Topics in Effective Field Theory

PhD-course, Rome 2020 by B. Bellazzini

We have seen in L3 the RG-flow for ϕ^4-theory from a scale Λ_0 to Λ, obtained by demanding Λ-independence of physical quantities such as scattering amplitudes. But our EFT's are never that simple: infinitely many operators appear of which only a finite # is relevant/marginal, e.g. $(\partial\varphi)^2$, φ^2, φ^4, φ^6, $(\partial\partial\varphi)^2(\partial\varphi)^2$, ... (assuming $\varphi\to-\varphi$ for simplicity). We saw what happens when $(\partial\varphi)^2$-operator is perturbed by a classically marginal op. φ^4 (in D=4): loop corrections turn φ^4 into marginally irrelevant.

Let's see this calculation again, in a different way, that will generalize to other perturbations

Fast & Slow modes (i.e. heavy & light modes)

Let's consider the free kinetic energy for ϕ at some energy $E=\Lambda_0$

$$(1) \quad S^{KT}_{\Lambda_0}[\phi] = \int d^4x \, \frac{1}{2}(\partial\phi)^2 \Big|_{\Lambda_0}$$

and integrate an energy-shell up to $E=\Lambda$, by splitting $\hat{\phi}_k$ into slow modes ($\hat{\varphi}_k$) and fast modes ($\hat{\chi}_k$) [also known as light & heavy modes]

$$(2) \quad \boxed{\hat{\phi}_k = \hat{\varphi}_k + \hat{\chi}_k \quad \text{where} \begin{cases} \hat{\chi}_k = 0 & \text{for } |k|<\Lambda \\ \hat{\varphi}_k = 0 & \text{for } |k|>\Lambda \end{cases}}$$

[hence $\phi(x) = \varphi(x) + \chi(x)$ in position space too]

The free action, being quadratic in the field, is simple under the split (because mixed terms $\chi_k \varphi_{-k} = 0$ have zero overlap)

L4/2

(3) $S_{\Lambda_0}^{KT}[\phi] = \int_{|k|<\Lambda_0} \frac{d^4k}{(2\pi)^4} k^2 \hat\phi_k \hat\phi_{-k} = \int_{|k|<\Lambda} \frac{d^4k}{(2\pi)^4} k^2 \hat\varphi_k \hat\varphi_{-k} + \int_{\Lambda<|k|<\Lambda_0} \frac{d^4k}{(2\pi)^4} k^2 \hat\chi_k \hat\chi_{-k}$

which is nothing but the statement that $S_{\Lambda_0}^{KT}[\phi] = S_\Lambda^{KT}[\varphi] + S_{\Lambda_0}^{KT}[\chi]$ so that the path-integral over the heavy modes is trivial (factorized)

(4) $Z[J] = \int_{|k|<\Lambda_0} [D\hat\phi] \exp\left[i S_{\Lambda_0}^{KT}[\phi] + i\int J\phi\, d^4x\right] =$

$$= \int_{|k|<\Lambda} [D\hat\varphi_k] \exp\left[i S_\Lambda^{KT}[\varphi] + i\int J\varphi\, d^4x\right] \overbrace{\int_{\Lambda<|k|<\Lambda_0} [D\hat\chi_k] \exp\left[i S_{\Lambda_0}^{KT}[\chi]\right]}^{\text{irrelevant constant } N}$$

notice that we were sourcing only slow modes: $\hat J_k = 0$ for $|k|>\Lambda$

$\rightarrow \int J\phi\, d^4x = \int \hat J_k \varphi_{-k} \frac{d^4k}{(2\pi)^4} = \int J\varphi\, d^4x$

Up to an overall constant (UV-contribution to cosmological constant if gravity is on) which is irrelevant for the correlation functions of φ, the action is the same: the free massless KT is invariant and it's called the Gaussian fixed point where all β-functions are zero, $\beta_i = 0$ at $g_i = 0$. Clearly, the only important thing was the quadratic action to reach this conclusion, so $\int \bar\psi i\partial\!\!\!/\psi$, $\int -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ are also Gaussian fixed points [a mass term is also quadratic so that $S_{\Lambda_0}^{free}[\phi] \rightarrow S_\Lambda^{free}[\varphi]$ even with $m\phi^2$ added, as we will see below, but since the $m^2/\Lambda_0^2 \rightarrow m^2/\Lambda^2$ is not invariant the $\beta_{m^2} \neq 0$ and the massive Gaussian theory is not scale invariant, as of course we know from L3 already]

What if we add a perturbation $\delta\mathcal{O}$ to the free massless KT (the Gaussian fixed point)?

