Topics in Effective Field Theory

PhD-course, Rome 2020 by B. Bellazzini

– Scale Invariance, and Dim-analysis as Spurion-Analysis

One important symm. of the free massless field theories is the invariance under Scale Transformations

of any spin, ϕ in Eq.(2) is a generic field

$$(1) \qquad x^\mu \longrightarrow x'^\mu = \lambda x^\mu \qquad (\lambda = e^\alpha > 0,\ \alpha \in \mathbb{R})$$

$$(2) \qquad \phi(x) \longrightarrow \phi'(x') = \frac{1}{\lambda^\Delta}\phi(x(x')) = \frac{1}{\lambda^\Delta}\phi(x'/\lambda) \qquad (\Delta > 0)$$

where Δ is a positive real number known as Scaling Dimension of ϕ, suitably chosen such that $S[\phi] = S[\phi']$.

Indeed, under (2) the action becomes

$$(3) \qquad S[\phi'] = \underbrace{\int d^D x'}_{\text{(dummy variable)}} \mathcal{L}(\phi'(x'), \partial\phi'(x')) = \int d^D x\, \underbrace{\lambda^D}_{\left|\frac{\partial x'}{\partial x}\right|} \mathcal{L}\Big(\frac{1}{\lambda^\Delta}\phi(x), \underbrace{\frac{1}{\lambda^{\Delta+1}}}_{\frac{\partial x^\nu}{\partial x'^\mu} = \delta^\nu_\mu \frac{1}{\lambda}} \partial\phi(x)\Big)$$

So that each field carries a $1/\lambda^\Delta$ factor, each ∂_μ a $1/\lambda$, and d^4x a λ^4

$$(4) \qquad d^4x \longrightarrow \lambda^4 d^4x, \qquad \partial_\mu \longrightarrow \frac{1}{\lambda}\partial_\mu, \qquad \phi \longrightarrow \frac{1}{\lambda^{\Delta_\phi}}\phi$$

These transformation rules match exactly the change in the action under change of units of mass dimension $1/\lambda$ provided $\Delta_\phi = [\phi]$, or more explicitly

$$(4)\quad \boxed{\begin{cases} S_\phi = \int d^4x\, \frac{1}{2}(\partial\phi)^2 \longrightarrow \int d^4x\, \lambda^{4-2\Delta_\phi-2}\, \frac{1}{2}(\partial\phi)^2 & \Delta_\phi = 1 \\ S_\psi = \int d^4x\, \bar\psi i \not\partial \psi \longrightarrow \int d^4x\, \lambda^{4-2\Delta_\psi-1}\, \bar\psi i\not\partial\psi & \Delta_\psi = 3/2 \\ S_A = \int d^4x\, -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \longrightarrow \int d^4x\, \lambda^{4-2\Delta_A-2}\, -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} & \Delta_A = 1 \end{cases}}$$

L3/2

More generally, invariance under (2) for free spin-even $\Rightarrow \Delta = 1$, for free spin-odd $\Delta = 3/2$

Invariance of the action under (2) (+ invariance of the vacuum) means invariance of correlators $\phi(x)$ and $\phi'(x)$

$$(5)\quad \langle T\,\phi(x_1)\ldots\phi(x_n)\rangle = \langle T\,\phi'(x_1'(x))\ldots\phi'(x_n'(x))\rangle = \frac{1}{\lambda^{n\Delta}}\langle T\,\phi(\tfrac{1}{\lambda}x_1)\ldots\phi(\tfrac{1}{\lambda}x_n)\rangle$$

That e.g. for the 2pt scalar function $f_2((x_1-x_2)^2) = 1/\lambda^{2\Delta}\, f_2\left(\frac{(x_1-x_2)^2}{\lambda^2}\right)$ which is solved by power-law behavior

$$(6)\quad \langle T\,\phi(x_1)\,\phi(x_2)\rangle = \frac{\text{const}}{(x_1-x_2)^{2\Delta}} \qquad \left(\text{in Minkowski} \propto \frac{1}{\left((x_{12}^0)^2 - \vec{x}_{12}^{\,2} + i\epsilon\right)^\Delta}\right)$$

In the free scalar theory $\propto 1/x_{12}^2$ where $x_{12}^\mu = x_1^\mu - x_2^\mu$.

