Topics in Effective Field Theory

PhD-course, Rome 2020 by B. Bellazzini

EFT for massless spin-1 & spin-2: the road to YM & GR

In the following lectures we will study the EFT for massless particles of spin-1 and spin-2 in Minkowski space to show that (under reasonable assumptions about the IR) their leading operators are those of Yang-Mills & Einstein's General Relativity.
To this ends, let's review in this lecture the defining properties of particles with spin ≥ 1 & $m=0$, according to two pillars of modern physics: QM + relativity.
We assume that the IR EFT respects the following:

Poincaré Transformations, locality (+ cluster decomposition) + causality:

Let's study first the massive case $\boxed{m^2>0}$

spacetime translations + Lorentz = Poincaré

Poincaré

(1)

- $\exists$ unitary op. $U(\Lambda, a) = U(a)\,U(\Lambda) = \exp[-i a_\mu P^\mu]\,\exp[i\, \omega_{\mu\nu} J^{\mu\nu}]$
- $U(\Lambda, a)|0\rangle = |0\rangle$
- 1-particle states $(p^2 = m^2 > 0,\ p^0 > 0)$ $\quad |p, \overset{\text{spin}}{\sigma}\rangle = \sqrt{2E_p}\, a^\dagger(p,\sigma)|0\rangle \ldots$
- S-matrix operator invariant, $U S U^\dagger = S \Rightarrow$ amplitudes transform covariantly as tensor products of 1-particle states

Locality + causality

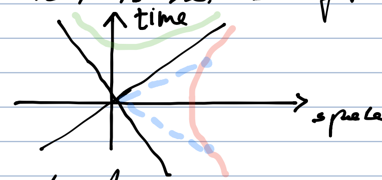
(2)

- $H_I(t) = \int d^3x\, \mathcal{H}_I(t, \vec{x})$ with
- $U(\Lambda, a)\, \mathcal{H}_I(x)\, U(\Lambda, a)^\dagger = \mathcal{H}_I(\Lambda x + a)$
- $[\mathcal{H}_I(x_1), \mathcal{H}_I(x_2)] = 0 \qquad (x_1 - x_2)^2 \leq 0$

$\Rightarrow$ $S = T\exp[-i\int dt\, H_I(t)]$ is Poincaré invariant

(3) $\Rightarrow$ $S = T\exp[-i\int d^4x\, \mathcal{H}_I(x)]$ is manifestly invariant

except for the time ordering which is not so for spacelike separated points



but, because of $[H,H]_{\text{spacelike}} = 0$,

the ordering of those is not important.

cluster decomp.

• The $H_I(x)$ is made of creation & annihilation operators

$$
(4)\quad \begin{cases} a(p,\sigma) \quad a^\dagger(p,\sigma) \quad \text{with} \quad [a(p,\sigma), a^\dagger(p',\sigma')] = (2\pi)^3\delta^3(p-p') \\ \qquad\qquad\qquad\qquad\qquad\quad\;\; [a,a] = 0 \\ \sqrt{2E(p)}\, a^\dagger(p,\sigma)|0\rangle \equiv |p,\sigma\rangle \\ \langle p'\sigma'|p\,\sigma\rangle = (2\pi)^3\delta^3(p-p')\, 2E(p) \end{cases}
$$

(Lorentz invariant scalar product)

which we packed into local fields → thanks to polarizations $u_\ell^\sigma(\vec p)$ & $v_\ell^\sigma(\vec p)$

Little group index

$$
(5)\quad \begin{cases} \text{(A)}\quad \phi_\ell(x) = \sum_\sigma \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}} \left\{ e^{-ipx} a(\vec p,\sigma)\, u^\sigma_\ell(\vec p) + e^{ipx} b^\dagger(\vec p,\sigma)\, v^\sigma_\ell(\vec p) \right\} \\ \text{(B)}\quad U(\Lambda,a)\,\phi_\ell(x)\,U(\Lambda,a)^\dagger = D_{\ell\ell'}(\Lambda^{-1})\,\phi_{\ell'}(\Lambda x + a) \end{cases}
$$

Lorentz index

(p on-shell $p^0 = E^p = \sqrt{\vec p^{\,2} + m^2}$)

irreducible Lorentz representation (not necessarily unitary)
(or reducible in two opposite-parity irreps if P-invar.)

so that

$$
(6)\quad [\phi_\ell(x), \phi^\dagger_{\ell'}(x')]_\pm = 0 \qquad (x-x')^2 \le 0
$$

to have a Lorentz invariant H_I and S-matrix.

Remember that $SO(3,1) \sim SU(2)_A \otimes SU(2)_B$

$\rightarrow D = (A, B)$

$\dim D = \dim A \, \dim B = (2A+1)(2B+1)$

$$
\begin{cases} A = (\vec J + i\vec K)/2 \\ B = (\vec J - i\vec K)/2 \\ \vec J = \vec A + \vec B \qquad j = A+B, \dots, |A-B| \end{cases}
$$

$\phi = (0,0)$, scalar	$\psi_L = (1/2, 0)$, left-handed spin-1/2	$A_\mu = (1/2, 1/2)$ vector	$h_{\mu\nu} = (1,1)$ symmetric 2-tensor
$D(\Lambda) = 1$	$D(\Lambda) = \exp[i\frac{\vec\sigma}{2}\cdot\vec\theta + \frac{\vec\sigma}{2}\cdot\vec\alpha]$	$D(\Lambda) = \Lambda^\nu{}_\mu$	$D(\Lambda) = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu$
$u = v = 1$	$u^\pm = \sqrt{p_\mu\sigma^\mu}\,\xi^\pm \quad v^\pm = u^\mp$	$u^\sigma(p) = \varepsilon^\sigma_\mu(p)$, $v^\sigma(p) = \varepsilon^\sigma_\mu(p)^*$	$u = \varepsilon^\sigma_{\mu\nu}(p)$, $v = (\varepsilon^\sigma_{\mu\nu}(p))^*$