(5) $\delta S_{\Lambda_0} = \int d^4x\, g\, \mathcal{O}(x)$ where $\mathcal{O}(x) = \frac{\partial^n \phi^m}{\Lambda_0^{n+m-4}}$

Clearly, it's no longer true that $S_{\Lambda_0}[\varphi+\chi] = S_{\Lambda_0}[\varphi] + S_{\Lambda_0}[\chi]$ and therefore is no longer true that integrating out the modes $\hat\chi_k$ in the shell $\Lambda<|k|<\Lambda_0$ does nothing.

Instead, we have the following:

$$(6)\quad \boxed{S[\phi] = S^{KT}_{\Lambda_0}[\varphi+\chi] + \delta S_{\Lambda_0}[\varphi+\chi] = S^{KT}_{\Lambda}[\varphi] + S^{KT}_{\Lambda_0}[\chi] + \delta S_{\Lambda}[\varphi] + \delta S_{\Lambda_0}[\chi] + \text{"non-linear terms"}}$$

where the non-linear terms appear whenever $\mathcal{O}(x)$ in (5) has more than 2 ϕ legs so that non-trivial overlap between $\hat{\chi}$'s & $\hat{\varphi}$'s are kinematically allowed by momentum conservations.

For example:

$$(7)\quad \mathcal{O}(x) = \phi^2(x) \rightarrow \delta S[\phi] = \int d^4x\, (\varphi+\chi)^2 = \delta S[\varphi] + \delta S[\chi] + \underbrace{2\int d^4x\, \varphi(x)\chi(x)}_{2\int \frac{d^4k}{(2\pi)^4} \hat{\varphi}_k \hat{\chi}_{-k} = 0} = \underbrace{\delta S[\varphi] + \delta S[\chi]}_{\text{no non-linear effect for } \phi^2\text{-pert.}}$$

whereas

$$(8)\quad \mathcal{O}(x) = \phi^4(x) \rightarrow \delta S[\phi] = \int d^4x\, (\varphi+\chi)^4 = \delta S[\varphi] + \delta S[\chi] + \int d^4x \left[\underbrace{4(\varphi\chi^3 + \chi\varphi^3) + 6\varphi^2\chi^2}\right]$$

where the last term is non-vanishing because non-trivial overlap is possible

$[|k_{1,2}|^2 < \Lambda,\ \Lambda < |k_{3,4}| < \Lambda_0,$ with $k_1 + k_2 = k_3 + k_4$ is satisfied e.g. by $k_1 = -k_2$ $k_3 = -k_4$ with $|k_1|^2 < \Lambda$ and $\Lambda < |k_3| < \Lambda_0$ $]$

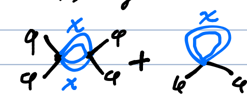
If we perform the path-integral over the fast ("heavy") modes:

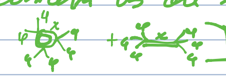
$$(9)\quad \boxed{\underset{\text{eq. (6)}}{Z[J]} = \int_{|k|<\Lambda} [D\hat{\varphi}]\, \exp\Big\{\underbrace{iS^{KT}_{\Lambda}[\varphi] + i\delta S^{KT}_{\Lambda}[\varphi]}_{\text{classical effect perturbation}} + i\int J\varphi\Big\} \cdot \underbrace{\int_{\Lambda<|k|<\Lambda_0} [D\hat{\chi}]\, \exp\Big\{iS^{KT}_{\Lambda_0}[\chi] + \text{non-linear terms}(\varphi)\Big\}}_{\text{quantum effects from int. out } \hat{\chi}}}$$

we see that our reasoning on the effect of perturbations in L3, where we simply added $\delta S[\varphi]$ to $S^{KT}[\varphi]$ to determine the scaling (i.e. marginality vs irrelevance vs relevance) was just an approximation where the non-linear (φ-dependent) effects in (9) where neglected (because e.g. small loop expansion in the coupling). Let's take them into account: the $S^{KT}[\chi]$ is quadratic so the quantum effects in (9) can be calculated in perturbation theory doing the usual gaussian integral with respect to χ while φ is an external spectator fields.

