Topics in Effective Field Theory

PhD-course, Rome 2020 by B. Bellazzini

Sensitivity to UV thresholds & the Hierarchy Problem

We have seen that a heavy field of mass $M \gg E$ (E being the energy we are interested in, e.g. the max energy available at a collider) leaves its imprint into operators made of the low-frequency modes, generating in general relevant, marginal and irrelevant operators. We want to study now how the low-energy predictions are sensitive to little changes of the UV scale $M \to M + \delta M$. At the tree-level clearly $\frac{\delta \mathrm{Obs}}{\mathrm{Obs}} \sim (E/M)^{\Delta-4}$ so one is expected to be very sensitive for the relevant operators $\Delta < 4$, only marginally sensitive for $\Delta = 4$, and insensitive for $\Delta > 4$. Let's look at it at one-loop, focusing on a few examples.

— QED with heavy ψ —

(1) $$\mathcal{L}^{QED}_{\Lambda \gg M} = -\frac{1}{4 g_0^2} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{\partial} - M)\psi + q \bar{\psi}\gamma^\mu \psi A_\mu$$ (+ other light fields like the electron)

g_0: gauge coupling

q: electric charge: with a single field ψ is not physical, it can be reabsorbed into q via $A_\mu \to A_\mu / q$. With more fields, the ratio of charges is physical $q_i \bar{\psi}\gamma^\mu\psi A_\mu \to \left(\frac{q_i}{q}\right) \bar{\psi}\gamma^\mu \psi A_\mu$.

We are interested in $E \ll M$ so that we integrate ψ out at one-loop:

(2) $$Z = \int [dA]\, e^{-\frac{i}{4 g_0^2}\int F_{\mu\nu}^2 d^4x}\, Z[A] \qquad Z[A] = \int [d\psi][d\bar{\psi}] \exp\left\{ i \int d^4x\, \bar{\psi}(i\not{\partial} - M)\psi + q J_\mu(\psi) A_\mu \right\}$$

where $Z[A]$ can be considered the functional generator for J^μ-currents of ψ & $\bar{\psi}$ in the spectator background of A_μ-field upon we haven't integrated yet.

L6/2

From the perspective of ψ, the A_μ is just an external source $\frac{1}{Z[0]} \frac{\delta^n Z[A]}{\delta A_{\mu_1}(x_1) \dots \delta A_{\mu_n}(x_n)} \propto \langle T j_{\mu_1}(x_1) \dots j_{\mu_n}(x_n) \rangle$
which brings down the functional derivatives, e.g.

(3) $Z[A] = \text{const} - \frac{g^2}{2!} \int d^4x\, d^4y\, A_\mu(x) \langle T j^\mu(x) j^\nu(y) \rangle A_\nu(y) + o(A^4 j^4)$

Since j^μ is conserved exactly, $\partial_\mu j^\mu = 0$, the $\langle T j^\mu(x) j^\nu(y) \rangle$ is transverse

(4) $\boxed{\langle T \hat{j}_\mu(k) \hat{j}_\nu(-k) \rangle = i (\eta^{\mu\nu} k^2 - k^\mu k^\nu) \Pi(k^2)}$

[the j^μ is neutral $\Rightarrow$ Ward-Takahashi ident. $\partial_\mu \langle T j^\mu j^\nu \rangle = 0$]

Now, the $(\eta^{\mu\nu} k^2 - k^\mu k^\nu)$ is the perfect tensor structure to generate corrections to $F_{\mu\nu}^2$, that is to change the gauge coupling constant. Indeed,

(5) $-\frac{1}{4} \int d^4x\, (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 \underset{\text{by parts}}{=} -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \hat{A}_\mu(k) (k^2 \eta^{\mu\nu} - k^\mu k^\nu) \hat{A}_\nu(-k)$

so that from (2) we get

$\frac{1}{g^2_{eff}} = \frac{1}{g^2(\Lambda)}$ for $k^2 = \Lambda^2$

(6) $Z = \int [dA] \{ \text{const} - \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \hat{A}_\mu(k) \hat{A}_\nu(-k) (k^2 \eta^{\mu\nu} - k^\mu k^\nu) \underbrace{\left[\frac{1}{g^2(\Lambda_0)} + g^2 \Pi(k^2) \right]}_{\text{running coupling}} + o(\partial A^4) \}$