The case $m=0$ is different for A_μ and $h_{\mu\nu}$:

Massless $s=1,2$

The local fields

(7)
$$\boxed{\begin{aligned} A_\mu &= \sum_{\sigma=\pm}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\,e^{-ipx}\,a(p,\sigma)\,\varepsilon^\sigma_\mu(p) + h.c. \\ h_{\mu\nu} &= \sum_{\sigma=\pm}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E(p)}}\,e^{-ipx}\,a(p,\sigma)\,\varepsilon^\sigma_{\mu\nu}(p) + h.c. \end{aligned}}$$

do not transform like in (B), they are not a Lorentz vector and a 2-tensor respectively. In fact, they transform as:

(8)
$$\boxed{(B')\ \begin{cases} U(\Lambda,a)\,A_\mu(x)\,U(\Lambda,a)^\dagger = (\Lambda^{-1})_\mu{}^\nu\cdot A_\nu(\Lambda x+a) + \partial_\mu\Omega(x,\Lambda) \\ U(\Lambda,a)\,h_{\mu\nu}(x)\,U(\Lambda,a)^\dagger = (\Lambda^{-1})_\mu^{\bar\mu}(\Lambda^{-1})_\nu^{\bar\nu}\cdot h_{\bar\mu\bar\nu}(\Lambda x+a) + \partial_\mu\Omega_\nu(x,\Lambda)+\partial_\nu\Omega_\mu(x,\Lambda) \end{cases}}$$

The reason is that the little groups for massive & massless states are different, reflecting the fact that @ $m=0$ one has only 2 d.o.f., ie. neither 3 nor 5 @ $m\neq0$ and $s=1,2$ respectively. Let's look into the little groups:

Little Group at $m\neq 0$

(9) $\begin{cases} p^2=m^2\neq0 \\ p^0>0\end{cases} \Rightarrow \exists\ L(p)\in SO(3,1)$ such that $\underline{p=L(p)k}$ for a reference $k^\mu=\begin{pmatrix} m\\0\\0\\0\end{pmatrix}=(m,\vec{0})$

(10) $|p,\sigma\rangle \equiv U(L(p))|k\,\sigma\rangle$ The one-particle states

(11) $|p,\sigma\rangle \xrightarrow{\Lambda} U(\Lambda)U(L(p))|k\sigma\rangle = U(L(\Lambda p))\,U(\underbrace{L(\Lambda p)^{-1}\Lambda L(p)}_{W(\Lambda,p)})|k\,\sigma\rangle$: Wigner Rotation

where $k\xrightarrow{L(p)} p \xrightarrow{\Lambda} \Lambda p \xrightarrow{L(\Lambda p)^{-1}} k$

$\Rightarrow W(\Lambda,p)\in$ subgroup of $SO(3,1)$ that leaves k invariant $=$ Little Group

(12) For $p^2=m^2>0 \rightarrow k=\begin{pmatrix} m\\0\\0\\0\end{pmatrix} \Rightarrow \boxed{\text{Little Group}\big|_{m^2>0} = SO(3)\sim SU(2)}$ $\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & R\end{array}\right)\begin{pmatrix} m\\ \vec{0}\end{pmatrix}=\begin{pmatrix} m\\ \vec{0}\end{pmatrix}$

(13) $U(W(\Lambda,p))\,|k,\sigma\rangle = |k,\sigma'\rangle\, \mathcal{L}_{\sigma'\sigma}(W)$ where $\mathcal{L}$ is a irrep of $SU(2)$, that is

must be a linear comb. of states with same k — unitary & finite dimensional

σ is the usual spin-index associated to 3D rotations. $\left|\ S\text{-matrix: } S(p_1^{\sigma_1}\ldots p_m^{\sigma_m}) = \mathcal{L}_{\bar\sigma_1\sigma_1}\ldots S(\Lambda p_1^{\bar\sigma_1}\ldots)\right|$

(14) $$\boxed{|p,\sigma\rangle \xrightarrow{\Lambda} U(L(\Lambda p))\,|k\,\sigma'\rangle\, \mathcal{L}_{\sigma'\sigma}(W) = |\Lambda p,\sigma'\rangle\cdot \mathcal{L}_{\sigma'\sigma}(W)}$$ Transformation rule massive particles

In words, the one-particle states carry a little-group (unitary & finite dim.) represent.

