Topics in Effective Field Theory

PhD-course, Rome 2020 by B. Bellazzini

Lecture 7th

We have seen in L6 that Lorentz-invariance of the S-matrix means that

(1) $$S(p_1\sigma_1, p_2\sigma_2, \dots \to p_1'\sigma_1', p_2'\sigma_2', \dots) = S(\Lambda p_1\bar\sigma_1, \Lambda p_2\bar\sigma_2, \dots \to \Lambda p_1'\bar\sigma_1', \Lambda p_2'\bar\sigma_2', \dots)\, \mathcal{L}_{\bar\sigma_1\sigma_1}(W(\Lambda,p)) \dots \mathcal{L}^*_{\bar\sigma_1'\sigma_1'}(W(\Lambda,p))$$

where $\mathcal{L}_{\bar\sigma\sigma}(W(\Lambda,p))$ are little-group transformations. Boosting $p \to \Lambda p$ generates a "rotation" given by $\mathcal{L}$.

For massless particles the little-group matrix is just a phase $\mathcal{L}_{\sigma'\sigma}(W(\Lambda,p))\big|_{m=0} = \delta_{\sigma'\sigma}\, e^{i\sigma\theta(W,p)}$.

The way one constructs S-matrix elements transforming as in (1) is via the polarizations $u_\ell^\sigma(p)$ (and $v_\ell^\sigma(p)$ for antiparticles, if particles = antiparticle one can choose $v_\ell^\sigma = u_\ell^{\sigma\,*}$) that are such that

(2) $$D_{\ell\ell'}(\Lambda)\, u^\sigma_{\ell'}(p) = u_\ell^{\sigma'}(\Lambda p)\, \mathcal{L}_{\sigma'\sigma}(W(\Lambda,p))$$

little group index

Lorentz index

where $D_{\ell\ell'}(\Lambda)$ is a irrep of Λ, that is ℓ & ℓ' are Lorentz indexes whereas σ & σ' are little group index

In this way is enough to contract the $u_\ell^\sigma(p)$ with some amputed Lorentz tensor $M^\ell(p_1,\dots,p_1',\dots)$ built out of the p_i, p_i', to get the correct transformation rule (1). This is achieved with the LSZ prescription acting on local fields $\phi_\ell(x) = \sum_\sigma \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}} \left\{ e^{-iE_pt + i\vec p\vec x}\, a(p,\sigma)\, u_\ell^\sigma(p) + \dots \right\}$ which transform covariantly under Lorentz, namely

(3) $$U(\Lambda)\,\phi_\ell(x)\,U^\dagger(\Lambda) = D_{\ell\ell'}(\Lambda^{-1})\,\phi_{\ell'}(\Lambda x)$$

Lorentz index contracted

$\Rightarrow$ pick the little-group $\mathcal{L}$

Example: massive spin-1 in $A_\mu(x)$ $m \neq 0$ $\quad u_\ell^\sigma(p) = \varepsilon_\mu^{\sigma=\pm,0}(p)$ $\quad \mathcal{M}(p_1\sigma_1,\dots) = \varepsilon^\sigma_\mu(p)\cdot \mathcal{M}^\mu(p_1\dots)$ where

$M^\mu = \text{amputed}\big|_{LSZ} \langle 0|T\hat A_\mu(p)\dots\rangle$

Remarkably, we have seen that (3) & (2) cannot be realized for m=0 in A_μ & $h_{\mu\nu}$

We have seen that the vector and tensor for massless particles transform like (3) only up to a gauge shift:

$$(4)\quad \begin{cases} U(\Lambda) A_\mu(x) U^\dagger(\Lambda) = (\Lambda^{-1})_\mu{}^\nu A_\nu(\Lambda x) + \partial_\mu \Omega \\ U(\Lambda)\, h_{\mu\nu}(x)\, U^\dagger(\Lambda) = (\Lambda^{-1})_\mu{}^{\bar\mu} (\Lambda^{-1})_\nu{}^{\bar\nu}\, h_{\bar\mu\bar\nu}(\Lambda x) + \partial_\mu \Omega_\nu + \partial_\nu \Omega_\mu \end{cases}$$

because the polarizations shift

$$(5)\quad \begin{cases} \varepsilon_\mu^{\sigma=\pm}(p) \longrightarrow (\Lambda^{-1})_\mu{}^\nu\, \varepsilon_\nu^{\sigma=\pm}(\Lambda p) + p_\mu f(p) \\ \varepsilon_{\mu\nu}^{\sigma=++,--}(p) = \varepsilon_\mu^\sigma \varepsilon_\nu^\sigma \rightarrow (\Lambda^{-1})_\mu{}^{\bar\mu} (\Lambda^{-1})_\nu{}^{\bar\nu}\, \varepsilon^\sigma_{\bar\mu\bar\nu}(\Lambda p) + p_\mu f_\nu(p) + p_\nu f_\mu(p) \end{cases}$$