From a direct calculation of $\langle \bar{\psi} \gamma^\mu \psi \bar{\psi} \gamma^\nu \psi \rangle$ we get $\gamma^\mu \bigcirc \gamma^\nu$

(7) $\langle T \hat{j}^\mu(k) \hat{j}^\nu(-k) \rangle = - \int \frac{d^4p}{(2\pi)^4} \text{Tr}\left[\gamma^\mu \frac{i}{\not{k} + \not{p} - M} \gamma^\nu \frac{i}{\not{p} - M} \right] = i (\eta^{\mu\nu} k^2 - k^\mu k^\nu) \Pi(k^2)$ where

(8) $\Pi(k^2) = -\frac{1}{2\pi^2} \int_0^1 dx\, x(1-x) \log\left(\frac{M^2 - x(1-x) k^2}{\Lambda_0^2} \right) + \text{counterterm}$

such that the matching (initial) condition is $\Pi(k^2 = \Lambda_0^2) = 0$ $\left(\frac{1}{g^2_{eff}} \Big|_{\Lambda = \Lambda_0} = \frac{1}{g_0^2} \right)$

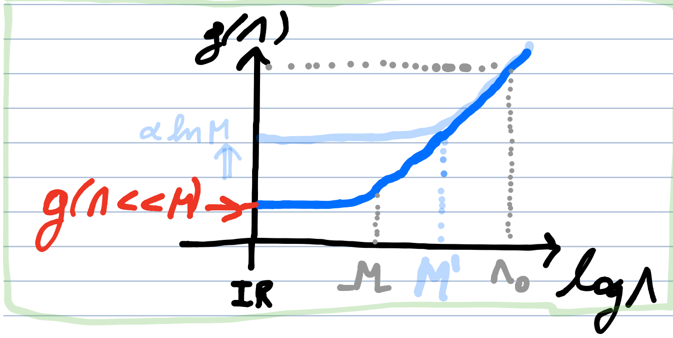
(9) $\Pi(k^2) = -\frac{1}{(2\pi^2)} \int_0^1 dx\, x(1-x) \log\left(\frac{M^2 - x(1-x) k^2}{M^2 - x(1-x) \Lambda_0^2} \right)$

The running coupling is therefore $\Lambda \frac{d}{d\Lambda} \frac{1}{g^2(\Lambda)} = -\frac{2}{g^3} \beta_g = g^2 \Lambda \frac{d}{d\Lambda} \Pi(k^2 = \Lambda) \to \boxed{\beta_g = -\frac{1}{2} g^3 g^2 \Lambda \frac{d}{d\Lambda} \Pi(\Lambda^2)}$

(10) $\beta_g = -\frac{1}{2} q^2 g^3 \Lambda \frac{d}{d\Lambda}\Pi(\Lambda^2) = -\frac{q^2 g^3}{2\pi^2}\int_0^1 dx\, \frac{x^2(1-x)\Lambda^2}{M^2 - x(1-x)\Lambda^2}$ $\xrightarrow{M\ll\Lambda}$ $\frac{q^2 g^3}{12\pi^2}$

$\xrightarrow{M\gg\Lambda}$ $\frac{-q^2 g^3}{60\pi^2}\frac{\Lambda^2}{M^2}$ ← once past M, it doesn't run any longer $\Lambda/M \to 0$

so that for $\Lambda \gg M$ (when the particle behaves like massless) $\beta^{QED} = \frac{q^2 g^3}{12\pi^2}$, whereas $\beta^{QED} \xrightarrow[\Lambda\ll M]{} 0$

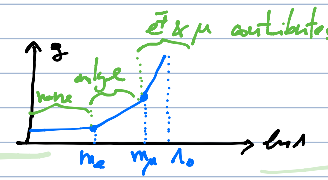


The IR coupling $g(\Lambda \ll M)$ depends only logarithmically on M: $g(\Lambda \ll M) \simeq g(\Lambda \simeq M)$ with $g(M) = g(\Lambda_0) + \beta^{QED}\ln(M/\Lambda_0)$

(11) $\underset{g_{IR}}{\uparrow}\, g(\Lambda \ll M) \simeq g(\Lambda \simeq M) = \underset{g_{UV}}{g_0} + \underset{\text{positive}}{\beta^{QED}} \ln(M/\Lambda_0)$

so that the sensitivity to the mass M is only logarithmic; it is usually measured by the dimensionless $M\frac{d}{dM} g_{IR} = \beta^{QED} \ll 1$