If the interaction perturbing the $S^{KT}_{\Lambda_0}[\phi]$ was

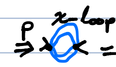
(10) $$\delta S^{I}_{\Lambda_0} = -\int d^4x \, \frac{\lambda}{4!}\phi^4 = -\int d^4x \left[\dots + \frac{\lambda}{4!}\left(4\varphi\chi^3 + 6\varphi^2\chi^2 + 4\varphi^3\chi\right)\right]$$

The gaussian integral in χ in (9) generates corrections to the classical operator such as φ^4 and φ^2: 

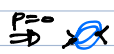
[other diagrams that we will not concern us are e.g. ]

- φ^4 corrections

From (9) + (10), two λ-insertions give

(11) $$\left(\frac{-i\lambda 6}{4!}\right)^2 \int_{|p|<\Lambda} \frac{d^4p}{(2\pi)^4} \, \varphi_p^2$$  $$\varphi^2_{-p}$$ where  $$= \int_{\Lambda<|k|<\Lambda_0} \frac{d^4k}{(2\pi)^4} \frac{i}{(k^2-m^2+i\epsilon)} \frac{i}{[(k+p)^2-m^2+i\epsilon]}$$

Since we are interested in φ^4-vertex corrections (rather than φ with derivatives $\propto p_i$) we can set $p=0$

(12)  $$\underset{\text{Wick rot.}}{=} \frac{-i}{16\pi^2}\int_{\Lambda^2}^{\Lambda_0^2} dk^2 \frac{k^2}{(k^2+m^2)^2} = \frac{-i}{16\pi^2}\left[\log\left(\frac{\Lambda_0^2+m^2}{\Lambda^2+m^2}\right) + m^2\left(\frac{1}{\Lambda_0^2+m^2} - \frac{1}{\Lambda^2+m^2}\right)\right] \simeq \frac{-i}{8\pi^2}\log(\Lambda/\Lambda_0) + O(m^2/\Lambda^2)$$

Since $\lambda(\Lambda)$ is the coefficient of $-\int d^4x \frac{\varphi^4(x)}{4!}$ (when φ is canonical normalized, or more physically the $2\to 2$ scattering amplitude) we get: $-\int\left(\frac{\lambda}{4}\right)^2\frac{4!}{8\pi}\ln(\Lambda/\Lambda_0)\,\frac{\varphi^4}{4!} = -\int d^4x \, \frac{3\lambda^2}{16\pi}\ln(\Lambda/\Lambda_0)\,\frac{\varphi^4(x)}{4!}$

(13) $$\boxed{\lambda(\Lambda) = \lambda(\Lambda_0) + \frac{3\lambda^2}{16\pi^2}\log(\Lambda/\Lambda_0) \longrightarrow \beta_\lambda = \frac{3\lambda^2}{16\pi^2}}$$

[notice that we made the approximation $1-x \simeq \exp(-x)$ in order to extract the corrections to the connected vertices]

- ϕ^2-corrections

The $\int -\frac{\lambda}{4!} 6\varphi^2\chi^2$ term in (10) generates as well corrections to the mass:

(14) $$\left(-\frac{i\lambda}{4}\right)\int d^4x \, \varphi^2(x)$$  where  $$= \int_{\Lambda<|k|<\Lambda_0}\frac{d^4k}{(2\pi)^4}\frac{i}{[k^2-m^2+i\epsilon]} \underset{\text{Wick}}{=} \frac{1}{16\pi^2}\int_{\Lambda^2}^{\Lambda_0^2} dk^2\frac{k^2}{k^2+m^2}$$

so that

corrections to the mass $-\int m^2(\Lambda_0)\frac{\varphi^2(x)}{2}d^4x$ is (recall $\exp(+iS)=1+iS+\ldots$)

$$(15)\quad m^2(\Lambda)=m^2(\Lambda_0)+\frac{\lambda}{2\cdot 16\pi^2}\int_{\Lambda^2}^{\Lambda_0^2}dK^2\frac{K^2}{K^2+M^2}$$