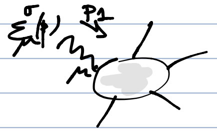
These transformations can be compatible with a Lorentz-invariant S-matrix, eq.(1), only if

on-shell gauge invariance, or Ward identities

$$(6)\quad \begin{aligned} &p_\mu M^\mu(p,\ldots) = 0\,, \\ &p_\mu M^{\mu\nu}(p,\ldots) = 0 \end{aligned}$$

where M^μ & $M^{\mu\nu}$ are the amputated Matrix elements, e.g.

$$(7)\qquad M(\gamma^\sigma X \to Y) = \varepsilon^\sigma_\mu(p_1)\, \mathcal{M}^\mu(p_i)$$



(For outgoing γ send $\varepsilon_\mu \to \varepsilon^*_\mu$)

where $M^\mu = \langle \text{out} | \hat{J}^\mu(p) | \text{in} \rangle$ and $M^{\mu\nu} = \langle \text{out} | \hat{T}^{\mu\nu}(p) | \text{in} \rangle$ are the linear couplings of the type $\int A_\mu J^\mu d^4x$, $\int d^4x\, h_{\mu\nu}(x) T^{\mu\nu}(x)$ in $\mathcal{L}_I(x)$.

—Soft-limits: charge & momentum conservation —— ∘ —— ∘ ——

We can now use Lorentz-invariance in the form of the Ward identities (6) for $m=0$ particles of spin-1 and spin-2 to derive charge conservation and energy-momentum conservation respectively. To do so, we compare a scattering amplitude $M(X \to Y)$ to one with an extra soft-photon attached, $p_\gamma \to 0$, say in the initial state $M(\gamma(\underset{\hookrightarrow 0}{p_\gamma}) X \to Y)$. Diagrammatically, it corresponds to:

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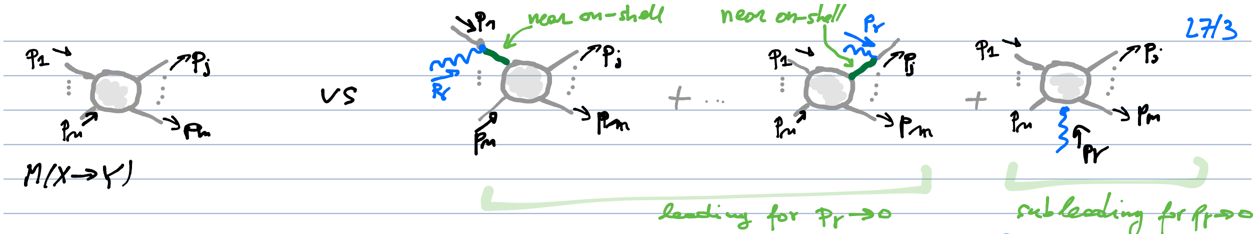
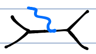
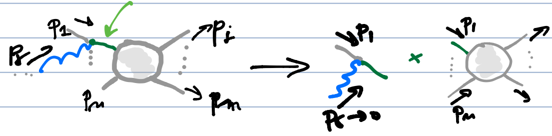


Clearly, the diagram where the photon is attached to internal lines, e.g.  or , gives a subleading contribution relative to the diagrams where γ is on the external lines because it is not putting one of the legs (near) on-shell, resulting in an enhancement by

(8) denominator: $\boxed{\dfrac{1}{(p_i \pm p_\gamma)^2 - m_i^2} \longrightarrow \dfrac{\pm 1}{2p_i\cdot p_\gamma}}$ the sign depending on p_i being in (+) or out (−) state

The line to which γ attaches in the in- or out-going state goes nearly on-shell and the amplitudes thus factorizes

(9) 

$$= \sum_{\sigma',\ell'} M^{\ell'}(X\to Y)\, u^{\sigma'}_{\ell'}(p_1)\cdot \frac{(\pm i)}{2p_1\cdot p_\gamma} \langle p_1\sigma'|J^\mu(0)|p_1\sigma\rangle\, \varepsilon^{\sigma_\gamma}_\mu(p_\gamma)$$

where we used that $\langle p'\sigma'|J^\mu(x)|p\sigma\rangle = \langle p'\sigma'|e^{+iPx} J^\mu(0)\, e^{-iPx}|p\sigma\rangle = e^{-i(p-p')x}\langle p'\sigma'|J_\mu(0)|p\sigma\rangle$

(or more explicitly in spinorial QED $i\sum_{\sigma'} M^{\ell}(X\to Y)\overbrace{\dfrac{u^{\sigma'}_{\ell}(p_1)\,\bar u^{\sigma'}_{\ell}(p_1)}{(2p_1\cdot p_\gamma)}}^{\text{propagator}}\ \overset{\text{vertex}}{\gamma^\mu}\ \overset{\text{external pol.}}{u^{\sigma}(p_1)\ \varepsilon^{\sigma_\gamma}_\mu(p_\gamma)}\cdot e_1 \leftarrow$ electric charge)