* Remarks:



- generalization to more species is simple
  the β-funct. undergoes a series of steps as a mass threshold is reached
- Heavy particles decouple when $M \to \infty$ except for setting the initial condition on the coupling which depends logarithmically on M
- QED coupling is marginally irrelevant: it becomes smaller as E is decreased. But it takes an exponentially long run to send it to zero, so we keep it around. [moreover, the running stop at $E = m_e$, so QED is actually not IR-free, which is why we can see with our eyes]
  Viceversa, g grows towards the UV: treating g as small perturbation around the free theory breaks down when $g(E \simeq \Lambda_{UV}) \simeq 4\pi = \underbrace{g_{IR}}_{\ll 1} + \beta \ln\frac{\Lambda_{UV}}{\Lambda_{IR}} \Rightarrow \boxed{\Lambda_{UV} = \Lambda_{IR}\exp[4\pi/\beta] \text{ LANDAU Pole}}$
- We performed the calculation in perturbation theory but most of the results are general even for strongly coupled scale-invariant theories with a global $U(1)$ conserved current that is coupled

to A_μ as perturbation (weak gauging, $g \ll g^*$ the coupling of the scale-inv. theory). Indeed, a conserved current j^μ has always scale-dimension $\Delta_J = 3$ (in $D=4$, $D-1$ in general, it follows from Ward identity $\partial_\mu \langle T J^\mu \mathcal{O} \rangle = i\,\delta^4(x-y) \langle \mathcal{O} \rangle$) $\Rightarrow \langle \hat{J}_\mu \hat{J}_\nu \rangle_{CFT} = -i(\eta^{\mu\nu}k^2 - k^\mu k^\nu) \frac{b^{CFT}}{16\pi^2} \ln \frac{k^2}{\Lambda^2}$

the 2pt-function is fixed up to one number b^{CFT}, so that $\beta^{CFT}_{QED} = g^3 \frac{b^{CFT}}{16\pi^2}$

Above we have basically calculated  J_μ J_ν when $CFT|_{gaussian}$ gives $b^{CFT}_{gauss} = 4/3$.

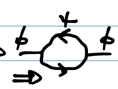
– UV-sensitivity & non-decoupling for scalars —— • ——

The UV d.o.f. set the initial conditions for the IR values of the EFT parameters so that one may be worried that relevant operators with $\Delta \leq 4$ are polynomially sensitive to UV masses, just by dimension analysis. We have seen this already at tree level (Eq. 15 in L3/6) and now we see an example at 1-loop for scalars. Consider a Yukawa Model with a heavy fermion:

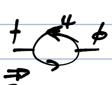
$$(12) \quad \mathcal{L}_Y = \frac{1}{2}(\partial\phi)^2 - m_\phi^2 \frac{\phi^2}{2} + \bar{\psi}(i\not{\partial} - M)\psi + y\phi\bar{\psi}\psi \Big|_{\Lambda_0} + \cdots$$

We integrate out ψ and look at the change in the ϕ-mass

$$(13) \quad \mathcal{L}^\phi_{EFT}\Big|_{\Lambda \ll M} = \frac{1}{2}(\partial\phi)^2 - m_\phi^2 \frac{\phi^2}{2} + \text{loops}\Big|_\Lambda$$

loops $\supset$  correction to $\delta m_\phi^2 \frac{\phi^2}{2}$

$$(14) \quad \phi \; \psi \; \phi \;(p=0) = -y^2 \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\left(\frac{\not{k}+M}{k^2 - M^2 + i\epsilon}\right)^2\right] \frac{\phi^2}{2} \Rightarrow \delta m_\phi^2 = -\frac{4y^2}{16\pi^2} \int_{\Lambda^2}^{\Lambda_0^2} dk^2 \frac{(k^2 - M^2)k^2}{(k^2+M^2)^2}$$



I: $$-\frac{4y^2}{16\pi^2}\left[\Lambda_0^2 - \Lambda^2 - 3M^2 \log\frac{\Lambda_0^2 + M^2}{\Lambda^2 + M^2} - 2M^4\left(\frac{1}{\Lambda_0^2 + M^2} - \frac{1}{\Lambda^2 + M^2}\right)\right]$$