Now, we have been a little sloppy with defining our perturbations ϕ^2 & ϕ^4 of the Gaussian fixed point. Indeed, things like $\int[d\varphi]\,\varphi^2\exp[iS]$ require to know the behavior at coincident points (deep UV, sensitive to Λ_0) so that the result is quadratically divergent when $\Lambda_0\to\infty$. There is nothing wrong with this but there is a much better "basis of operators" to use: we define "Normal ordered operators" as following

$$(16)\quad :\phi^2:\,\equiv\phi^2-\underbrace{\langle\phi^2(x)\rangle}_{\substack{\text{subtracting something}\\ \propto\text{ to identity op.}}},\qquad :\phi^4:\,\equiv\phi^4-\underbrace{6\phi^2\langle\phi^2\rangle}_{\propto\text{ to }\phi^2}+\underbrace{3\langle\phi^2\rangle^2}_{\propto\text{ identity op.}},\ \ldots$$

Going to this basis one is removing all contractions of two legs that have both ends on the same vertex. In the case we are considering these contractions are scale dependent because the Gaussian fixed point includes modes only up to a scale Λ

$$(17)\quad :\phi^2:_\Lambda\equiv\phi^2-\int_0^{\Lambda^2}\frac{dK^2}{16\pi^2}\qquad :\phi^4:_\Lambda\equiv\phi^4-6\phi^2\int_0^{\Lambda^2}\frac{dK^2}{16\pi^2}+3\left(\int_0^{\Lambda^2}\frac{dK^2}{16\pi^2}\right)^2,\ \ldots$$

This means that adding $-\lambda:\phi^4:_{\Lambda_0}$ to the $\mathcal{L}^{free}_{\Lambda_0}$ one is also shifting the mass term

$$(18)\quad \boxed{m^2_{\Lambda_0}\to m^2_{\Lambda_0}-\frac{\lambda}{2}\int_0^{\Lambda_0^2}\frac{dK}{16\pi^2}}$$

(we actually started with $m^2_{\Lambda_0}=0$ as perturbing the Gaussian fixed point when $\lambda=m=0$)

& recalling that quadratic terms don't generate overlap between $\hat{\chi}$ & $\hat{\varphi}$ in $\phi=\varphi+\chi$, the net result when integrating out χ-modes is

$$(19)\quad \mathcal{L}_\Lambda(\varphi)=\frac{(\partial\varphi)^2}{2}-\overbrace{\left(m^2_{\Lambda_0}-\frac{\lambda}{2}\int_0^{\Lambda_0^2}\frac{dK^2}{16\pi^2}\right)\frac{\varphi^2}{2}-\frac{\lambda(\Lambda)}{4!}\varphi^4}^{\text{classical term}}-\underbrace{\frac{\delta\lambda}{4!}\varphi^4-\frac{\delta m^2}{2}\varphi^2}_{\text{non-linear, quantum effects from int. out }\chi}$$

already calculated above $\delta\lambda\propto\frac{\lambda^2}{16\pi^2}$

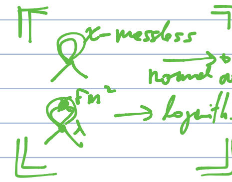
$$=\frac{(\partial\varphi)^2}{2}-\left(m^2_{\Lambda_0}-\frac{\lambda}{2}\int_{\Lambda^2}^{\Lambda_0^2}\frac{dK^2}{16\pi^2}\right)\frac{:\varphi^2:_\Lambda}{2}-(\lambda+\delta\lambda)\frac{:\varphi^4:_\Lambda}{4!}-\frac{\delta m^2}{2}:\varphi^2:_\Lambda+\text{cst}'\ \text{(+two loop corr.)}$$

$\varphi^4=\,:\varphi^4:_\Lambda+6:\varphi^2:_\Lambda\int^{\Lambda^2}\frac{dk}{16\pi^2}+\text{const}$

that is, the subtraction of the tadpole loops from massless χ-modes

$$(20)\quad m^2(\Lambda) = m^2(\Lambda_0) + \frac{\lambda}{2\cdot 16\pi^2}\overbrace{\int_{\Lambda^2}^{\Lambda_0^2} dK^2 \frac{K^2}{K^2+m^2}}^{\propto\, \delta m^2} \underbrace{-1\,\Big)}_{\text{from normal ordering}}$$

$$= m^2(\Lambda_0) - \frac{\lambda}{2\cdot 16\pi^2} m^2 \log\left(\frac{\Lambda_0^2+m^2}{\Lambda^2+m^2}\right)$$