Now, by Lorentz covariance of $\langle p\sigma'|J^\mu(0)|p\sigma\rangle = e^{i(\sigma-\sigma')\theta}\Lambda^\mu{}_\nu\langle \Lambda p\,\sigma'|J_\mu(0)|\Lambda p\,\sigma\rangle$ we see that it must be proportional to p^μ (the only available 4-vector) and $\delta_{\sigma\sigma'}$,

(10) $\boxed{\langle p_i\,\sigma_i'|J_\mu(0)|p_i\sigma_i\rangle = e_i\, p_\mu}$

S-matrix definition of electric charge
(the matrix element eq when $p_\gamma \to 0$, i.e. $p\to p'$)

(For the case of QED with a spin-1/2 fermion $J^\mu = \bar\psi_i\gamma^\mu\psi_i e_i$, so that $\langle p\sigma'|J^\mu|p\sigma\rangle = \bar u^{\sigma'}(p_i)\gamma^\mu u^\sigma(p_i)\, e_i = p^\mu\delta^{\sigma\sigma'}e_i$).

This happens for every leg the photon attaches to:

$$(11)\quad M(\gamma(\underset{\to 0}{p_\gamma})\,X\to Y) = \left[\sum_{\text{incoming}} \frac{e_i}{2p_i\cdot p_\gamma}\, p^i_\mu\cdot \varepsilon^{\sigma_\gamma\,\mu}(p_\gamma) - \sum_{\text{outgoing}} \frac{e_j}{2p_j\cdot p_\gamma}\, p^j_\mu\cdot\varepsilon^{\sigma_\gamma}_\mu(p_\gamma\right] M(X\to(Y))$$

Lorentz invariance requires the Ward identity $\varepsilon^\mu \to \varepsilon^\mu + p_\gamma^\mu$ with M being invariant whereas from (11) we get

$$(12)\quad M(\gamma(\underset{\hookrightarrow 0}{p_\gamma})X\to Y) = M(\gamma(\underset{\hookrightarrow 0}{p_\gamma})X\to Y) + \frac{1}{2}\cdot\Big[\sum_{\text{incoming}} e_i - \sum_{\text{outgoing}} e_j\Big] M(X\to Y)$$

which thus implies charge conservation:

$$(13)\quad \boxed{\sum_{\text{incoming}} e_i - \sum_{\text{outgoing}} e_i = 0}$$

<u>charge conservation</u>

Consistency conditions of a Lorentz-invariant soft amplitude allowed us to derive charge conservation (charges are those e_i defined in 10, in the IR for very soft photon $\overset{p_\gamma\to 0}{\underset{p\ \ e_i\ \ p}{\uparrow\wr}}$)

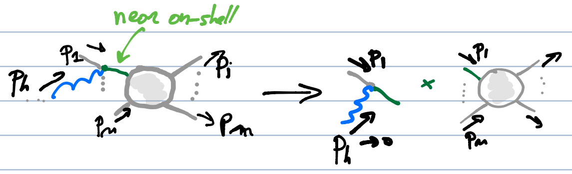
Clearly, at the field theory level this is just the statement $\partial_\mu J^\mu = 0$ $Q|p_i\sigma\rangle = e_i|p\sigma\rangle$

Remark: the non-minimal coupling to identically conserved currents $J^\mu_{id} = \partial_\alpha J^{\alpha\mu}$ with $J^{\alpha\mu} = -J^{\mu\alpha}$ does not contribute to the electric charges e_i, e.g. $\int A_\mu J^\mu_{id} \underset{\text{by parts}}{=} -\frac{1}{2} F_{\alpha\mu} J^{\mu\alpha}$, and therefore it gives a vanishing matrix element when $p_\gamma \to 0$. This is in agreement with $Q = \int d^3x\, J^0$ which receive no correction either: $\delta Q_{id} = \int d^3x\, \partial_\alpha J^{\alpha 0} \underset{\text{antisym.}}{=} \int d^3x\, \partial_i J^{i0} \underset{J^{i0}\text{ vanish fast enough @ infinity}}{=} 0$

Soft graviton scattering

A very similar story can be told for gravitons: let's add a very soft graviton to an amplitude $M(X\to Y)$, i.e. $M(h(\underset{\hookrightarrow 0}{p_h})X\to Y)$ and check consistency between factorization of the amplitude and its Lorentz-covariance (i.e. Ward-id)

near on-shell

(14) 

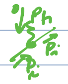
$$= \sum_{\sigma', \ell'} M^{\ell'}(X\to Y)\, u^{\sigma'}_{\ell'}(p_1)\cdot\frac{(\pm i)}{2p_1\cdot p_h}\langle p_1\sigma'|T^{\mu\nu}(0)|p_1\sigma\rangle\, \varepsilon^{\sigma_h}_{\mu\nu}(p_h)$$

(where we used that $\langle p_1\sigma'|T^{\mu\nu}(x)|p_1\sigma\rangle = e^{-i(p-p')x}\langle p'\sigma'|T^{\mu\nu}(0)|p\sigma\rangle$)

Like before, the Lorentz covariance of the $\langle p\sigma'|T_{\mu\nu}(0)|p\sigma\rangle$ defines "gravitational couplings"