Now, interactions & mass terms in general break scale invariance explicitly, because an explicit dimensionful term appears in the action, e.g. $m^2\frac{\phi^2}{2}$, $(F_{\mu\nu}F^{\mu\nu})^2\frac{1}{M^4}$, $\bar\psi\gamma^\mu\psi F_{\mu\nu}\frac{1}{\Lambda^3}$, … but also $\lambda\phi^4$ (since $\lambda = \lambda(\Lambda)$) say things like $[m^2] = 2$ or $[1/\Lambda^2] = -2$ …

The very fact that we assign mass-dimensions to these "coupling" that deform the original scale-invariant theory is what we would call a spurion analysis of Scale Transformation: by promoting m^2, $1/\Lambda^2$, … to fields (eventually frozen to their vev's) that transform in a way to formally restore scale invariance we can keep track of the impact of these deformations. Let's see it for a scalar theory 13/3

$$(7)\qquad \int d^4x \left\{ \frac{1}{2}(\partial\phi)^2 \longrightarrow \int d^4x \left\{ \frac{1}{2}(\partial\phi)^2 \right.\right.$$

$$-m^2\frac{\phi^2}{2} \longrightarrow -\lambda^2 m^2 \frac{\phi^2}{2}$$

$$-g_4\phi^4 \longrightarrow -\lambda^0 g_4 \phi^4$$

$$-\frac{1}{\Lambda^2}\phi^6 \longrightarrow -\frac{1}{\lambda^2}\frac{1}{\Lambda^2}\phi^6$$

$$\frac{1}{\Lambda^6}(\partial\partial\phi)^2(\partial\phi)^2 \longrightarrow \frac{1}{\lambda^6}\frac{1}{\Lambda^6}(\partial\partial\phi)^2(\partial\phi)^2$$

$$+\frac{1}{\Lambda^{n+m-4}}\partial^n\phi^m \longrightarrow \frac{1}{\lambda^{n+m-4}}\frac{1}{\Lambda^{n+m-4}}(\partial^n\phi^m)$$

(← we will come back to the quantum correct.)

So that the spurion Assignment under scale transf. is simply

$$m^2 \to 1/\lambda^2 m^2, \quad g_4 \to g_4, \quad \frac{1}{\Lambda} \to \lambda\frac{1}{\Lambda}, \quad \ldots$$

(8) i.e. $\Delta S = \int d^4x \; g_i \mathcal{O}_i(\partial, \phi) \longrightarrow \Delta_{g_i} = 4 - \Delta_{\mathcal{O}_i}$ (Like imaginary charge conservation, $\lambda = e^{\alpha}$, $\alpha \in \mathbb{R}$)

where classically $\Delta_g^{cl.} = n + m\Delta_\phi$ (this will receive quantum corrections that are important only when $\Delta_{\mathcal{O}}^{cl} \simeq 4$, i.e. $\Delta_g^{cl} \simeq 0$).

We have seen in all examples that the EFT in the IR (in fact in the UV too) was well described by a relativistic free field theory where interactions (and IR masses) are treated as small perturbations, so that we can immediately understand their impact at low-energy by the Scale-invariance spurion analysis. For an observable Obs. (e.g. a scattering ampl. M or a correlation funct.) which depends as well on a dimensionful kinematic parameter such as the characteristic Energy E, $[E]=1$, one has two spurions at his/her disposal to build formally scale-invariant quantities such as

(9) $\Delta S = g \underbrace{\int d^4x\, \mathcal{O}(x)}_{-4+\Delta_{\mathcal{O}}} \rightarrow \begin{cases} \Delta_g = 4-\Delta_{\mathcal{O}} \\ \Delta_E = 1 \end{cases} \rightarrow \frac{\delta Obs}{Obs}(E,g) = g\cdot E^{-\Delta_g}$

(in the regime $M_{IR} \ll E \ll M_{UV}$, see below, where one neglects the other kinematical spurions)

Example: N-body scatt. amplitude

$$i(2\pi)^4\delta^4(\Sigma K_i)\,\delta M_N(\overset{\text{momenta}}{K_i}, \overset{\text{coupling}}{g}) = i\int d^4x_1\, e^{iK_1 x_1} \cdots \Box_{x_1} \cdots \langle T\phi(x_1)\ldots\phi(x_N) \int d^4x\, g\, \mathcal{O}(x)\rangle$$

$$= i\int d^4x_1'\, e^{i\frac{K_1}{\lambda}x_1'} \cdots \Box_{x_1'} \cdots \langle T\phi(\tfrac{x_1'}{\lambda})\ldots\phi(\tfrac{x_N'}{\lambda}) \int d^4x'\, g\, \mathcal{O}(\tfrac{x'}{\lambda})\rangle \cdot \lambda^{-4N+2N-4}$$

