{\alpha\beta}$ with $D=4$ spacetime dimensions

• Plugging the $\partial_\beta h^{\mu\beta} = -\Gamma^\mu_{\beta\sigma}\eta^{\beta\sigma}$ in (15) again: $-\Gamma^\nu_{\alpha\mu}\eta^{\beta\mu} - \Gamma^\mu_{\alpha\beta}\eta^{\beta\nu} = \partial_\alpha h^{\mu\nu} - \eta^{\mu\nu}\frac{\partial_\alpha h}{D-2}$

that with all lowered indices is:

(15') $-\Gamma_{\alpha\mu\nu} - \Gamma_{\alpha\nu\mu} = \partial_\alpha h_{\mu\nu} - \eta_{\mu\nu}\frac{\partial_\alpha h}{D-2}$ $\Gamma_{\alpha\mu\nu} \equiv \Gamma^\beta_{\alpha\mu}\eta_{\beta\nu} = \Gamma_{\mu\alpha\nu}$ sym. w.r.t. first two indices

• $(15')_{\alpha\mu\nu} + (15')_{\mu\nu\alpha} - (15')_{\nu\alpha\mu} = -2\Gamma_{\mu\alpha\nu} = -(\partial_\alpha \bar{h}_{\mu\nu} + \partial_\mu \bar{h}_{\nu\alpha} - \partial_\nu \bar{h}_{\mu\alpha})$

where $\bar{h}_{\mu\nu} \equiv -(h_{\mu\nu} - \eta_{\mu\nu}\frac{h}{D-2})$ $\bar{h} = -h(1 - \frac{D}{D-2}) = \frac{2}{D-2}h$

from where we extract

(17) $\boxed{\Gamma^\alpha_{\mu\nu} = \frac{1}{2}(\partial_\mu \bar{h}_\nu{}^\alpha + \partial_\nu \bar{h}_\mu{}^\alpha - \partial^\alpha \bar{h}_{\mu\nu})}$ that is, $\Gamma^\alpha_{\mu\nu}$ = linearized Christoffel symbol for $\bar{h}_{\mu\nu}$

• Plug (17) & $\frac{1}{2}\partial_\alpha\bar{h} = \Gamma^\beta_{\alpha\beta}$ in (16) to get ($\bar{h}_{\mu\nu} \equiv -(h_{\mu\nu} - \eta_{\mu\nu}\frac{h}{D-2})$)

(16') $\boxed{0 = \Box\bar{h}_{\mu\nu} + \partial_\mu\partial_\nu\bar{h} - \partial_\mu\partial_\alpha\bar{h}_{\nu\alpha} - \partial_\alpha\partial_\nu\bar{h}_{\mu\alpha} = 0}$ (it's equivalent to (20) of L7/6 with $T_{\mu\nu}=0$ since the trace $=0$ gives $\Box\bar{h} = \partial_\mu\partial_\nu\bar{h}^{\mu\nu}$)

Which is the correct e.o.m for $m=0$ spin-2 particle. (It is easy to check gauge-invariance $\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$. Notice that working with $\bar{h}_{\mu\nu} = -(h_{\mu\nu} - \frac{h}{D-2}\eta_{\mu\nu})$ or with $h_{\mu\nu}$ is ok either way given that both have non-vanishing overlap $\langle 0|\bar{h}_{\mu\nu}(0)|p,\sigma\rangle = -\varepsilon^\sigma_{\mu\nu}(p)$

The trace part is clearly gauge dependent: one can work with $\xi_\mu{}^\mu = 0$ as we have done so far, since under Lorentz $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\Omega_\nu + \partial_\nu\Omega_\mu$ up to the overall $\Lambda^{-1}{}_\mu{}^{\bar\mu}\Lambda^{-1}{}_\nu{}^{\bar\nu}$ factors, so that one can choose $2\partial_\mu\Omega^\mu = -h$ & set $h=0$

The next step is thus to find a conserved $T_{\mu\nu}$ to couple to $h^{\mu\nu}$ in the 1st order formalism (such that it will remain conserved even with this interaction).

In order to extract $T_{\mu\nu}$ from $\mathcal{L}^{1st}_{free}$ we perform as usual the local version ($\epsilon_\mu = \epsilon_\mu(x)$) of the symmetry of interest, spacetime translations in our case:

(18) $x^\mu \to x^\mu + \epsilon^\mu(x)$ (local translations = infinit. Diff's)

Now, if were to introduce an auxiliary metric $H_{\mu\nu}$ that transforms covariantly, we knew we could make the theory invariant even under diffs (17); it would just require to covariantize $\mathcal{L}^{1st}_{free}$:

For example:

(19) $S[\phi,\partial\phi]=\int d^4x\left(\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi+\dots\right)\Rightarrow S_{covar.}=\int d^4x\,\sqrt{|H|}\,\frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi\,H^{\mu\nu}+\dots$ Diff invariant