& $\eta_{\mu\nu}$ is also consistent with Lorentz but it is contracted with $\varepsilon^{\sigma=\pm}_{\mu\nu} = \varepsilon^{\sigma}_{\mu}\varepsilon^{\sigma}_{\nu}$ where $\varepsilon^{\sigma}_{\mu}\varepsilon^{\sigma\mu}=0$

(15) $$\langle p_i' \sigma_i' | T_{\mu\nu}(0) | p_i \sigma_i \rangle = g_i \, p^i_\mu p^i_\nu \, \delta_{\sigma\sigma'}$$

IR gravitational coupling g_i of particle p_i to soft graviton $p_h \to 0$ 

(16) $$M(h(p_h) X \to Y) = \Big[\sum_{\text{incoming}} \frac{g_i}{2p_i \cdot p_h} p^i_\mu p^i_\nu \, \varepsilon^{\sigma_h}_{\mu\nu}(p_h) - \sum_{\text{outgoing}} \frac{g_i}{2p_j \cdot p_h} p^j_\mu p^j_\nu \, \varepsilon^{\sigma_h}_{\mu\nu}(p_h) \Big] M(X \to Y)$$

$p_h \to 0$

which is consistent with $\varepsilon_{\mu\nu}(p_h) \to \varepsilon_{\mu\nu}(p_h) + p_\mu \Omega_\nu + p_\nu \Omega_\mu \ \forall \Omega_\mu$, i.e. Ward id (6), if

(17) $$\boxed{\sum_{\text{incoming}} g_i \, p^i_\mu - \sum_{\text{outgoing}} g_j \, p^j_\mu = 0}$$

for any choice of p_i^μ consistent with a non-vanishing $M(X(p_i) \to Y(p_j))$, in particular for any choice of p_i^μ such that $\sum_{\text{incoming}} p_i^\mu - \sum_{\text{outgoing}} p_j^\mu = 0$.

The only non-trivial ($g_i \neq 0$) solution to (17) is thus

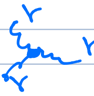
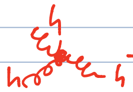
(18) $$\boxed{g_i = i\text{-independent} = \frac{1}{M_{PL}} \iff (17) = \sum_{\text{incoming}} p_i^\mu - \sum_{\text{outgoing}} p_j^\mu = 0}$$

Equivalence Principle via Consistency

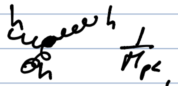
which is nothing but the statement that the graviton couplings g_i in (15) are universal. That is, consistency of soft scattering of gravitons with non-vanishing g_i + energy-momentum conservation imply the equivalence principle, the universality of the gravitational minimal coupling g_i = universal constant in the IR, called $1/M_{PL}$ (usually the $1/M_{PL}$ the inverse of the (reduced) Planck mass is factored out, $\int h_{\mu\nu}(x) \frac{T^{\mu\nu}(x)}{M_{PL}} d^4x$ in $\mathcal{L}_2$)

Clearly, one was right to identify $T_{\mu\nu}$ with the energy-momentum tensor $P^\mu = \int d^3x \, T^{0\mu}$.

Non-minimal coupling to $R_{\mu\nu\alpha\beta}$ or to identically conserved currents $\theta^{\mu\nu} = \partial_\alpha \theta^{\alpha\mu\nu}$ with $\theta^{\alpha\mu\nu} = -\theta^{\mu\alpha\nu}$, give vanishing contributions to the g_i, that is they vanish faster in the soft-graviton limit $p_h \to 0$.

Now, contrary to the photon who has charge zero $e_\gamma = 0$ and no  appears the graviton has energy & momentum so that one expects  $\neq 0$.

Another way to look at it is that $T^{\mu\nu}$ must be conserved, exactly, and so it must be the total $T^{\mu\nu}$, not just the one of other fields, and must thus include the $T^{\mu\nu}$ from h itself. Like for the other fields the first non-vanishing contribution to $T^{\mu\nu}$ from $h_{\mu\nu}$ is quadratic, $\mathcal{O}(\partial_\mu^2, h_{\mu\nu}^2)$, so that $h_{\mu\nu}T^{\mu\nu}$ gives indeed rise to to self interactions.



For example, the free kinetic term for the $h_{\mu\nu}$ associated to spin-2 and $m=0$

$$(19)\qquad \mathcal{L}_h^{(2)} = -\frac{1}{4}\left(2\partial_\mu h^\mu{}_\nu \partial^\nu h - 2\partial_\alpha h^\alpha{}_\nu \partial_\beta h^{\beta\nu} + \partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta} - \partial_\mu h \partial^\mu h\right) \qquad h \equiv h_\mu{}^\mu = h_{\mu\nu}\eta^{\mu\nu}$$

$$\eta_{\mu\nu} = (+1,-1,-1,-1)$$

This is found by writing the most general $\mathcal{O}(\partial_\mu^2, h^2)$ term which gives an invariant action under Lorentz $h_{\mu\nu} \to (\Lambda^{-1})_\mu{}^{\bar\mu}(\Lambda^{-1})_\nu{}^{\bar\nu} h_{\bar\mu\bar\nu}(\Lambda x) + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$; since all Lorentz indices are contracted we just need to check gauge-invariance. [I leave it as exercise]