At the level of $a(p,\sigma)$ & $a^+(p,\sigma)$ it reads

(15) $$\begin{cases} U(\Lambda)\, a^+(p,\sigma)\, U^+(\Lambda) = a^+(\Lambda p,\sigma')\, \mathcal{L}_{\sigma'\sigma}(W(\Lambda,p))\cdot \dfrac{\sqrt{E_p'}}{\sqrt{E_p}} \qquad (E_p' = \Lambda^0{}_\nu p^\nu = p'^0) \\ U(\Lambda)\, a(p,\sigma)\, U^+(\Lambda) = a(\Lambda p,\sigma')\, \mathcal{L}^*_{\sigma'\sigma}(W(\Lambda,p))\sqrt{\dfrac{E_p'}{E_p}} = \underline{\mathcal{L}_{\sigma\sigma'}(W^{-1})\, a(\Lambda p,\sigma')\sqrt{E_p'/E_p}} \end{cases}$$

Now, to make it compatible with the properties (B) $$\begin{cases} U(\Lambda)\,\phi_\ell(x)\,U(\Lambda)^+ = D_{\ell\ell'}(\Lambda^{-1})\,\phi_{\ell'}(\Lambda x) \\ \phi_\ell(x) = \sum_\sigma \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left\{ e^{-ipx}\, a(\vec p,\sigma)\, u_\ell^\sigma(\vec p) + \ldots\right\} \end{cases}$$

the polarizations $u_\ell^\sigma(\vec p)$ have one Lorentz-index "ℓ"

and one little-group index "σ":

(16) $$\begin{cases} \phi_\ell(x) = \sum_\sigma \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left\{ e^{-ipx}\, a(p,\sigma)\, u_\ell(p) + \ldots\right\} \xrightarrow[(15)]{\Lambda} \sum_{\sigma\sigma'} \int \frac{d^3p}{(2\pi)^3\, 2E_p}\sqrt{2E_p'}\left\{ e^{-ipx}\, \mathcal{L}_{\sigma\sigma'}(W^{-1})\, a(\Lambda p,\sigma')\, u_\ell^\sigma(p) + \ldots\right\} \\ U(\Lambda)\,\phi_\ell(x)\,U(\Lambda)^+ = \sum_\sigma \int \frac{d^3p'}{(2\pi)}\frac{1}{\sqrt{2E_p'}}\, e^{-ip'x'}\, \mathcal{L}_{\sigma\sigma'}(W^{-1})\, a(p,\sigma')\, u_\ell^\sigma(\Lambda^{-1}p') + \ldots \Big|_{(x'=\Lambda x)} \end{cases}$$

Lorentz invariant volume elem. change variable $\vec p' = \Lambda \vec p$

$(p\cdot x = \Lambda p \cdot \Lambda x)$

(17) $\mathcal{L}_{\sigma\sigma'}(W^{-1}(\Lambda,p))\cdot u_\ell^\sigma(p) = D_{\ell\ell'}(\Lambda^{-1})\, u_{\ell'}^{\sigma'}(\Lambda p)$

that is

(18) $$\boxed{D_{\ell\lambda'}(\Lambda)\, u_{\lambda'}^\sigma(p) = u_\ell^{\sigma'}(\Lambda p)\, \mathcal{L}_{\sigma'\sigma}(W(\Lambda,p))}$$

Lorentz act on the left whereas little group on the right

Examples: massive spin-1 $\quad \Lambda_\mu{}^\nu\, \varepsilon_\nu^\sigma(p) = \varepsilon_\mu^{\sigma'}(\Lambda p)\, R_{\sigma'\sigma}(\Lambda,p) \qquad (D^{s=1}(\Lambda) = \Lambda^\mu{}_\nu)$

The transformation rule (18) is actually a constructive definition:

- set $\vec{p}=0$ and take $\Lambda = L(q)$ $L(q)k=q$ $k=\binom{m}{\vec{0}}$ $\Rightarrow L(p)=1$ in

$$W(\Lambda,p)=W(L(q),k)=L^{-1}(\Lambda p)\Lambda L(p) = L^{-1}(q)L(q)=1$$

(19) $$\boxed{u^\sigma_\ell(q) = D_{\ell\ell'}(L(q))\,u^\sigma_{\ell'}(k)}$$

The polarization at q is obtained by the Lorentz tr. of $u^\sigma_\ell(k) = u^\sigma_\ell(k|_{ref})$
$\Rightarrow$ It's enough to find $u^\sigma_\ell(\vec{0})$ in the rest frame & then $\quad \overset{\parallel}{u^\sigma_\ell(\vec{0})}$
to boost it to the desired $\vec{q}$.

Example: $\varepsilon^\sigma_\mu(\vec{p}) = \Lambda_\mu{}^\nu\, \varepsilon^\sigma_\nu(\vec{0})$ where $\Lambda^\mu{}_\nu k^\nu = \Lambda^\mu{}_\nu \begin{pmatrix} m \\ 0 \\ 0 \end{pmatrix}^\nu = p^\mu$

e.g. $\Lambda = R_\theta^{-1} B_\beta R_\theta$ where R_θ orient z-axis along $\vec{p}$, B_α boost it such that $m\gamma = E_p$, $B_\beta = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix}$.

- set $\vec{p}=\vec{k}=0$ & take $\Lambda =$ rotation $= R$ $\quad R\vec{p}=\vec{0}$ $\Rightarrow W=R$

(20) $\Rightarrow$ $$\boxed{D_{\ell\ell'}(R)\,u^\sigma_{\ell'}(\vec{0}) = u^{\sigma'}_\ell(\vec{0})\,\mathcal{L}_{\sigma'\sigma}(R)}$$

Recalling that particles at rest have definite spin along the z-direction:
$U(R_z(\theta))|\vec{0},\sigma\rangle = e^{i\sigma\theta}|\vec{0},\sigma\rangle \Rightarrow \mathcal{L}_{\sigma'\sigma}(R_z(\theta)) = \delta_{\sigma'\sigma}\, e^{i\sigma\theta}$, the condition is

(21) $$\boxed{D_{\ell\ell'}(R_z(\theta))\,u^\sigma_{\ell'}(\vec{0}) = e^{i\sigma\theta}\,u^\sigma_\ell(\vec{0})}$$

which is an eigenvalue equation, for the irreps of rotations. (infinitesimally, $D(J^z)_{\ell\ell'}u^\sigma_{\ell'}(\vec{0}) = \sigma\, u^\sigma_\ell(\vec{0})$)