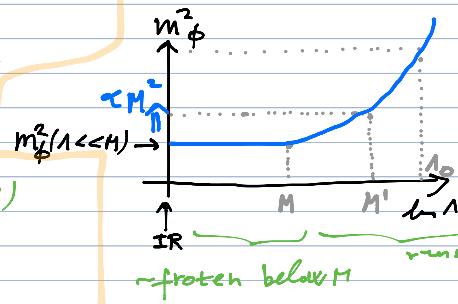
HARD CUTOFF (with matching $\delta m_\phi^2(\Lambda = \Lambda_0) = 0$)

II: $$-\frac{4y^2}{16\pi^2} 3M^2 \log \Lambda^2/\Lambda_0^2$$

DIM-REGU.

where we are going to show two regulators, hard cutoff (I) and dimens. regularization (II) in order to show that the question of sensitivity to UV thresholds is a physical issue independent of renormalization schemes [side remark: in terms of the physical mass $m_R^2 = m^2(\Lambda=0)$, the RG-flow is cutoff independent, namely $m^2(\Lambda) = m_R^2 - \Lambda^2 - 3M^2 \log \frac{M^2}{M^2+\Lambda^2} - 2M^4\left(\frac{1}{M^2} - \frac{1}{\Lambda^2+M^2}\right)$, which is Λ_0-independent]

From the hard-cutoff (I) we get quadratic sensitivity



$$(15) \quad M\frac{d}{dM} m_\phi^2(\Lambda) \underset{(I)}{=} -\frac{4y^2}{16\pi^2}\left(-6M^2 \log(\Lambda_0/\Lambda)^2\right)$$

(neglecting M relative to Λ & Λ_0)

Notice that the mass is frozen below M,

$$(16) \quad \Lambda\frac{d}{d\Lambda} m_\phi^2(\Lambda) = -\frac{8y^2}{16\pi^2}\begin{cases}\Lambda^2 - 3M^2 & M \ll \Lambda \\ M^2 \cdot \left(\frac{\Lambda}{M}\right)^4 & \Lambda \ll M\end{cases}$$

The IR value $m_\phi^2(\Lambda \ll M) \simeq m_\phi^2(\Lambda \simeq M)$ is quadratically sensitive to M.

The same sensitivity is visible in dimensional regularization (II):

$$(17) \quad M\frac{d}{dM} m_\phi^2(\Lambda) \underset{(II)}{=} -\frac{4y^2}{16\pi^2}\, 6M^2 \log(\Lambda/\Lambda_0)^2 \qquad (\Lambda < \Lambda_0)$$

* Remarks:

- this UV sensitivity means that the initial conditions must be chosen very accurately in order to land on a small coefficient of a relevant operator. In the example above, $m^2(\Lambda) = m^2(\Lambda_0) - \frac{4y^2}{16\pi^2}\left(\Lambda_0^2 - \Lambda^2 - 3M^2 \log\frac{\Lambda_0^2+M^2}{\Lambda^2+M^2} + \dots\right)$ one sees that keeping fixed Λ_0 but varying M there must be an accurate cancellation between the $m^2(\Lambda_0)$ (the initial condition) and M^2, both are much larger than m^2_ϕ.

- One can actually use $\Lambda_0 \frac{d m_\phi^2}{d\Lambda_0}$ rather than $M\frac{d m_\phi^2}{dM}$ to see the sensitivity to the initial conditions at Λ_0.

Indeed, one can regard Λ_0 (or even Λ) as physical thresholds of a sort at $E = \Lambda_0$, one that cuts off the integral and makes finite the amplitudes in the UV

(string theory makes these amplitudes finite, so a lattice!)

Of course a sharp momentum cutoff is a very crude or brutal threshold, so some times it does not give a reasonable estimate of the sensitivity.

When the hard cutoff does not mimic well the physical properties of a threshold we should not use $\Lambda_0 \frac{\partial}{\partial \Lambda_0}$, but rather $M \frac{\partial}{\partial M}$. This is e.g. the case for gauge theories where a hard cutoff breaks gauge invariance explicitly (hence changing the # of IR d.o.f.) something that a physical threshold never does. [for example, charged matter renormalize the gauge coupling but do not generates the longitudinal polarization of the photon unless U(1) is broken spont.]

In these cases, is much better to use dim-reg (or use the correct $M\frac{d}{dM}$ estimator) since it represents the mildest regulator (that does not even introduce a physical threshold) which automatically respects all properties one requires for the UV.