If we were to couple the $h_{\mu\nu}$ to the $T_{\mu\nu}$ of a free massless scalar, $\mathcal{L}_\phi^{(2)} = \frac{1}{2}(\partial\phi)^2$ and $\mathcal{L}_I = \frac{1}{M_{Pl}} h_{\mu\nu} T_\phi^{\mu\nu} = \frac{1}{M_{Pl}} h_{\mu\nu}\left(\partial^\mu\phi\partial^\nu\phi - \frac{1}{2}(\partial\phi)^2\eta^{\mu\nu}\right)$, we would get an inconsistent e.o.m

$$(20)\qquad \boxed{\frac{\delta S}{\delta h_{\mu\nu}} = 0 \Longleftrightarrow \frac{1}{2}\left(\partial_\alpha\partial_\beta h^\alpha{}_\nu - \partial_\alpha\partial^\alpha h_{\beta\nu} - \partial_\beta\partial_\nu h + \partial_\alpha\partial_\nu h^\alpha{}_\beta\right) - \frac{1}{2}\eta_{\beta\nu}\left(\partial_\alpha\partial_\gamma h^{\alpha\gamma} - \partial_\alpha\partial^\alpha h\right) = \frac{T^\phi_{\beta\nu}}{M_{Pl}}}$$

where the left-hand side is identically conserved

Known as linearized ↙ Bianchi identity

$$(21)\qquad \partial_\beta[\text{L.H.S. }(20)]^\beta = \frac{1}{2}\left(\cancel{\Box\partial_\alpha h^\alpha{}_\nu} - \cancel{\Box\partial_\beta h^\beta{}_\nu} - \Box\partial_\nu h + \cancel{\partial_\nu(\partial_\alpha\partial_\beta h^{\alpha\beta})}\right) - \frac{1}{2}\partial_\nu\left(\cancel{\partial_\alpha\partial_\gamma h^{\alpha\gamma}} - \partial_\nu\Box h\right) \Big|_{\text{identically}} = 0$$

whereas the R.H.S. is not vanishing on the e.o.m for ϕ:

contradiction!

$$(22)\qquad \partial_\beta T^{\beta\,\phi}_\nu = \Box\phi\partial_\nu\phi + \cancel{\partial^\beta\phi\partial_\beta\partial_\nu\phi} - \cancel{(\partial_\nu\partial_\beta\phi)\partial^\beta\phi} \Big|_{\text{e.o.m.: } -\Box\phi - \frac{1}{M_{Pl}}[\partial_\mu(h^{\mu\nu}\partial_\nu\phi) - \partial_\alpha(\partial^\alpha\phi h)]} \neq 0$$

Clearly what's happening is that one should use the full $T_{\mu\nu}$ that includes $h_{\mu\nu}$ too in order to make it fully conserved on-shell. But adding $T^{(2)}_{\mu\nu} = T^{(2)}_{\mu\nu}(h^2)$ from $\mathcal{L}_h^{(2)}$ is still not enough because the new coupling $h_{\mu\nu}T^{(2)}_{\mu\nu}$ would further correct $T_{\mu\nu} = T^{(2)}_{\mu\nu}(h^2) + T^{(3)}_{\mu\nu}(h^3) + \ldots$ and so on.

What one needs to do is resumming an infinite series of higher and higher $h_{\mu\nu}$-corrections to $T^{\mu\nu}$ so that the RHS of (20) be conserved on-shell

$$(23) \quad \boxed{\partial_\mu T^{\mu\nu}\Big|_{e.o.m.} = 0 .}$$

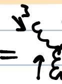
This program has been carried out by various people (Feynman, Weinberg, Deser, Wald, ...) and it gives rise to the fully non-linear theory of General Relativity (GR), which does emerges as the unique theory of (interacting) massless spin-2 theory that includes $V = 1/r$ type of force the minimal coupling to matter $h_{\mu\nu} T^{\mu\nu}_{matt.}$ $\leadsto$ [if one set all minimal coupling to $T^{\mu\nu}$ to zero, there exists an alternative where one couples $h_{\mu\nu}$ non-minimally only, via $R_{\mu\nu\rho\sigma}$ which is gauge invariant, e.g. $R_{\mu\nu\rho\sigma} R^{\rho\sigma\alpha\beta} R_{\alpha\beta}{}^{\mu\nu}$ or $R_{\mu\nu\rho\sigma} \bar\psi \bar\sigma^{\mu\nu}\psi \, \bar\psi \bar\sigma^{\rho\sigma}\psi$, but it decays fast at long distances, see L6].
We are going to show the quickest way to GR following the 1st order formalism à la Deser (gr-qc0411023) but before doing so we show how to solve the analogous problem for self-interacting theory of (more "flavors" of) $m=0$ spin-1 particles and see how Yang-Mills theory emerges.