The resulting β_{m^2}-function is

$$(21)\quad \boxed{\Lambda \frac{d}{d\Lambda} m^2(\Lambda)\Big|_{\Lambda=\Lambda_0} = +\frac{\lambda\, m^2}{16\pi^2}\frac{\Lambda_0^2}{\Lambda_0^2+m^2} \xrightarrow[\Lambda_0 \gg m]{} \frac{\lambda^2}{16\pi^2} m^2} \quad \leftarrow \text{anomalous dimension of } \phi^2$$

The $\gamma_{m^2} = \lambda^2/16\pi^2$ is the anomalous-dimension of ϕ^2 at the Gaussian fixed point.

Let's write the RG-flow for dimensionless couplings: $g_2 \equiv m^2/\Lambda^2 \qquad g_4 \equiv \lambda$

$$(22)\quad \begin{cases} \beta_2 = \Lambda \frac{d}{d\Lambda} g_2 = (\underbrace{-2}_{\text{classical}} + \underbrace{\frac{g_4}{16\pi}}_{\text{quantum}})\, g_2 \\ \beta_4 = \Lambda \frac{d}{d\Lambda} g_4 = \frac{3 g_4^2}{16\pi^2} \end{cases}$$

← quantum only because we are in $D=4$, where g_4 is classically scale inv.

More generally, in D-spacetime dimensions $\quad [\phi] = \frac{D-2}{2} \quad [\phi^2] = D-2 \quad [\phi^4] = 2(D-2)$

$$\rightarrow \delta S^{cl} = \int d^Dx\, -\frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \Rightarrow [m^2] = D - 2\frac{(D-2)}{2} = 2 \qquad [\lambda] = D - 2(D-2) = 4-D$$

so that for dimensionless couplings $g_2(\Lambda) = \frac{m^2}{\Lambda^2}$, $g_4(\Lambda) = \frac{\lambda}{\Lambda^{4-D}}$, the classical running is just dictated by the mass dimension as we have seen in L3:

$$(23)\quad S_{\Lambda_0} = \int d^Dx\, \frac{1}{2}(\partial\phi)^2 - \Lambda_0^2 g_2 \frac{\phi^2}{2} - \frac{g_4 \phi^4}{4!}\Lambda_0^{4-D} \Rightarrow S_\Lambda = \int d^Dx\, \frac{1}{2}(\partial\phi)^2 - \frac{\phi^2 \Lambda^2}{2}\left(g_2 \frac{\Lambda_0^2}{\Lambda^2}\right) - \frac{\phi^4 \Lambda^{4-D}}{4!}\left[g_4\left(\frac{\Lambda_0}{\Lambda}\right)^{4-D}\right]$$

$$(24)\quad \begin{cases} \Lambda \frac{d g_2}{d\Lambda} = -2 g_2 \qquad \Lambda \frac{d}{d\Lambda} g_4 = (D-4)\, g_4 \\ \Lambda \frac{d}{d\Lambda} g_{\mathcal{O}} = (\Delta - D)\, g_{\mathcal{O}} \end{cases} \qquad \left(\delta S^{cl}_{\Lambda_0} = \int g\, \Lambda_0^{D-\Delta}\, \mathcal{O}\, d^Dx \rightarrow \delta S^{cl}_{\Lambda} = \int d^Dx\, g\left(\frac{\Lambda_0}{\Lambda}\right)^{D-\Delta} \Lambda^{D-\Delta}\, \mathcal{O}(x) \right)$$

Classical Running

Comparing with (22) we see that the non-linear effects give an extra contribution that corrects the classical scaling, justifying the name anomalous dimension for γ_{m^2}

For the λ-coupling, the classical scaling vanishes at $D=4$ & therefore the quantum corrections are important (making it, marginally irrelevant). On the other hand, for a strongly relevant deformation such as the mass, for which $|\Delta - D| = |-2| \gg$ loop corrections (if in perturb. theory) the quantum effects are negligible. Likewise, strongly irrelevant operator with $\Delta \gg D$ remains so. It's interesting to study now an intermediate case, where classical & quantum effects balance each other, generating a new fixed point:

Wilson-Fisher Fixed Point in $D=4-\varepsilon$ dimensions

From (24) we see that in $D \neq 4$ there is classical scaling contribution to g_4

$$(25)\quad \boxed{\begin{cases} \beta_2 = \left(-2 + \dfrac{g_4}{\ell_D}\right) g_2 \\ \beta_4 = \underbrace{-\varepsilon g_4}_{\text{push towards relev.}} + \underbrace{\dfrac{3}{\ell_D} g_4^2}_{\text{push towards irrelevant}} \end{cases}}$$

where $D = 4-\varepsilon$ with $\varepsilon > 0$ and $\ell_D \equiv \dfrac{d\, \pi^{D/2}}{\Gamma(D/2+1)\, 2(2\pi)^D}$ is the D-dimensional loop factor $\left\{ \begin{array}{l} \ell_4 = 16\pi^2 \\ \ell_3 = 4\pi^2 \end{array} \right.$

Let's imagine we could expand in small $\varepsilon \ll 1$, rather than integer ε, so that εg_4 could cancel against $\frac{3}{\ell_D} g_4^2$ still within the regime of perturbation theory:

$$(26)\quad \boxed{g_4^* = \varepsilon \cdot \frac{\ell_D}{3} \ll 1 \qquad g_2^* = 0}$$

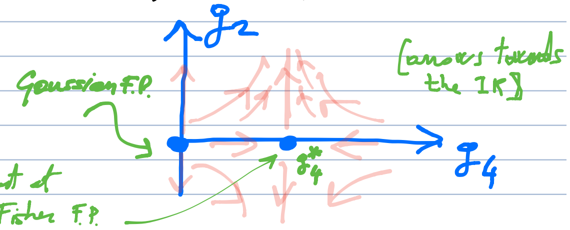
makes $\beta_{2,4} = 0 \Rightarrow$ The Wilson-Fisher fixed point

What's very interesting about this is that for g_4 very small, relative to ε, g_4 is relevant at the Gaussian F.P. (rather than irrelevant at $D=4$) and it becomes an irrelevant direction at the Wilson-Fisher F.P. (the mass remains relevant at both F.P.'s)

already diagonal thanks to normal ordering

$$(27)\quad \left.\frac{\partial \beta_i}{\partial g_j}\right|_{g=g^*} = \begin{pmatrix} -2 + \dfrac{g_4^*}{\ell_D} & \dfrac{g_2^*}{\ell_D} \\ 0 & \underbrace{-\varepsilon + \dfrac{6}{\ell_D} g_4^*}_{2\varepsilon} \end{pmatrix} = \begin{pmatrix} -2+\varepsilon/3 & 0 \\ 0 & \varepsilon \end{pmatrix} \Longrightarrow \begin{cases} \beta_2 \simeq \left(-2 + \frac{\varepsilon}{3}\right) g_2 \\ \beta_4 \simeq \varepsilon \left(g_4 - g_4^*\right) \end{cases}$$

or, solving (27) near g_*: $\begin{cases} g_2 = \left(\dfrac{\Lambda_0}{\Lambda}\right)^{2-\varepsilon/3} \bar{g}_2 \\ g_4 - g_4^* \propto \left(\Lambda/\Lambda_0\right)^{\varepsilon} \to 0 \end{cases}$ irrelevant at Wilson-Fisher F.P.



* Some Remarks:

- the ϕ^2-perturbation is relevant also for the Wilson-Fisher F.P., so to hit it one has to tune the initial mass closer and closer to zero as $\Lambda/\Lambda_0 \to 0$ in order to reach it. So, the continuum limit $\Lambda_0 \to \infty$ gives an interacting theory ($g = g_4^* \neq 0$ in contrast to Gaussian in $D=4$) if $\bar{g} = g_2(\Lambda_0)$, that is $m^2 = m^2(\Lambda_0)$ is chosen suitably as $\Lambda_0 \to \infty$: $\bar{g}_2(\Lambda_0) = g_2(\Lambda)\left(\frac{\Lambda}{\Lambda_0}\right)^{2-\varepsilon/3}(1+O(\frac{\Lambda}{\Lambda_0}))$, keeping $g_2(\Lambda)$ fixed (the low-energy renormalized quantity). On the other hand, the g_4 is irrelevant in the IR so there is a focusing effect, the IR value of g_4 is very close to g_4^* irrespectively of its starting value in the IR (except $g_4(\Lambda_0)=0$).
[more generally, irrelevant operators are focused to values determined by the relevant parameters]