Rule of thumb: a hard cutoff Λ_0 can be used to estimate sensitivity if it does not break important symmetries, including gauge redundancies, that we wish the UV physics to respect.

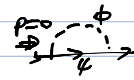
— Symmetry - Protection : spurions

Relevant operators have coefficients that may be UV-sensitive as we have seen above, but there are nice exceptions because of symmetries, even approximate ones. Let's see this again in the Yukawa model where we integrate out a heavy scalar:

$$(18)\quad \mathcal{L}_Y = \left[\frac{1}{2}(\partial\phi)^2 - M_\phi^2\frac{\phi^2}{2} + \bar\psi(i\cancel{\partial} - m_\psi)\psi + y\phi\bar\psi\psi\right]\Big|_{\Lambda \gg M_\phi^2 \gg m_\psi}$$

Integrating out ϕ at $\Lambda \ll M_\phi$ we get

(19) $\mathcal{L}^{(4)}_{EFT} = \left[\bar\psi(i\slashed{\partial} - m_\psi)\psi + \text{loops}\right]_\Lambda$

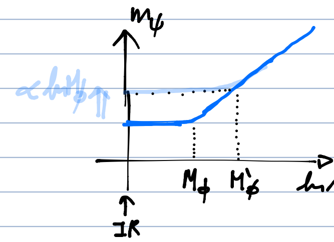
and focusing on the mass correction  $= \frac{-y^2}{2!}\int\frac{d^4k}{(2\pi)^4}\, i\frac{\slashed{k}-m_\psi}{k^2-m_\psi^2}\,\frac{i}{k^2-M_\phi^2} = \frac{y^2 i}{2\cdot 16\pi^2}\int dk^2\,\frac{k^2}{(k^2+m_\psi^2)(k^2+M_\phi^2)}$

and for small m_ψ

(20) $$\boxed{\delta m_\psi = \frac{-y^2}{2\cdot 16\pi^2}\, m_\psi \log\left(\frac{\Lambda_0^2+M_\phi^2}{\Lambda^2+M_\phi^2}\right)}$$

We see the striking contrast in the sensitivity: the UV mass threshold M_ϕ enters only logarithmically, the positive mass dimension of the relevant $\bar\psi\psi$ being saturated by the IR scale m_ψ, rather than UV $M_\phi, \Lambda, \Lambda_0$. The fermion mass, despite being relevant def., is only logarithmically sensitive like a marginal operator would be

(21) $$\Lambda\frac{d}{d\Lambda}m_\psi = \frac{y^2}{4\cdot 16\pi^2}\, m_\psi\frac{\Lambda^2}{\Lambda^2+M_\phi^2} \to \frac{y^2}{4\cdot 16\pi^2}\begin{cases} m_\psi & M_\phi \ll \Lambda \\ m_\psi\frac{\Lambda^2}{M_\phi^2} & M_\phi \gg \Lambda\end{cases}$$



This UV-insensitivity comes from a symmetry that emerges when $m_\psi \to 0$

(22) $\begin{cases}\psi_L \to +\psi_L \\ \psi_R \to -\psi_R \\ \phi \to -\phi\end{cases}$ (i.e. $\psi \to -\gamma_5\psi$) "Discrete Chiral Symmetry"

↳ left & right-handed $\psi_{L,R}$ transform differently

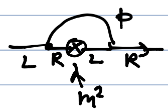
The kinetic term with $m_\psi = 0$ is clearly invariant under (22)

(23) $\bar\psi i\gamma^\mu\partial_\mu\psi = \psi_L^\dagger\bar\sigma^\mu i\partial_\mu\psi_L + \psi_R^\dagger i\sigma^\mu\partial_\mu\psi_R \xrightarrow[(22)]{} \bar\psi i\slashed{\partial}\psi$ $\qquad \psi = \begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix}$

and likewise interactions are, $\phi\bar\psi\psi = \phi(\psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L) \to (-\phi)(-\bar\psi\psi) = \phi\bar\psi\psi$, whereas the mass term, a relevant term, actually breaks it

(24) $$\boxed{m_\psi\bar\psi\psi \to -m_\psi\bar\psi\psi}$$

Now, the RG-flow respects the symmetry when $m_\psi = 0$, and in fact it can generate symmetry breaking operators at low-energy only if m_ψ is inserted in the loops



which we see now is logarithmically UV-sensitive!