-Bootstrapping Yang-Mills Theory: 1st order formalism ——— o ———

Let's formulate the problem of finding YM-theory via consistency conditions in a way that it will be usefull for the problem of finding GR that we attack later.

For a theory of just 1-single $m=0$ spin-1, the interactions start from $O(\partial^4, A^4)$ like $(F_{\mu\nu}F^{\mu\nu})^2$  because any trilinear vertex  either breaks gauge-invariance or bose symmetry, by direct inspection

$$(24) \quad M(p_1^{\sigma_1} p_2^{\sigma_2} p_3^{\sigma_3}) =$$



$$\propto (\varepsilon_1\cdot\varepsilon_2)(p_1\cdot\varepsilon_3) - (\varepsilon_1\cdot\varepsilon_3)(p_1\cdot\varepsilon_2) + (\varepsilon_2\cdot\varepsilon_3)(p_2\cdot\varepsilon_1)$$

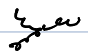
$(\varepsilon_i \equiv \xi^{\sigma_i}_\mu(p_i))$

is the unique gauge-invariant, $\varepsilon_i \to \varepsilon_i + p_i$ 3pt-vertex at order $O(\partial)$

[the way this can be found is simply writing the most general object out of $\varepsilon_1\, \varepsilon_2\, \varepsilon_3$ and p_1, p_2 $(p_3 = p_1 + p_2)$ recalling that $p_i \cdot \varepsilon_i = 0$, $\varepsilon_i \cdot \varepsilon_i = 0$, which gives 3 terms only $(\varepsilon_1\cdot\varepsilon_2)(\varepsilon_3 \cdot p_1)$, $(\varepsilon_1\cdot\varepsilon_3)(\varepsilon_2\cdot p_1)$, $(\varepsilon_2\cdot\varepsilon_3)(\varepsilon_1\cdot p_2)$ with relative coefficients fixed by invariance under $\varepsilon_i \to \varepsilon_i + p_i$]

L7/8

The catch is that $M(p_1,p_2,p_3)$ is not symmetric under $p_i \leftrightarrow p_j$ as it should instead be for integer-spin particles: $1 \leftrightarrow 2$ in (24) gives $M(213) = (\varepsilon_1\cdot\varepsilon_2)(\underset{-p_1\cdot\varepsilon_3}{\underbrace{p_2\cdot\varepsilon_3}}) - (\varepsilon_2\cdot\varepsilon_3)(p_2\cdot\varepsilon_1) + (\varepsilon_1\cdot\varepsilon_3)(p_1\cdot\varepsilon_2) = -M(123)$

So, for having self-interacting spin-1 particles with a non-trivial  we need more species such that

(25) $M(p_a p_b p_c) = f_{abc} \cdot (24)$ with f_{abc} antisymmetric so that it compensate the $(-)$ above from the exchange $A \leftrightarrow B$ and it restores the bose symmetry. So $a,b,c = 1,\dots,N$ different spin-1 m=0

Rather than writing their kinetic terms as just $-\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}$ we are going to use the first order formalism where $A^{a=1,\dots,N}_\mu$ & $F^{a=1,\dots,N}_{\mu\nu}$ are independent variables and on-shell only one has $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$

[the reason being that the current associated to a "rotation" between the $F^a_{\mu\nu}$ & A^a_μ remains the same expression even after turning-on interactions; keep reading and see below]

(26) $$\boxed{S^{(2)}[A^a_\mu, F^a_{\mu\nu}] = \int d^4x \, +\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} - \frac{1}{2}F^a_{\mu\nu}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)}$$

Free-theory 1st order formalism A_μ

Since $F_{\mu\nu}$ enters quadratically & without derivatives, it's a non-propagating d.o.f.: if we were to perform the path-integral in $F^a_{\mu\nu}$ first we would just plug back the solutions of the e.o.m w.r.t. $F^a_{\mu\nu}$:

(27) $$\frac{\delta S^{(2)}[A,F]}{\delta F^a_{\mu\nu}} = 0 \iff F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu \Rightarrow S^{(2)}[A^a_\mu, F^a_{\mu\nu}]\Big|_{\text{on-shell}} = \int d^4x \, -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 = \int d^4x \, -\frac{1}{4}(F^a_{\mu\nu})^2\Big|_{F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu}$$

So (26) is as good as starting point as $\int -\frac{1}{4}(F^a_{\mu\nu})^2\Big|_{(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)} d^4x$.