$(H^{\mu\nu}\to H^{\mu\nu}+\nabla^\mu\epsilon^\nu+\nabla^\nu\epsilon^\mu)$

$$\Rightarrow\ \boxed{\delta S_{covariant}=0=-\int d^4x\,\frac{1}{2}\underbrace{(\nabla^\mu\epsilon^\nu+\nabla^\nu\epsilon^\mu)}\sqrt{|H|}\left[T_{\mu\nu}-\frac{2}{\sqrt{|H|}}\frac{\delta S_{cov}}{\delta H^{\mu\nu}}\right]=0}$$

from the usual Noether current definition $\delta S=-\int d^4x\,\partial_\mu\epsilon^i\,J^i_\mu$, just now "$i$" = spacetime index $\nu$, i.e. $J^i_\mu=T_\mu{}^\nu$, and we took it symmetric $T_{\mu\nu}=T_{\nu\mu}$ and covariant.

so that going back to the original action $H_{\mu\nu}=\eta_{\mu\nu}$

$$(20)\quad \boxed{T_{\mu\nu}=+\frac{2}{\sqrt{|H|}}\frac{\delta S_{cov}[H_{\mu\nu},\phi,\nabla\phi]}{\delta H^{\mu\nu}}\bigg|_{H^{\mu\nu}=\eta^{\mu\nu}}\quad (\text{Rosenfeld } T_{\mu\nu} \text{ definition})}$$

In our case, the eq. (16') involves $\bar{h}_{\mu\nu}=-(h_{\mu\nu}-\frac{h}{D-2}\eta_{\mu\nu})$ so that the correct equation of the type $\Box h_{\mu\nu}\dots=T_{\mu\nu}$ (hence $\Box h\dots=T$) is obtained from $\Box\overbrace{(h_{\mu\nu}-\frac{1}{D-2}h\,\eta_{\mu\nu})}^{\bar{h}_{\mu\nu}}+\dots=T_{\mu\nu}-\frac{\eta_{\mu\nu}T}{D-2}$ which is equivalent to couple the $h^{\mu\nu}$ to $(T_{\mu\nu}-\frac{\eta_{\mu\nu}T}{D-2})\equiv\tau_{\mu\nu}$.

A nice trick to calculate directly $\tau_{\mu\nu}$ is varying the action w.r.t. the external metric density $H^{\mu\nu}\sqrt{|H|}=\psi^{\mu\nu}$:

$$(21)\quad \frac{2}{\sqrt{|H|}}\frac{\delta S^{cov}}{\delta H^{\mu\nu}}=\frac{2}{\sqrt{|H|}}\int\frac{\delta S^{cov}}{\delta\psi^{\alpha\beta}}\frac{\delta\psi^{\alpha\beta}}{\delta H^{\mu\nu}}=\frac{2}{\sqrt{|H|}}\left[\frac{\delta S^{cov}}{\delta\psi^{\mu\nu}}\sqrt{|H|}-\frac{1}{2}\frac{\delta S^{cov}}{\delta\psi^{\alpha\beta}}H^{\alpha\beta}\sqrt{|H|}\,H_{\mu\nu}\right]=2\frac{\delta S^{cov}}{\delta\psi^{\mu\nu}}-\frac{\delta S^{cov}}{\delta\psi^{\alpha\beta}}H^{\alpha\beta}H_{\mu\nu}$$

and therefore

$$(22)\quad T_{\mu\nu}=2\frac{\delta S^{cov}}{\delta\psi^{\mu\nu}}-\frac{\delta S^{cov}}{\delta\psi^{\alpha\beta}}H^{\alpha\beta}H_{\mu\nu}\ \Rightarrow\ T=(2-D)\frac{\delta S^{cov}}{\delta\psi^{\mu\nu}}H^{\mu\nu}\ \Rightarrow\ T_{\mu\nu}=2\frac{\delta S^{cov}}{\delta\psi^{\mu\nu}}-\frac{T}{(2-D)}H_{\mu\nu}\quad \text{i.e.}$$

$$(23)\quad \boxed{2\frac{\delta S^{cov}}{\delta\psi^{\mu\nu}}\bigg|_{H^{\mu\nu}=\eta^{\mu\nu}}=\left(T_{\mu\nu}-\frac{T}{(D-2)}H_{\mu\nu}\right)\bigg|_{H^{\mu\nu}=\eta^{\mu\nu}}=T_{\mu\nu}-\frac{T}{D-2}\eta_{\mu\nu}=\tau_{\mu\nu}}$$