More generally, if we promote m to a spurionic field $m \longrightarrow -m$ charged under the symmetry the low-energy EFT must be invariant under (22) + ($m \to -m$) and built out of ψ & m only, the low-energy fields: $m\bar\psi\psi$ is neutral & it can thus appear, whereas $\bar\psi\psi M_\phi$ is charged and it can't appear generated by the RG-flow. Let's see another example

Parity for fermions

Another symmetry of $\mathcal{L}_Y$ in (18) is a $\mathbb{Z}_2$ symmetry called parity:

(25) $\psi_L \longleftrightarrow \psi_R \qquad (\phi \to \phi)$

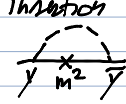
This sym. forbids the generation of parity-odd operators such as

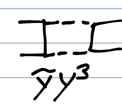
(26) $\bar\psi\gamma^5\psi$, $(\bar\psi\gamma^5\psi)(\bar\psi\psi)$, $(\bar\psi\gamma^5\gamma^\mu\psi)(\bar\psi\gamma^\mu\psi)$ all change sign under (25)

In order to generate P-breaking operators let's add also

(27) $\delta\mathcal{L}^{UV}_{\not P} = i\tilde y\,\phi\bar\psi\gamma^5\psi\Big|_{\Lambda_0} = i\tilde y\,\phi(\psi_L^\dagger\psi_R - \psi_R^\dagger\psi_L)\Big|_{\Lambda}$.

The coupling $\tilde y$ promoted to a spurion $\tilde y \xrightarrow[P]{} -\tilde y$ restores the symmetry so that $\bar\psi\gamma^5\psi$, $\bar\psi\gamma^5\psi\bar\psi\psi$ and $\bar\psi\gamma^5\gamma^\mu\psi\bar\psi\gamma^\mu\psi$ requires at least on $\tilde y$-insertion





Example: U(1) chiral symmetry

Let's consider now a 2-component fermion ψ. One way to understand why its mass is small (relative to any UV-scale) is to endow ψ with a U(1) symmetry

(28) $\mathcal{L}^{EFT}_\psi = \bar\psi i\gamma\psi - \cancel{m(\psi^{\dagger 2}+\psi^2)} + \frac{g_*^2}{\Lambda^2}\psi^{\dagger 2}\psi^2 + \dots \Big|_\Lambda \qquad \psi \xrightarrow{U(1)} e^{i\alpha}\psi$

The mass $m(\phi_1^2+\phi_2^2)$ breaks the symmetry, so that again m-running is $\propto m$ itself, if the breaking was small in the UV it remains so in the IR.

A very nice representative of this idea is the relation between Lepton number & neutrino masses in the SM: consider the theory at $\Lambda \ll M_W$

Fermi Theory

$$\mathcal{L}^{SM}_{\Lambda \ll M_W} \supset \bar{e}\, i\not{\partial} e + \bar{\mu} i \not{\partial} \mu - m_e \bar{e} e - m_\mu \bar{\mu}\mu + \bar{\nu} i \not{\partial} \nu + \frac{G_F}{\sqrt{2}} (\bar{e}\gamma^\alpha \nu_e)(\bar{\nu}_\mu \gamma_\alpha \mu) \tag{29}$$

(← neglecting τ for simplicity)

This EFT is invariant under $U(1)^2 = U(1)_{e,\nu_e} \otimes U(1)_{\mu,\nu_\mu}$ ← lepton #

$$\boxed{\begin{cases} e \to e^{i\alpha} e & \mu \to e^{i\beta}\mu \\ \nu_e \to e^{i\alpha}\nu_e & \nu_\mu \to e^{i\beta}\nu_\mu \end{cases}} \tag{30}$$

A small Majorana mass $m_{\nu_i}(\nu_i^2 + \nu_i^{+2})$ is justified because it breaks $U(1)$'s, as long as one has a good reason to think that Lepton-# is a good symmetry in the UV, broken by small effects. Without such a good reason one can say that small Majorana mass is technically natural. So, small Majorana masses for neutrinos are technically natural in the Fermi theory, but they are actually fully natural, not just technically, in the UV-completion of Fermi theory, the SM, because the SM has a reason why lepton-# is an approximate symmetry: the SM matter content with the observed gauge interactions does not allow for any relevant or marginal lepton-# violating interaction, even if one wanted to write down one (while respecting the gauge invariance). The lowest dimensional op. in the SM that breaks lepton-# is irrelevant

$$\boxed{y^2\frac{(LH)^2}{\Lambda}} \quad \Delta = 5 \quad \left(\begin{matrix} Y(L) = -1/2 \\ Y(H) = 1/2 \end{matrix} \quad Q = T^3 + Y\right) \tag{31}$$