There is a "sort of rotation" that leaves the action (26) invariant:

(28) $$\boxed{\begin{cases} A^a_\mu \to A^a_\mu - f^{abc}\epsilon^b A^c_\mu + o(\epsilon^2) \\ F^a_{\mu\nu} \to F^a_{\mu\nu} - f^{abc}\epsilon^b F^c_{\mu\nu} + o(\epsilon^2) \end{cases}}$$

with f^{abc} a fully anti-symmetric set of numbers.
(ϵ^b infinitesimal param. of transformation. For rotations $f^{abc} = \epsilon^{abc}$ and $\epsilon^b = \omega^b$ is the axis of rotation)

$$\Rightarrow \delta S^{(2)} = \int d^4x \, \cancel{\frac{1}{2}F^a_{\mu\nu}(-f^{abc}\epsilon^b F^c_{\mu\nu})} - \frac{1}{2}\cancel{F^c_{\mu\nu}\epsilon^b} f^{abc}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) - \frac{1}{2}F^a_{\mu\nu}\cancel{f^{abc}}\epsilon^b(\partial_\mu A^c_\nu - \partial_\nu A^c_\mu)\Big|_{f^{abc}\text{ full antisymmetry}} = 0$$

We want to couples the A^a_μ to themselves via the coupling to the associated conserved Noether current.

The Noether current is found in the usual way: promote ϵ^b to $\epsilon^b(x)$ and look to the response of the action:

Noether current definition

$$(29) \quad \boxed{\delta S_1 \Big|_{\epsilon=\epsilon(x)} \equiv -\int d^4x\, \partial_\mu \epsilon^b J^b_\mu}$$

(there could also be surface terms $\epsilon^b(x)\partial_\mu K^b_\mu$ which give total derivatives when $\epsilon \to$ const, but we can include them in J^b_μ integrating by parts)

because the terms without ∂'s on ϵ^b don't notice that $\epsilon=\epsilon(x)$ is not a symmetry; moreover $\partial_\mu J^b_\mu \big|_{\text{on-shell}}=0$ because on the e.o.m. all $\delta S=0$, including those in (29) with $\epsilon=\epsilon(x)$ arbitrary.

By doing the transformations (28) with $\epsilon=\epsilon(x)$

$$(30) \quad \begin{cases} A^a_\mu \to A^a_\mu - f^{abc}\epsilon^b(x) A^c_\mu + \dots \\ F^a_{\mu\nu} \to F^a_{\mu\nu} - f^{abc}\epsilon^b(x) F^c_{\mu\nu} + \dots \end{cases} \Rightarrow \delta S_1 \Big|_{\epsilon=\epsilon(x)} = \int d^4x\, F^a_{\mu\nu} A^c_\nu f^{abc} \partial_\mu \epsilon^b(x)$$

$$(31) \quad \boxed{J^b_\mu = -f^{abc} F^a_{\mu\nu} A^c_\nu}$$

Noether current of transf. (28)

[which is indeed conserved on-shell,
$$\partial_\mu J^b_\mu = -f^{abc}[\underset{\substack{\| \\ 0 \\ \text{on-shell}}}{(\partial_\mu F^a_{\mu\nu})} A^c_\nu + \underbrace{F^a_{\mu\nu}\partial_\mu A^c_\nu}_{F^a_{\mu\nu}F^c_{\mu\nu}\, 1/2 \text{ on-shell}}] = 0$$
]

So, we are now in full analogy with GR: we are going to couple A^a_μ to J^a_μ which is itself made of A^a_μ and $F^a_{\mu\nu}$, generating [illegible] self-interactions.

$$(32) \quad S^{int}[A,F] = \int d^4x \left\{ \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} - \frac{1}{2} F^a_{\mu\nu}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) + \frac{g}{2} A^a_\mu J^a_\mu \right\}$$

Crucially, the interaction term added, $A^a_\mu J^a_\mu$, respects the global symmetry (28) if the f^{abc} are not only antisymmetric but respect also the Jacobi identity:

$$(33) \quad A^a_\mu J^a_\mu = +A^a_\mu f^{abc} F^b_{\mu\nu} A^c_\nu \xrightarrow[(28)]{} (\underbrace{f^{abc} f^{exy} + \dots}_{\text{Jacobi}}) \epsilon^x F^y_{\mu\nu} A^c_\nu A^b_\mu = 0$$

This means that there is still a conserved Noether current even after adding the interactions. Since the interaction involves no derivatives of F or A (here is the advantage of first order formalism), the current is just the same one as in the free-theory: this will make the equations consistent with the Bianchi identity. Namely, The e.o.m.

$$(34)\quad \delta S' = \int d^4x\, \tfrac{1}{2}\left[F^a_{\mu\nu} - (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)\right]\delta F^a_{\mu\nu} + [\partial_\mu F^a_{\mu\nu}]\delta A^a_\nu + \frac{g}{2}\Big([A^a_\mu A^c_\nu f^{abc}]\delta F^b_{\mu\nu} + \delta A^a_\mu [f^{abc} F^b_{\mu\nu} A^c_\nu] + [f^{abc} F^b_{\mu\nu}]\delta A^c_\nu\Big)$$
$$\overset{\Vert}{0}$$

are

$$(35)\quad \begin{cases} F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \\ \partial_\mu F^a_{\mu\nu} = -g f^{abc} F^b_{\mu\nu} A^c_\mu = g J^a_\nu \end{cases}$$

Yang-Mills equations

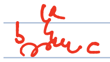
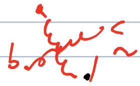
This is perfectly consistent because $\partial_\nu \cdot [\text{LHS } 35]^\nu = 0$ identically (Bianchi identity) by antisymmetry of $F_{\mu\nu}$, $\partial_\mu \partial_\nu F_{\mu\nu} = 0$, while $\partial_\nu \cdot [\text{RHS-}(35)]^\nu \big|_{\text{on-shell}} = 0$ because $\partial_\mu J^b_\mu \big|_{\text{on-shell}} = 0$ by Noether theorem.