So, covariantizing $\mathcal{L}^{1-st}_{free}$ in Eq. (13), $\mathcal{L}^{1-st}_{free}=h^{\mu\nu}(\partial_\alpha\Gamma^\alpha_{\mu\nu}-\partial_\nu\Gamma^\alpha_{\mu\alpha})+\eta^{\mu\nu}(\Gamma^\beta_{\mu\nu}\Gamma^\alpha_{\beta\alpha}-\Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\beta\nu})$

$$(24)\quad \mathcal{L}^{1-st,cov}_{free}=\sqrt{|H|}\,h^{\mu\nu}(\nabla_\alpha\Gamma^\alpha_{\mu\nu}-\nabla_\nu\Gamma^\alpha_{\mu\alpha})+H^{\mu\nu}\sqrt{|H|}\,(\Gamma^\beta_{\mu\nu}\Gamma^\alpha_{\beta\alpha}-\Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\beta\nu})$$

the resulting $\tau_{\mu\nu}$ we need to couple is (*)

(*) there would be an extra factor of 2 from the definition that we are neglecting because it cancels against $\frac{1}{2}h^{\mu\nu}\tau_{\mu\nu}$ ($\vec{J}\cdot\vec{q}$ with the indep. $\vec{q}$)

$$(25)\quad \boxed{\tau_{\mu\nu}=(\Gamma^\beta_{\mu\nu}\Gamma^\alpha_{\beta\alpha}-\Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\beta\nu})+\partial^\alpha\tau_{\alpha\mu\nu}}\quad \text{with}\quad \tau_{\alpha\mu\nu}=-\tau_{\mu\alpha\nu},\ \tau_{\alpha\mu\nu}=\tau_{\alpha\nu\mu}$$

$$\tau_{\alpha\mu\nu}\sim\mathcal{O}(h\Gamma)$$

The last term is a superpotential ambiguity of any $\tau_{\mu\nu}$ (or $T_{\mu\nu}$), which is never important, neither to define the global charges $P_\mu=\int d^3x\,T_{\mu 0}$, nor for the local e.o.m. since we can absorb it

Specifically, one can perform a field redefinition $h_{\mu\nu} \to h_{\mu\nu} + \Delta_{\mu\nu}(h\Gamma)$ such that $\partial^\alpha \tau_{\alpha\mu\nu}$ is removed

[this is done writing the Einstein tensor $G_{\mu\nu}$ as $\partial_\alpha G_{\alpha\mu\nu}$, see Gravitation & Cosmology ch.7 by Weinberg)

Discarding the irrelevant superpotential term we get from the coupling $\int \tau_{\mu\nu} h_{\mu\nu}$:

(26)
$$\boxed{\begin{aligned} S^{1-st}_{cubic} &= \int d^4x\, h^{\mu\nu}(\partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha}) + (\eta^{\mu\nu} + h^{\mu\nu})(\Gamma^\beta_{\mu\nu}\Gamma^\alpha_{\beta\alpha} - \Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\beta\nu}) \\ &= \int d^4x\, \overbrace{(\eta^{\mu\nu} + h^{\mu\nu})}^{g^{\mu\nu}} \underbrace{\left[(\partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha}) + (\Gamma^\beta_{\mu\nu}\Gamma^\alpha_{\beta\alpha} - \Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\beta\nu})\right]}_{= R_{\mu\nu}(\Gamma)} + \underbrace{\text{total derivative}}_{(\eta^{\mu\nu}(\partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha}))} \\ &= \int d^4x\, g^{\mu\nu} R_{\mu\nu}(\Gamma) = S_{palatini} \end{aligned}}$$

where we recognize the Palatini formulation of GR, where $g^{\mu\nu} \equiv \eta^{\mu\nu} + h^{\mu\nu}$ & the connection $\Gamma^\alpha_{\mu\nu}$ inside the Ricci tensor are varied independently, delivering the Einstein's equations in vacuum

(27)
$$\left\{ \begin{aligned} &\frac{\delta S^{1-st}_{cubic}}{\delta g^{\mu\nu}} = 0 \\ &\frac{\delta S^{1-st}_{cubic}}{\delta \Gamma^\alpha_{\mu\nu}} = 0 \end{aligned} \right. \Rightarrow \left\{ \begin{aligned} &R_{\mu\nu}(\Gamma) = 0 \\ &\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \end{aligned} \right.$$

← (just repeat the steps from 15 to 18 above where $h^{\mu\sigma} \to (h^{\mu\sigma} + \eta^{\mu\sigma})$)

One interesting point is that the full self-interacting theory is diffeomorphism-invariant

(28)
$$g^{\mu\nu} \to g^{\mu\nu} + \nabla^\mu \epsilon^\nu + \nabla^\nu \epsilon^\mu$$

which is the infinitesimal diff-transformation that supersedes the linear-infinitesimal Diff's ($h^{\mu\nu} \to h^{\mu\nu} + \partial^\mu \epsilon^\nu + \partial^\nu \epsilon^\mu$), very much the new gauge invariance of the YM-theory when $\partial_\mu \Omega$ is replaced by $U^{-1}\partial_\mu U$.