$\Longrightarrow$ $\boxed{m_\nu = \frac{y^2 v^2}{\Lambda} \text{ very small if } \Lambda \text{ is very large (see-saw m)}}$

The smallness of Lepton-# violation, hence small neutrino masses, is explained by the irrelevance of the $\Delta=5$ operator in (31): the larger the cutoff to SM that breaks L, the smaller the Majorana mass.

* Remark: the masses of the charged fermions of the SM are also technically natural, "protected" by a chiral sym. $\psi \to e^{i\alpha}\psi$ like for the neutrinos: ($m_e \bar{e}e$)

However, the SM does not offer (contrary to the neutrinos) a fully natural solution because the smallness of Yukawa's is not explained:

IR	$\to$ SM
$m\bar{\psi}\psi$	$yH\bar{\psi}_L\psi_R$
small relevant	small marginal, $m \propto y v$

In technicolor models (now dead) one has instead of $yH\bar{\psi}_L\psi_R$ an irrelevant operator $\frac{\bar{T}_L T_R \bar{\psi}_L\psi_R}{\Lambda_{UV}^2}$ with new techniquarks T_L, T_R that condense $\langle \bar{T}_L T_R\rangle = \Lambda^3 \ll \Lambda_{UV}$ generating small masses

Example: $g-2$

Spurion analysis is useful to suppress not only the relevant and marginal op., but also the leading irrelevant ones. The $g-2$ of the electron (and μ) is very (very) well measured, and it nicely match the SM predictions.

However some low-energy UV-physics that could complete the SM could generate another dim-5 operator that changes $g-2$:

$$(32) \quad \frac{e}{\Lambda}\bar{\psi}\sigma^{\mu\nu}\psi F_{\mu\nu} \Rightarrow \Delta_g^{UV} \propto m_\psi/\Lambda$$

which would require $\Lambda \ggg m_\psi$ to be compatible with measurements of $g-2$. But, actually, the coupling (32) breaks also chiral symmetry since $\bar{\psi}\sigma^{\mu\nu}\psi = \psi_L^\dagger\sigma^{\mu\nu}\psi_R + \ldots \xrightarrow{\text{chiral}} -\bar{\psi}\sigma^{\mu\nu}\psi$ so that to generate (32) one needs to insert one spurion m_ψ:

(33) $\frac{e m_\psi}{\Lambda^2}\bar{\Psi}\sigma^{\mu\nu}\Psi F_{\mu\nu} \to \Delta g^{UV} \propto m_\psi^2/\Lambda^2$ which is a tremendous $\left(\frac{m_\psi}{\Lambda}\right) \ll 1$ gain: so that new physics can be much lower.

Estimating Size of Operators via Spurions

Let's consider a real scalar field ϕ and 2 Weyl 2-component ψ_i:

(34) $\mathcal{L} = \frac{(\partial\phi)^2}{2} + \psi_i^\dagger i\bar{\sigma}\psi_i + \phi\left(y_1\psi_1^2 + y_{12}\psi_1\psi_2 + y_2\psi_2^2 + h.c.\right)$

In the limit y's $\to 0$ there is a $[U(1)]^2$ symmetry

(reminder: $\psi_1^2 = \psi_{1\alpha}\psi_1^\alpha = \psi_1^\alpha\psi_1^\beta\epsilon_{\alpha\beta}$; $\psi^\dagger\psi$ not Lorentz invariant)

(35) $\psi_i \xrightarrow{U(1)} e^{-i\alpha_i}\psi_i$

	$U_1(1)$	$U_2(1)$
y_1	2	0
y_2	0	2
y_{12}	1	1

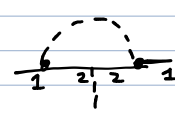
We can assign the following spurionic charges

So, how large can we expect y_1 given non-vanishing y_2 and y_{12}?