[[explicitly: $\partial_\nu J^b_\nu = \partial_\nu(-f^{abc} F^b_{\mu\nu} A^c_\mu) = -f^{abc}\big(\underbrace{\partial_\nu F^b_{\mu\nu}}_{g f^{bde} F^d_{\rho\mu} A^e_\rho A^c_\mu} A^c_\mu + F^b_{\mu\nu}\underbrace{(\tfrac{\partial_\nu A^c_\mu - \partial_\mu A^c_\nu}{2})}_{\to\ \frac{1}{2}(F^c_{\nu\mu} + g f^{cde} A^d_\mu A^e_\nu)}\big)$

$= g(-f^{abc} f^{bde} + \frac{f^{adb} f^{ebc}}{2}) F^d_{\rho\mu} A^e_\rho A^c_\mu = -g \frac{F^d_{\rho\mu} A^e_\rho A^c_\mu}{2}[f^{abc} f^{bde} - f^{abe} f^{bdc} + f^{adb} f^{ebc}] \underset{\text{Jacobi}}{=} 0$]]

We can now integrate $F^a_{\mu\nu}$ out and get the usual canonical form of YM

$$(36)\quad S = \int d^4x\, \tfrac{1}{4}(F^a_{\mu\nu})^2 - \tfrac{1}{2} F^a_{\mu\nu}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^c_\nu A^b_\mu)\Big|_{F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^c_\nu A^b_\mu}$$
$$= \int d^4x\, -\tfrac{1}{4}(F^a_{\mu\nu})^2 \Big|_{F^a_{\mu\nu} = \text{on-shell}}$$

The YM theory has emerged as the theory of (several species of) $m=0$ spin-1 particles that are self-interacting via a non-trivial (minimal) conserved current, that generates  $\propto f^{abc} O(\partial^1_\mu)$ as well  $\sim f^{abx} f^{xdc} O(\partial^0)$. Non-minimal coupling such as $(F^a_{\mu\nu} F^a_{\mu\nu})^2$ are possible, as they give rise to identically conserved currents and are suppressed in the IR because $O(\partial^4)$. For photons, where $f^{abc} = 0$, that's the first non-trivial interaction term, which is irrelevant.

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In the next lecture we repeat these steps for the $m=0$ spin-2 particle coupled to its own $T_{\mu\nu}$, and bootstrap GR like we did for YM.

Aside comment on massless particles with integer spin $s>2$

From looking at the soft emission of a particle of spin s, $h_{\mu_1\cdots\mu_s}$, coupled to a current $J_{\mu_1\cdots\mu_s}$, we would get

$$M(h(p_h)X \to Y) \underset{\hookrightarrow 0}{=} M(X\to Y)\left[\sum_{\text{incoming}} \frac{g_i\, p_{\mu_1}\cdots p_{\mu_s}\, \varepsilon^{\sigma}_{\mu_1}\cdots\varepsilon^{\sigma}_{\mu_s}}{2p_i\cdot p_h} - \sum_{\text{outgoing}} \frac{g_j\, p_{\mu_1}\cdots p_{\mu_s}\, \varepsilon^{\sigma}_{\mu_1}\cdots\varepsilon^{\sigma}_{\mu_s}}{p_j\cdot p_h}\right]$$

where we used that $\langle p_i\sigma_i'|J^{\mu_1\cdots\mu_s}(0)|p_i\sigma_i\rangle = g_i\,\delta_{\sigma\sigma'}\,p_i^{\mu_1}\cdots p_i^{\mu_s}$ up to $\eta_{\mu_i\mu_j}$'s that are irrelev. (because $\varepsilon^{\sigma}_{\mu_1}\cdot\varepsilon^{\sigma}_{\mu_1}=0$)

$$\Rightarrow \sum_{\text{incoming}} g_i\, p_i^{\mu_2}\cdots p_i^{\mu_s} - \sum_{\text{outgoing}} g_j\, p_j^{\mu_2}\cdots p_j^{\mu_s} = 0$$

which can't be compatible for generic p_i & $\sum_{in} p_i^i - \sum_{out} p_r^j = 0$ unless $g_i = 0$.

$\Rightarrow$ Massless higher spin can't have long range interaction ~~of~~ the type $h_{\mu_1\cdots\mu_s}J^{\mu_1\cdots\mu_s}$ that survive in the $p_h\to 0$ limit $\cancel{\underset{\overrightarrow{p_h}}{m}} \to 0$. Coupling via derivatives of $h_{\mu_1\cdots\mu_s}$ is not excluded in this way.