The solutions are well known: they are labelled by the total spin j:

$(2j+1)$ # of eigenvalues from $j = A+B, \dots, |A-B|$
in the irrep $D_{\ell\ell'} = D^{(j)}_{\ell\ell'}$ & $\sigma = +j, \dots, -j$

Example: massive spin-1:

$$(22)\quad \begin{cases} \epsilon^0_\mu(\vec 0) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \qquad R_z(\theta) = \left(\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline 0 & c_\theta & s_\theta & 0 \\ 0 & -s_\theta & c_\theta & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \\ \epsilon^\pm(\vec 0) = \frac{1}{\sqrt2}\begin{pmatrix}0\\1\\\pm i\\0\end{pmatrix} \to \begin{cases} R_z(\theta)\epsilon^\pm(\vec 0) = \frac{1}{\sqrt2}\begin{pmatrix}0\\e^{\pm i\theta}\\\pm i e^{\pm i\theta}\\0\end{pmatrix} = e^{\pm i\theta}\epsilon^\pm_\mu(\vec 0) \\ R_z(\theta)\,\epsilon^0(\vec 0) = \epsilon^0(\vec 0) \end{cases} \end{cases} \Rightarrow \begin{cases} \epsilon^{\sigma=0,\pm}_\mu(\vec p) = \Lambda^\nu{}_\mu\, \epsilon^{\sigma=0,\pm}_\nu(\vec 0) \\ p = \Lambda\begin{pmatrix}m\\\vec 0\end{pmatrix} \end{cases}$$

(In fact, in $(1/2, 1/2) \in SU(2)_A \otimes SU(2)_B$ there is $\overset{\text{spin-1}}{1} \oplus \overset{\text{spin-0}}{0}$ in $SU(2)_{A+B} = SU(2)_J$ so that it should be possible to build a spin-0 out of A_μ.
This corresponds to take the $\epsilon^{j=0}_\mu(\vec 0) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \Rightarrow \Lambda^\nu{}_\mu \begin{pmatrix}1\\0\\0\\0\end{pmatrix}_\nu = \begin{pmatrix}\gamma\\0\\0\\\gamma\beta\end{pmatrix}$ where $\Lambda^\nu{}_\mu\begin{pmatrix}m\\0\\0\\0\end{pmatrix} = \begin{pmatrix}E\\0\\0\\p\end{pmatrix}$
i.e. $\gamma = E/m$ and $\epsilon^{(j=0)}_\mu = \frac{p_\mu}{m} \Rightarrow A_\mu = \partial_\mu\phi$. This is a vector field but it generates only spin-0 states.)

Little Group at m = 0

Massless particles have $p^2 = 0$ and there is no frame where they are at rest.
We can obtain a p light-like from another reference light-like vector $k^\mu = \begin{pmatrix}E\\0\\0\\E\end{pmatrix}$
All the steps we have done above for the one-particle state can be repeated, the only difference that the Wigner $W(\Lambda,p)$ belongs to a different little group:

$$|p,\sigma\rangle \xrightarrow{\Lambda} U(\Lambda)\,U(L(p))\,|k\,\sigma\rangle = U(L(\Lambda p))\,U(\underbrace{L(\Lambda p)^{-1}\Lambda L(p)}_{W(\Lambda,p)})\,|k\,\sigma\rangle \quad : \text{Wigner Rotation}$$

where $k \xrightarrow{L(p)} p \xrightarrow{\Lambda} \Lambda p \xrightarrow{L(\Lambda p)^{-1}} k$

the subgroup of $SO(3,1)$ that preserves $k^\mu = \begin{pmatrix}E\\0\\0\\E\end{pmatrix}$ is $ISO(2)$, the Euclidean group of the plane, $SO(2) \rtimes T_{1,2}$, that is 2D-translations & 1 rotation. The rotation is around the z-axis $R_z\begin{pmatrix}E\\0\\0\\E\end{pmatrix} = \begin{pmatrix}E\\0\\0\\E\end{pmatrix}$. The most general element of $ISO(2)$ is $S(\alpha,\beta)\,R_z(\theta) = W^\mu{}_\nu \in ISO(2)$

$$(23)\quad R^\mu_{z\,\nu}(\theta) = \left(\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline 0 & c_\theta & s_\theta & 0 \\ 0 & -s_\theta & c_\theta & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \text{and} \quad S^\mu{}_\nu(\alpha\beta) = \begin{pmatrix} 1+\xi & \alpha & \beta & -\xi \\ \alpha & 1 & 0 & -\alpha \\ \beta & 0 & 1 & -\beta \\ \xi & \alpha & \beta & 1-\xi \end{pmatrix}^\mu{}_\nu \qquad \xi = (\alpha^2+\beta^2)/2$$

Now, translations are non-compact $\Rightarrow$ unitary irreps of those would be infinite dimensional (called continuous spin rep)

or must be trivially realized $\mathcal{L}(\xi)=1$ (the two generators of translations annihilate the vacuum)

$\Rightarrow$ $m=0$ + unitarity + finite # d.o.f. imply $\Rightarrow$ only $SO(2)\sim U(1)$ rotations around z-axis acts non-trivially on the states:

$\boxed{\text{spin-1}}$: $\sigma=\pm 1$, $D(\Lambda)=\Lambda$

(24) $\mathcal{L}_{\sigma\sigma'}(W)=\mathcal{L}_{\sigma\sigma'}(R_z)=e^{i\sigma\theta}\delta_{\sigma\sigma'}$

$$\begin{cases} |p\,\sigma\rangle \xrightarrow{\Lambda} U(\Lambda)|p,\sigma\rangle = |\Lambda p,\sigma\rangle\, e^{i\sigma\theta(\Lambda,p)} \\ a(p,\sigma) \longrightarrow U(\Lambda)\,a(p,\sigma)\,U^{\dagger}(\Lambda) = a(\Lambda p,\sigma)\, e^{-i\sigma\theta(\Lambda,p)} \end{cases}$$

(25) $U(\Lambda)A_\mu(x)U(\Lambda)^{\dagger}=\int\frac{d^3q}{(2\pi)^3}\frac{\sqrt{2E_p'}}{2E_p}\left\{e^{-ipx}a(\Lambda p,\sigma)\,e^{-i\sigma\theta(\Lambda,q)}\,\varepsilon^{\sigma}_{\mu}(q)+\ldots\right\}$ ← only a phase because $\mathcal{L}(S)=1$ and $\mathcal{L}(SR)=e^{-i\sigma\theta}$

$$=\int\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{p'}}}\left\{e^{-ip'x'}a(p',\sigma)\,e^{-i\sigma\theta(\Lambda,\Lambda^{-1}p')}\,\varepsilon^{\sigma}_{\mu}(\Lambda^{-1}p')+\ldots\right\}\Big|_{x'=\Lambda x}$$

one would try to set $e^{-i\sigma\theta(\Lambda,p)}\varepsilon^{\sigma}_{\mu}(p)=\Lambda^{-1}{}_{\mu}{}^{\nu}\,\varepsilon^{\sigma}_{\nu}(\Lambda p)$, so that $U(\Lambda)A_\mu U(\Lambda)^{\dagger}=\Lambda^{-1}{}_{\mu}{}^{\nu}A_\nu(\Lambda x)$

In a nicer form: what one would like is

(26) $\Lambda_{\mu}{}^{\nu}\,\varepsilon^{\sigma}_{\nu}(p)=\exp[i\sigma\theta(\Lambda,p)]\,\varepsilon^{\sigma}_{\mu}(\Lambda p)$. ⇐ turns out to be impossible, see below

It's actually impossible to build such Lorentz covariant polarizations.

Let's go step by step as before:

- set $p=k$ & take $L(q)k=q$ $\left(k\xrightarrow{L(q)}q\right)$ $\Rightarrow W=0\Rightarrow\theta=0$

(27) $\underset{(26)}{\Rightarrow}$ $\varepsilon^{\sigma}_{\mu}(q)=L(q)_{\mu}{}^{\nu}\,\varepsilon^{\sigma}_{\nu}(k)$ $\qquad$ $\left(L(p)=R^{z}_{\theta}\cdot B^{z}_{|\beta}\right)$

$L(p)\cdot k=R^{z}_{\theta}\begin{pmatrix}E\\0\\0\\E\end{pmatrix}=\begin{pmatrix}E\\\vec{k}\end{pmatrix}$ $|\vec{k}|=E$

so that again would be enough to know $\varepsilon^{\sigma}_{\mu}(k)$ at the reference vector in order to know the $\varepsilon^{\sigma}(p)$ at another p, just act with the canonical $L(p)$ on $\varepsilon^{\sigma}(k)$.

• set $p=k$ and take a little group transformation $\Lambda = R^z \Rightarrow W = R^z$

in this way (26) gives $\Lambda = R^z$ and the usual eigenvalue problem (but for $\sigma = \pm 1$ only)

(28) $$\begin{cases} R^z{}_\mu{}^\nu(\theta)\, \varepsilon^\sigma_\nu(k) = e^{i\sigma\theta}\, \varepsilon^\sigma_\mu(k) \\ \sigma = \pm 1 \end{cases} \rightarrow \boxed{\varepsilon^\pm_\mu(k) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix}_\mu}$$ (same $\pm$ solutions as for $m \neq 0$)

However, it fails the transformations associated to the ISO(2) translations $S(\alpha,\beta)$:

• set $p=k$ and take a little-group transformation $\Lambda = S(\alpha,\beta)$ (i.e. $\theta=0$, the ISO(2)-transl. act trivially on one-particle states)

(29) $$S(\alpha,\beta)_\mu{}^\nu\, \varepsilon^\sigma_\nu(k) \overset{?}{=} \varepsilon^\sigma_\mu(k)$$ no little-group factor since $\mathcal{D}(S)=1$

But in fact we have by direct inspection something different:

(30) $$S(\alpha,\beta)^\mu{}_\nu\, \varepsilon^{\nu\sigma}(k) = \left(\begin{array}{c|cc|c} 1+\zeta & \alpha & \beta & -\zeta \\ \hline \alpha & 1 & 0 & -\alpha \\ \beta & 0 & 1 & -\beta \\ \hline \zeta & \alpha & \beta & 1-\zeta \end{array}\right)^\mu{}_\nu \cdot \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix}^\nu \frac{1}{\sqrt{2}} = \left(\begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} + \begin{pmatrix} \alpha+i\beta \\ 0 \\ 0 \\ \alpha+i\beta \end{pmatrix}\right)\frac{1}{\sqrt{2}} = \varepsilon^{\mu\sigma}(k) + \frac{\alpha+i\beta}{\sqrt{2}}\frac{k^\mu}{|\vec{k}|}$$

that is

(31) $$\boxed{S(\alpha,\beta)_\mu{}^\nu\, \varepsilon^\sigma_\nu(k) = \varepsilon^\sigma_\mu(k) + \frac{\alpha+i\beta}{\sqrt{2}}\frac{k_\mu}{|\vec{k}|} \neq \varepsilon^\sigma_\mu(k)}$$ that we wanted for general covariance