- Even though we were in the unphysical situation of non-integer ε, there is great evidence that the Wilson-Fisher F.P. survives @ $\varepsilon = 1$, corresponding to $D=3$: it's the fixed point of 3D-Ising model as well as boiling water (they are very different microscopically but flow to the same IR fixed point since all irrelevant operators that tell the models apart are erased exponentially fast.

- Remarkably, if we look at the critical exponent of boiling water $\nu = \frac{1}{2-\gamma_*} \simeq \frac{1+\gamma_*}{4}$ $= 1/2 + \varepsilon/12$ which is remarkably in good agreement at $\varepsilon = 1$!

- One could be worry that ϕ^6-perturbation seems marginal in $D=3$ ($\varepsilon = 1$)
[recall $[\phi] = (D-2)/2 \Rightarrow [\phi^6] = 3(D-2)|_{D=3} = 3$] But this is true around the Gaussian F.P.
One should actually check the scaling dimension at the Wilson-Fisher F.P.:
a direct calculation shows that $\Lambda \frac{d}{d\Lambda} g_6 = \underbrace{(2D-6)}_{\text{classical}} g_6 + 15 g_4 g_6 + \ldots = (2-2\varepsilon) g_6 + \frac{15}{\ell_D} g_4 g_6 + \ldots$
so that g_6 is actually irrelevant

$$\overset{!}{\underset{g_4^* = \frac{\varepsilon \ell_D}{3}}{\simeq}} (2 - 2\varepsilon + 5\varepsilon) g_6 + \ldots = \underbrace{(2+3\varepsilon)}_{>0} g_6 + \ldots \quad \text{irrelevant!}$$

(in fact, more than around the Gaussian fixed point)

— Wave-function Renormalization

We have so far assumed that the kinetic term was canonically normalized and at 1-loop this normalization is not changed (by 1-loops) in ϕ^4-theory. Should the RG-flow change $(\partial\phi)^2$-operator (e.g. at 2-loops in ϕ^4, , and at 1-loop in ϕ^3) we would need to be careful in comparing physical observables. Say e.g.

(28) $S_\Lambda = \int d^4x \, \frac{1}{2} Z^2(\Lambda)(\partial\phi)^2 - \frac{\lambda(\Lambda)}{4!}\phi^4 + \dots$, the running coupling to perform the match would be $\bar\lambda(\Lambda) = \lambda(\Lambda)/Z^2(\Lambda)$. In ϕ^4-theory $Z = 1 + O(2\text{-loops})$.

There are two important examples where the running all comes from the wave-function: QED (thanks to the Ward-identity) and in Supersymmetry (to all loop order, thanks to the non-renormalization of the superpotential). For example in QED we just need to extract the corrections to

(29) $-\int F_{\mu\nu}^2/4 \, d^4x = \frac{1}{2}\int d^4x \, A_\mu(\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu)A_\nu$ from $= (\eta^{\mu\nu}k^2 - k^\mu k^\nu)\Pi(k^2)$
$= \langle T J^\mu(k) J^\nu(-k)\rangle$

with $\Pi(k^2) \simeq \frac{\#}{16\pi^2}\ln k^2/\Lambda^2$. We will see this example in detail in L5.

— β-function Universality

The first two coefficients in the β-function are independent of the precise definition of coupling as long as they can be related perturbatively as

(30) $\bar g(\Lambda) = g(\Lambda) + a\, g^2(\Lambda) + \dots = F(g)$

besides, $\beta_{\bar g} = 0 \Leftrightarrow \beta_g = 0$ (unless F not invertible, not a legal change of variables)

Indeed, $\beta_{\bar g} = \Lambda \frac{d}{d\Lambda}\bar g = F'(g)\,\beta_g$ where is understood $g = g(\bar g)$ on the r.h.s.