$\lambda x_i = x_i'$

$(\langle \phi(\tfrac{x}{\lambda})\ldots\phi(\tfrac{x}{\lambda})\mathcal{O}(\tfrac{x}{\lambda})\rangle$ "$\lambda^{N+\Delta_{\mathcal{O}}}\langle\phi(x)\ldots\phi(x)\mathcal{O}(x)\rangle)$

$$\overset{Eq(3)}{=} \int d^4x_1'\, e^{i\frac{K_1}{\lambda}x_1'} \cdots \Box_{x_1'} \cdots \langle T\phi(x_1')\ldots\phi(x_N') \int d^4x'\, g\, \mathcal{O}(x')\rangle \cdot \lambda^{N+\overbrace{\Delta_{\mathcal{O}}-4}^{-\Delta_g}}\lambda^{-2N}$$

$$= i(2\pi)^4\delta^4(\underbrace{\Sigma_i \tfrac{K_i}{\lambda})\,\lambda^{-4}}_{\text{these }\lambda\text{ cancel out}}\cdot\lambda^{4-N}\,\delta M_N(\tfrac{K_i}{\lambda}, \tfrac{g}{\lambda^{\Delta_g}}) = i(2\pi)^4\delta^4(\Sigma_i K_i)\,\lambda^{4-N}\,\delta M_N(\tfrac{K_i}{\lambda}, \tfrac{g}{\lambda^{\Delta_g}})$$

(10) $\delta M_N(K_i, g) = \lambda^{4-N}\,\delta M_N(K_i/\lambda,\ g/\lambda^{\Delta_g})$ which is solved by

(11) $\delta M_N(E, g, \theta_i) \propto E^{4-N}\cdot(E^{-\Delta_g}\cdot g) = E^{4-N}(E^{\Delta_{\mathcal{O}}-4}\cdot g)$

dimensionless angles from K_i/E

The $(4-N)$-factor is just kinematical, $[M_{1\to2}]=1$, $[M_{2\to2}=0]$, ..., it depends on the total # of legs, so it's there also for $g=0$, and it cancels in $\delta M/M$ (or we can just agree to factor it out)

The relevance of Eq.(9), that we rewrite here as

(12) $\boxed{\frac{\delta Obs}{Obs} = \hat{g}\left(\frac{E}{\Lambda}\right)^{\Delta_{\mathcal{O}}-4}}$ where $g \equiv \frac{1}{\Lambda^{\Delta_{\mathcal{O}}-4}}\hat{g}$ $\left(\frac{\delta Obs}{Ob} = E^{\Delta_{\mathcal{O}}-4}\cdot g\right)$

is that it makes obvious that higher-dimensional operators with $\Delta_{\mathcal{O}}>4$ give a vanishing contribution when $E\to0$ (or equivalently $\Lambda\to\infty$ at E fixed)

As a theory is deformed by $\Delta S = g\int d^4x \frac{\mathcal{O}(x)}{\Lambda^{\Delta-4}}$ the relative importance on the observables is controlled by both "couplings": g and also $\boxed{g_E = \left(\frac{E}{\Lambda}\right)^{\Delta-4} g}$.

For this reason the operators are classified as

(13)
- Irrelevant $\Delta > 4$ ← less important as E decreases (or Λ increased)
- Marginal $\Delta = 0$ ← equally important, energy-indep.
- Relevant $\Delta < 0$ ← more important as E decreases

(e.g. a mass is initially a small contribution $(m/E)^2 \ll 1$ but it eventually becomes big as $E \to 0$ or m)

Conversely, as E is increased the relevant operators become less important, while the irrelevant operators become gradually more important.

In the examples of L2 the $\frac{(F_{\mu\nu}F^{\mu\nu})^2}{\Lambda^4}$ deformation of the free theory gives $\mathcal{M}(E) \sim \frac{E^4}{\Lambda^4} \to 0$ as $E \to 0$ because $\Delta_{F^4} = 8$ is strongly irrelevant.