The polarizations $\varepsilon^{\sigma=\pm1}_\mu(k)$ are not Lorentz vectors but rather shift by $\propto k^\mu$

Under a general little-group transformation $W = S(\alpha,\beta)\, R^z(\theta)$

(32) $$\boxed{W_\mu{}^\nu\, \varepsilon^\sigma_\nu(k) = e^{i\sigma\theta}\left(\varepsilon^\sigma_\nu(k) + \frac{\alpha+i\beta}{\sqrt{2}}\frac{k_\mu}{|\vec{k}|}\right)}$$

Now, under a general Λ that sends $k \to p = \Lambda k$ we have in general

(32) $$W(\Lambda,p) = L^{-1}(\Lambda p)\,\Lambda\, L(p) \Rightarrow L(p) = \Lambda^{-1} L(\Lambda p)\, W(\Lambda,p)$$

(33) $$\Rightarrow \boxed{\varepsilon^\sigma_\mu(p) = L_\mu{}^\nu(p)\, \varepsilon^\sigma_\nu(k) = e^{+i\sigma\theta(\Lambda,p)} (\Lambda^{-1})_\mu{}^\nu \left(\varepsilon^\sigma_\nu(\Lambda p) + \frac{\alpha+i\beta}{\sqrt{2}}\frac{(\Lambda p)_\nu}{|\vec{k}|}\right)}$$

At the level of fields $A_\mu(x) = \sum_{\sigma=\pm} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2|p|}} \{ e^{-ipx} a(p,\sigma)\, \varepsilon_\mu^\sigma(p) + h.c. \}$

this implies:

(34) $$U(\Lambda) A_\mu(x) U(\Lambda)^\dagger = \int \frac{d^3q}{(2\pi)^3} \frac{\sqrt{2E_p'}}{2E_p} \{ e^{-ipx} a(\Lambda p, \sigma)\, \underbrace{e^{-i\sigma\theta(\Lambda,q)} \varepsilon_\mu^\sigma(q)}_{\to (\Lambda^{-1})_\mu{}^\nu [\varepsilon^\sigma(\Lambda p) + f(p^\sigma)(\Lambda p)]_\nu} + \ldots \}$$

$$= (\Lambda^{-1})_\mu{}^\nu \int \frac{d^3p'}{(2\pi)^3 \sqrt{2E_p'}} \{ e^{-ip'x'} a(p',\sigma) (\varepsilon_\nu^\sigma(p') + f(p') p'_\nu) + h.c. \} \Big|_{x' = \Lambda x}$$

(35) $$\boxed{U(\Lambda) A_\mu(x) U(\Lambda)^\dagger = (\Lambda^{-1})_\mu{}^\nu A_\nu(\Lambda x) + \partial_\mu \Omega(\Lambda, x)}$$

The spin-1 polarizations $\varepsilon_\mu^\pm(p)$ are Lorentz vectors only up to a gauge transformat.

This means that Lorentz invariant $\mathcal{L}_I(A_\mu)$ built with $A_\mu = (1/2, 1/2)$ out of massless spin-1 annihilation & creation operators must be gauge-invariant. A theory with $\mathcal{L} = \ldots + (A_\mu A^\mu)^2$ is a theory for massive spin-1 (or, if one insists on $m=0$, that break Lorentz explicitly despite appearance).

Remark: It's actually possible to write a manifestly Lorentz invariant theory for $m=0$ $s=1$ particles: just use the gauge invariant representation $F_{\mu\nu} = (1,0) \oplus (0,1)$ (assuming parity for simplicity) where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

(36) $$U(\Lambda) F_{\mu\nu}(x) U(\Lambda)^\dagger = (\Lambda^{-1})_\mu{}^{\bar\mu} (\Lambda^{-1})_\nu{}^{\bar\nu} F_{\bar\mu\bar\nu}(\Lambda x)$$ ← bona fide tensor

because $\partial_\mu \partial_\nu \Omega - \partial_\nu \partial_\mu \Omega = 0$

This is perfectly legitimate, in fact the EFT below the lightest charged particle is like that, $\mathcal{L} = \mathcal{L}(F_{\mu\nu})$, but it is not the most general option (see below), neither the one that survive at long distances (but still $r \ll 1/m_e$) because of the suppression as $E \to 0$ due to the ∂'s

A more general option is the coupling to (exactly) conserved currents:

(37) $$\mathcal{L}_I = -g A_\mu(x) J^\mu \quad \text{with } \partial_\mu J^\mu = 0$$

linear coup. to charged matter

Spin-2 $\sigma = \pm 2$

The story for $m=0$ $s=2$ is essentially identical, up to doubling the Lorentz indices.