Since

(31) $\boxed{\beta_g = b\, g^2 + b' g^3 + \dots}$

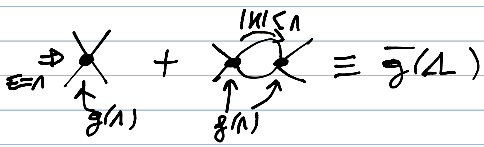
one can just invert $g = g(\bar g) = \bar g - a\bar g^2 + \dots$

(32) $\beta_g = b g^2 + b' g^3 + \dots = b\bar g^2 + (b' - 2ab)\bar g^3 + \dots$

$$\boxed{\begin{aligned}\beta_{\bar g} &= F'(g)\,\beta_g = (1 + 2a\bar g + \dots)(b\bar g^2 + (b' - 2ab)\bar g^3 + \dots) = b\bar g^2 + b'\bar g^3 + \cancel{2ab\bar g^3} - \cancel{2ab\bar g^3} + \dots \\ &= b\bar g^2 + b'\bar g^3 + \dots\end{aligned}}$$

which has the same coefficients b & b' than β_g in (31).

This explains why the RG-flow à la Wilson and à la Gellman–Low give the same β-functions in the perturbative regime. Take the definition of a coupling $\bar g(\Lambda)$ as the physical value of 2-to-2 amplitudes at $E = \Lambda$:

(33) 

$\equiv \bar g(\Lambda)$

But since the $|k| < \Lambda$ in the loop, by our original definition of $g(\Lambda)$, the calculation contains no large logarithmic correction and it can thus be expanded as:

no large log's: $g^2(\Lambda)\displaystyle\int_{|k|<\Lambda}\frac{d^4k}{(2\pi)^4}\frac{1}{(p+k)^2+m^2}\frac{1}{k^2+m^2}$, $p \simeq \Lambda$, $\simeq \dfrac{g^2(\Lambda)}{16\pi^2}\log \cdot \mathcal{O}(1)$

(34) $g(\Lambda) + a\, g^2(\Lambda) + \dots = \bar g(\Lambda)$

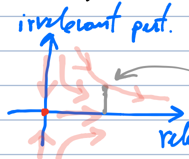
Which is precisely the redefinition of couplings we have seen above that leaves β_g first two coefficients invariant.

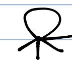
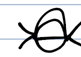
There is even a renormalization scheme where the β-function contains exactly only the first two terms, it's the Polchinski's equation. We will not, unfortunately, have time to discuss it

— General lessons: loops in EFT —

The general picture that has emerged is that relevant (irrelevant) operators give increasing (decreasing) contribution to the observables at low energy. Loops in perturbative

theorie can't turn a strongly relevant operator into an irrelevant one (and viceversa).
(Nevertheless they can decide the fate for classically marginal operators.)



It's interesting to look at this from a different perspective: classically, an irrelevant perturbation $\int g_O \frac{\delta O(x)}{M^{\Delta-4}} d^4x$ corrects $\frac{\delta obs}{obs} \simeq g_O \left(\frac{E}{M}\right)^{\Delta-4} \xrightarrow[\Delta>4]{} 0$. Loops, such as  or  seem to give large corrections, naively, $\left(\frac{g_6}{16\pi^2} \frac{\Lambda^2}{M^2} \text{ and } \frac{g_6^2}{(16\pi^2)^3} \frac{\Lambda^4}{M^4} \right)$ but the point is that for $\Lambda \simeq M$ these effect can be reabsorbed in the less irrelevant operators (φ^4 in the two diagrams) by suitably matching physical observables.
Then, for $\Lambda \ll M$, the true running below the matching scale $E \simeq M$, they contribute very little, or rather they generate more irrelevant operators ($\frac{g_6^2}{(16\pi^2)^3} \frac{K^4}{M^4} \leftrightarrow \frac{(\partial\varphi)^4}{M^4} \frac{g_6^2}{16\pi^2}$)
We could have gotten to the same conclusion if we used a more gentle regulator than the sharp cutoff, known as dimensional regularization where power-law divergences are not generated from the start, making matching easier on the practical level, (although many people misunderstand its role in EFT), because the scale introduced in dim-reg (replacing our Λ) enters only logarithmically. For power counting and dimensional analysis dim-reg is better suited.