We need essentially build out of y_2 and y_{12} a combination, a spurion, that transforms like y_1 (so that ψ_1^2 is going to be generated by loops of ψ_2 & ϕ).

Clearly, $y_{12}^2 y_2^*$ transforms like y_1 $\Rightarrow$ $y_1\psi_1^2\phi$ is expected with

(36) $\boxed{y_1 \simeq \frac{y_{12}^2 y_2^*}{16\pi^2}}$



If the y_1 is observed much smaller than this the theory would look unnatural.

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— Example: Parity-breaking QED — ∘ — ∘ —

Consider a U(1) gauge theory with a massless spin-1 boson B_μ coupled to 2 Weyl fermions χ_L and ψ_L via a global U(1)-current under which χ_L (ψ_L) has charge +1 (−1)

	B_μ	χ_L	ψ_L
U(1) charge	0	1	−1

Let's write down the most general Lagrangian for χ_L, ψ_L & B_μ made of marginal and relevant operators only (those that dominate at low-energy):

(37) $$\mathcal{L}_{\Delta\le 4} = -\frac{1}{4g^2}B_{\mu\nu}B^{\mu\nu} + \bar\psi_L i\not{\partial}\psi_L + \bar\chi_L i\not{\partial}\chi_L + B_\mu\left(\psi_L^\dagger\bar\sigma^\mu\psi_L - \chi_L^\dagger\bar\sigma^\mu\chi_L\right) - \left(m\,\chi_L\psi_L + h.c.\right)$$

(omitting $B_{\mu\nu}B_{\rho\sigma}\,\varepsilon^{\mu\nu\rho\sigma}$ which is a total derivative)

Now, we can rewrite (37) in Dirac 4-component notation with $\psi = \begin{pmatrix}\psi_L \\ \epsilon\chi_L^*\end{pmatrix} = \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}$

[recall that $(1/2,0) \xrightarrow{*} (0,1/2)$ i.e. $i\sigma^2\chi_L^* \xrightarrow{\text{boost}} \exp[-\frac{\vec\sigma}{2}\vec\alpha]\, i\sigma^2\chi_L^*$ like right-handed field whereas $\chi_L \xrightarrow{\text{boost}} \exp[+\frac{\vec\sigma}{2}\vec\alpha]\chi_L$ like left-handed, since $\sigma^2\sigma^{i*}\sigma^2 = -\sigma^i$]

(38) $$\mathcal{L}_{\Delta\le 4} = -\frac{1}{4g^2}B_{\mu\nu}B^{\mu\nu} + \bar\psi i\not{\partial}\psi + B_\mu\bar\psi\gamma^\mu\psi - m\bar\psi\psi$$

So, this theory is accidentally Parity-symmetric like QED, because we restricted to $\Delta\le 4$ operators only.

At dim-6, there are various operators that break Parity:

(39) $$\mathcal{L}_{\not P}^{\Delta=6} = (\bar\psi\gamma^\mu\psi)(\bar\psi\gamma^\mu\gamma^5\psi),\ (\bar\psi\psi)(\bar\psi\gamma^5\psi)$$ (the latter breaks also chiral symmetry)

Under chiral U(1) $\begin{cases}\chi_L \to e^{i\alpha}\chi_L \\ \psi_L \to e^{i\alpha}\psi_L\end{cases}$ (so that ψ_L & $\psi_R \equiv \epsilon\chi_L^*$ have opposite charges)

We can thus use m as spurion: $m \to e^{-2i\alpha}m \Rightarrow m\bar\psi\psi = m\,\psi_L\chi_L + h.c.$ invariant so that the basic invariant are $\bar\psi\gamma^\mu\psi$, $\bar\psi\gamma^\mu\gamma^5\psi$ and $m\bar\psi\psi$, $m\bar\psi\gamma^5\psi$ ⇒ ratio of coefficients

(40) $1 : \epsilon_{\not P} : \epsilon_{\not P}\frac{m^2}{\Lambda^2}$ corresponding to $\frac{(\bar\psi\gamma^\mu\psi)^2}{\Lambda^2} + \frac{\epsilon_{\not P}}{\Lambda^2}(\bar\psi\gamma^\mu\psi)(\bar\psi\gamma^\mu\gamma^5\psi) + \frac{\epsilon_{\not P}m^2}{\Lambda^4}(\bar\psi\psi)(\bar\psi\gamma^5\psi)$