(38) $$U(\Lambda) h_{\mu\nu}(x) U^\dagger(\Lambda) = (\Lambda^{-1})_\mu{}^{\bar\mu} (\Lambda^{-1})_\nu{}^{\bar\nu} h_{\bar\mu\bar\nu}(\Lambda x) + \partial_\mu \Omega_\nu + \partial_\nu \Omega_\mu$$

One way to see this simple fact is by choosing the $\sigma = \pm 2$ polarization as

(39) $$\varepsilon^{++}_{\mu\nu}(p) = \varepsilon^+_\mu(p)\, \varepsilon^+_\nu(p) \qquad \varepsilon^{--}_{\mu\nu}(p) = \varepsilon^-_\mu(p)\, \varepsilon^-_\nu(p)$$

$$\varepsilon^{\pm}_{\mu\nu}(k) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \pm i & 0 \\ 0 & \pm i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so that transform well under little-group rotations, the helicity taking value $\sigma = \pm 2$

(40) $$W_\nu{}^{\bar\nu} W_\mu{}^{\bar\mu} \varepsilon^\sigma_{\bar\mu}(k) \varepsilon^\sigma_{\bar\nu}(k)\Big|_{\sigma=\pm 1} = e^{i2\sigma\theta(\Lambda,p)} \left(\left(\varepsilon^\sigma_\mu(k) + \frac{\alpha+i\beta}{\sqrt{2}} \frac{k_\mu}{|\vec k|}\right)\left(\varepsilon^\sigma_\nu(k) + \frac{\alpha+i\beta}{\sqrt{2}} \frac{k_\nu}{|\vec k|}\right)\right)\Big|_{\sigma=\pm 1}$$

and more generally $$\varepsilon^\sigma_{\mu\nu}(p) = L_\mu{}^{\bar\mu}(p) L_\nu{}^{\bar\nu}(p)\, \varepsilon^\sigma_{\bar\mu}(k)\, \varepsilon^\sigma_{\bar\nu}(k) = e^{2i\sigma\theta(\Lambda,p)} (\Lambda^{-1})_\mu{}^{\bar\mu} (\Lambda^{-1})_\nu{}^{\bar\nu} \left(\varepsilon^\sigma_{\bar\mu}(\Lambda p) + \frac{\alpha+i\beta}{\sqrt{2}} \frac{(\Lambda p)_{\bar\mu}}{|\vec p|}\right) \cdot \left(\varepsilon^\sigma_{\bar\nu}(\Lambda p) + \frac{\alpha+i\beta}{\sqrt{2}} \frac{(\Lambda p)_{\bar\nu}}{|\vec p|}\right)$$

so that

(41) $$U(\Lambda) h_{\mu\nu}(x) U^\dagger(\Lambda) = \int \frac{d^3p}{(2\pi)^3} \frac{\sqrt{2E_{p'}}}{2E_p} \left\{ e^{-ipx} a(\Lambda p, \sigma)\, e^{-2i\sigma\theta(\Lambda,p)} \varepsilon^\sigma_{\mu\nu}(p) + \dots \right\} \to (\Lambda^{-1})_\mu{}^{\bar\mu} (\Lambda^{-1})_\nu{}^{\bar\nu} (\dots) .$$

$$x' = \Lambda x$$

$$= (\Lambda^{-1})_\mu{}^{\bar\mu} (\Lambda^{-1})_\nu{}^{\bar\nu} \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{p'}}} \left\{ e^{-ipx'} a(p', \sigma)\, \varepsilon^\sigma_{\bar\mu\bar\nu}(\Lambda p) + \dots \right\}\Big|_{x'=\Lambda x} + \partial_\mu \Omega_\nu + \partial_\nu \Omega_\mu$$

$$= (\Lambda^{-1})_\mu{}^{\bar\mu} (\Lambda^{-1})_\nu{}^{\bar\nu} h_{\bar\mu\bar\nu}(\Lambda x) + \partial_\mu \Omega_\nu + \partial_\nu \Omega_\mu$$

confirming (38). Analogously to $F_{\mu\nu}$ for the spin-1, one can look at the irrep

(42) $$(2,0)\oplus(0,2) = R_{\mu\nu\alpha\beta} = \frac{1}{2}\left[\partial_\mu \partial_\beta h_{\nu\alpha} - \partial_\mu \partial_\alpha h_{\nu\beta} - \partial_\nu \partial_\beta h_{\mu\alpha} + \partial_\nu \partial_\alpha h_{\mu\beta}\right]$$

which is a true 4-tens: $$0 = \partial_\mu \partial_\beta (\cancel{\partial_\nu \Omega_\alpha} + \cancel{\partial_\alpha \Omega_\nu}) - \partial_\mu \partial_\alpha (\cancel{\partial_\nu \Omega_\beta} + \cancel{\partial_\beta \Omega_\nu}) - \partial_\nu \partial_\beta (\cancel{\partial_\mu \Omega_\alpha} + \cancel{\partial_\alpha \Omega_\mu}) + \partial_\nu \partial_\alpha (\cancel{\partial_\mu \Omega_\beta} + \cancel{\partial_\beta \Omega_\mu})$$

$\Big(R_{\nu\beta} \equiv R^\mu{}_{\nu\mu\beta} \equiv \eta^{\mu\alpha} R_{\mu\nu\alpha\beta} = \frac{1}{2}\left[\partial_\beta \partial_\mu h^\mu{}_\nu + \partial_\nu \partial_\mu h^\mu{}_\beta - \Box h_{\nu\beta} - \partial_\nu \partial_\beta h\right]$ $\qquad h = h_\mu{}^\mu \equiv h_{\mu\nu}\eta^{\mu\nu}$

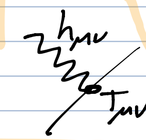
$R \equiv \eta^{\nu\beta} R_{\nu\beta} = \partial_\mu \partial_\nu h^{\mu\nu} - \Box h$

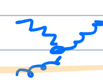
One difference with photons is that $R_{\mu\nu\rho\sigma} \sim O(\partial^2, h)$ so that the correct kinetic term of $O(\partial^2, h^2)$ can't be built out of invariant lagrangian density terms. The graviton kinetic term is gauge invariant only up to a total derivatives

$$\mathcal{L} = -\frac{1}{4}\left(2\partial_\mu h^\mu{}_\nu \partial^\nu h - 2\partial_\alpha h^\alpha{}_\nu \partial_\beta h^{\beta\nu} + \partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta} - \partial_\mu h\, \partial^\mu h\right) \Big)$$

Like for the photon, there are more general Lorentz (hence gauge) invariant interactions for the graviton which are not built out of $R_{\mu\nu\alpha\beta}$, those of the type

(43) $$\boxed{\int h_{\mu\nu} T^{\mu\nu} \Rightarrow \int (\partial_\mu \Omega_\nu + \partial_\nu \Omega_\mu) T^{\mu\nu} = \int 2\partial_\mu \Omega_\nu T^{\mu\nu} \Rightarrow \partial_\mu T^{\mu\nu} = 0}$$



where $h_{\mu\nu}$ couples linearly to a conserved symmetric 2-tensor $T^{\mu\nu}$, the energy-momentum tensor. As we will see in the next lectures, the graviton contributes to its own $T_{\mu\nu}$ giving rise to  self-interactions, and in particular there is no distance past which we can neglect the coupling to the source $T_{\mu\nu}$.

Sometimes, the linear coupling to the conserved currents, $A_\mu J^\mu$ & $h_{\mu\nu}T^{\mu\nu}$, is called the minimal coupling, to contrast it to coupling to trivial currents, e.g. $F_{\mu\nu}\bar\psi \sigma^{\mu\nu}\psi \sim 2A_\mu \partial_\nu(\bar\psi \sigma^{\mu\nu}\psi)$ where $\hat{J}^\mu = \partial_\nu(\bar\psi \sigma^{\mu\nu}\psi)$ is identically conserved, $\partial_\mu \hat{J}^\mu \big|_{\text{ident.}} = 0$ by antisym.. Analogous non-minimal couplings can arise from self-interactions, $(F_{\mu\nu}F^{\mu\nu})^2 = F_{\mu\nu}(F^{\mu\nu}F^2) \sim 2A_\mu\partial_\nu(F^{\mu\nu}F^2)$ $\Rightarrow \partial_\mu\partial_\nu(F^{\mu\nu}F^2) = 0$ identically, not just on the eq. of motions.

Similar non-minimal couplings exist for gravity too, e.g. $R_{\mu\nu\rho\sigma}R^{\rho\sigma\alpha\beta}R_{\alpha\beta}{}^{\mu\nu}$.

These non-minimal coupling to matter, e.g. $F_{\mu\nu}\bar\psi\sigma^{\mu\nu}\psi$, $R_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}$ ..., built directly in terms of gauge invariant $F_{\mu\nu}$ or $R_{\mu\nu\rho\sigma}$ give effects which at long distance are more suppressed than the one due to minimal coupling $A_\mu J^\mu$ and $h_{\mu\nu}T^{\mu\nu}$ because the latter involve less derivatives. For example, the force between two static sources in $F_{\mu\nu}\frac{\bar\Psi\gamma^{\mu\nu}\psi}{\Lambda}$ is the one of dipole-dipole type (rather than pole-pole in $J_\mu A^\mu$ [illegible]), giving rise to $V =$ [illegible] $\sim \frac{1}{r^3\Lambda^2}$ a much rapid fall-off with distance than the $1/r$ from Coulomb law. Non-minimal coupling to $R_{\mu\nu\alpha\beta}$ can give $V \sim \frac{1}{r^5\Lambda^4}$.

Remark: The case of massless higher spins $\geq 3/2$ is totally analogous: gauge-invariance is required by Lorentz-invariance, in particular the higher-spin polarizations for integer spin transform as:

$$(44)\qquad \varepsilon^{\sigma=\pm s}_{\mu_1\ldots\mu_n}(p) \to (\Lambda^{-1})_{\mu_1}{}^{\bar\mu_1}\cdots(\Lambda^{-1})_{\mu_n}{}^{\bar\mu_n}\,\varepsilon^{\sigma=\pm s}_{\bar\mu_1\ldots\bar\mu_n}(\Lambda p) + \partial_{(\mu_1}\Omega_{\mu_2\ldots\mu_n)}$$

symmetrize

[Aside comment: so far we insisted in spin ≥ 1 of mass zero, obtained by vector & tensor fields (up to gauge transf.). Nothing prevents to embed spin-0 particles inside true Lorentz vector and tensor: pick the trivial irrep of rotations around $R_z(\theta)$ in Eq.(28), $R^z(\theta)_\mu{}^\nu\,\varepsilon_\nu(k) = \varepsilon_\nu(k)$ corresponding to $\sigma = 0$, then clearly it's possible to find such $\varepsilon_\mu(k)$ with also $S_\mu{}^\nu(\alpha,\beta)\,\varepsilon_\nu(k) = \varepsilon_\nu(k)$ by just taking $\varepsilon_\nu(k) = k_\nu = \begin{pmatrix} E\\0\\0\\E\end{pmatrix}$ that by definition of little group is left-invariant by $W \in ISO(2)$ $W = S\cdot R_z$. With this choice $\varepsilon_\mu(p) = L(p)_\mu{}^\nu\varepsilon_\nu(k) = p_\mu \Rightarrow A_\mu = \partial_\mu\pi$, $h_{\mu\nu} = \partial_\mu\partial_\nu\pi, \ldots$ which are true Lorentz tensor, no gauge symmetry is required.]

In the next lecture we will explore the deep consequences that Lorentz invariant brings (via on-shell gauge invariance) to S-matrix elements.