The # of relevant parameters, $\Delta < 4$, is often finite because built out of a finite # of ∂_μ ($\Delta_\partial = 1$) and finite # of fields ($\Delta_\phi = 1$ or $3/2$).

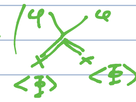
We have seen in L1 and L2 that integrating out heavy degrees of freedom generate at low-energy infinitely many local term $\mathcal{O}(x)$

(14) $$\mathcal{L}^{EFT} = \sum_i \hat{g}_i \frac{\mathcal{O}_i(x)}{\Lambda^{\Delta_i - 4}}$$

with more and more ∂_μ's and fields, but the low-energy theory remains predicting because the # of irrelevant op. below a certain Δ is finite so that matching $(E/\Lambda)^\Delta$ with the desired precision requires only a finite # of $\mathcal{O}_i$ and hence finite # of measurements of the $\hat{g}_i$. (compare it to renorm. th.)

[one should not have the impression that only irrelevant op. are generated when integrating-out heavy particles. Take for example the following UV-theory

(15) $\mathcal{L}^{UV}(\varphi,\Phi)=\frac{1}{2}(\partial\varphi)^2-\frac{1}{2}m^2\varphi^2+\frac{1}{2}(\partial\Phi)^2-\lambda(\Phi^2-v^2)^2+g^2\Phi^2\varphi^2$

with $\begin{cases}\lambda \gg g \\ \lambda v^2 \gg m^2\end{cases}$. Since $M_\Phi \propto \lambda v^2$ we can integrate it out: treating $g \ll \lambda$ as a small perturbation we see that the e.o.m. set Φ at its VEV v $\Rightarrow \mathcal{L}^{EFT}(\varphi)\simeq\frac{1}{2}(\partial_\mu\varphi)^2-\left(m^2+\frac{g^2 M_\Phi^2}{8\lambda}\right)\frac{\varphi^2}{2}+\dots$ ()

So that m^2 receives large corrections from the UV scale M_Φ]

Summarizing, in the relativistic theories of particles we are considering the structure is

(16) $\boxed{\text{scale-invariant kinetic term}} + \boxed{\text{relevant op.}} + \boxed{\text{leading irrelevant op.}}$

[for spin ≥ 1 the # d.o.f. changes from $m=0$ and $m=O(\epsilon)$ but as we will see it's simple to keep track of the extra d.o.f using Goldstone Boson & the equivalence theorem. So the logic behind (16) works e.g. also for a massive spin-1

(17) $-\frac{F_{\mu\nu}^2}{4}+\frac{m^2 A_\mu^2}{2} \xrightarrow{\text{equival.}} -\frac{F_{\mu\nu}^2}{4}+\frac{(D_\mu\pi)^2}{2}$ where $D_\mu\pi=\partial_\mu\pi-mA_\mu$ contains a "relevant interaction" $\longrightarrow m\,\partial_\mu\pi A_\mu$

if the extra d.o.f. are made manifest. Not much different than $(\partial\phi)^2$ perturbed by $\phi^2 m^2$.]

Sometimes the leading effect comes only from irrelevant operators, such as it is the case for breaking-terms of accidental symmetries, they are broken by op. with $\Delta>4$ e.g. lepton # in the SM. (e.g. $(LH)^2/\Lambda$ is the lowest dim. op. of SM that break Lept#)

Another case is when the # of IR d.o.f + Lorentz inv. forbid relevant/marginal interactions:

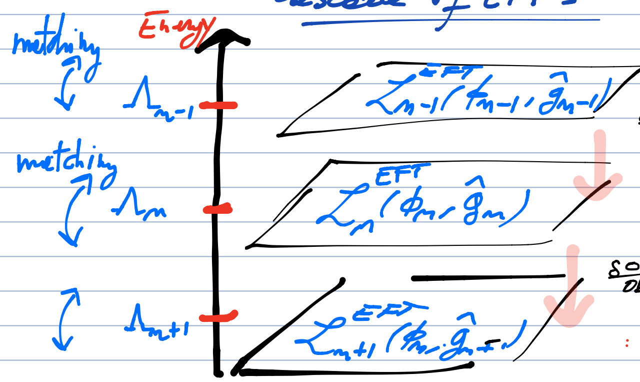
<u>Example</u>:

(18) $\begin{cases} \bullet \text{ only 1 massless Weyl fermion} \Rightarrow \mathcal{L}_{int}^{\Delta_{min}}=\frac{(\bar\psi\gamma^\mu\psi)^2}{\Lambda^2} \quad \Delta_{min}=6 \text{ lowest dim.} \\ \bullet \text{ only 1 photon-like} \Rightarrow \mathcal{L}_{int}^{\Delta_{min}}=\frac{(F_{\mu\nu}F^{\mu\nu})^2}{\Lambda^4} \quad (2 \text{ of them}) \quad \Delta_{min}=8 \\ \bullet \text{ only 1 graviton} \Rightarrow M_{Pl}^2 R\sqrt{-g} \underset{\text{than w/ }\partial^2}{\text{unique}} \Rightarrow \frac{\partial^2 h^3}{M_{Pl}}+\frac{\partial^2 h^4}{M_{Pl}^2} \to \mathcal{M}(hh\to hh) \underset{\text{IR}}{\xrightarrow{E^2/M_{Pl}^2}} 0 \end{cases}$

We would like to stress that an EFT seen as an approximately scale-invariant field theory has a limited range of validity, in both directions:

Cascade of EFT's

$$\boxed{\left(g_E = \left(\frac{E}{\Lambda}\right)^{\Delta-4} \hat{g} \right)}$$



$\delta O_{Obs}(E \ll \Lambda_{n-1}) \simeq 1$

$\hat{g}_{n-1}$ with $\Delta_{\hat{g}_{n-1}} > 0$ $(g_E \to 1)$

$(\text{e.g. } (m/E) \to 1$, int. out m too$)$

$\delta O_{Obs}(E \ll \Lambda_n) \simeq 1$

$\hat{g}_n$ with $\Delta_{\hat{g}_n} > 0$

⋮

- E decreases ⇒ relevant op's set IR cut-off *
- E increases ⇒ irrelevant op's set UV cutoff *

[* not necessarily physical threshold, just where the EFT with a given spectrum & quantum # is $O(1)$ away from the free scale-invariant theory of particles. They are sometimes called the strong-coupling scales since $g_E \to 1$ with g_E defined in (P3)]

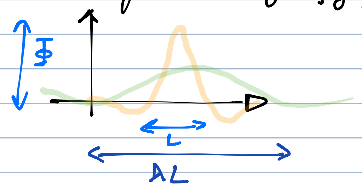
As E is decreased one goes from one EFT to another one, usually with different d.o.f, the new one has coupling $\hat{g}_n$ defined in terms of of $\hat{g}_{n-1}$ by matching physical observables (e.g. amplitudes & correlation functions).

Now, matching observables requires in general to include loops: this naively would seem to spoil our arguments based on scale invariance given that they involve a priori very large energies running in the loop?

The answer is yes & no, they break scale invariance explicitly but in no way worse than the of relevant & irrelevant deformations we have seen so far. This brings us to the Wilsonian RG picture where any QFT = EFT obtained by integrating out energy shells, not just entire massive fields. The advantage is conceptual and practical (no large separation of scales = no large breaking of scale inv.)

Before discussing the Wilsonian RG let's obtained again the useful separation in relevant, irrelevant & marginal operators from the path integral:

consider a field configuration of size Φ over a region L which contributes to the path-integral (orange fig.). The characteristic energy scale associated is $\kappa = 1/L$



$\updownarrow \frac{1}{\lambda^{\Delta}}\Phi$ In the path-integral, the configurations $\Phi_\lambda = \frac{1}{\lambda^{\Delta}}\Phi\left(\frac{x}{\lambda}\right)$ with $\lambda > 1$ are localized over λL (green fig.) and here characteristic momentum scale $1/\lambda L$. But Φ and the whole orbit Φ_λ contribute identically in the path-integral if one includes just the kinetic term and chose Δ to be the scaling dimension, because the K.T. (and the measure) are scale invariant: $S_{KT}[\Phi] = S_{KT}[\Phi_\lambda]$. Now, the other terms in the action, schematically

$$(19) \quad \Delta S[\Phi] = \sum_{n,m} \int \frac{\partial^n \phi^m(x)}{\Lambda^{n+m\Delta-4}} \hat{g}_{nm} \, d^4x$$

distinguish instead Φ from Φ_λ:

$$(20) \quad \Delta S[\Phi_\lambda] = \sum_{n,m} \frac{1}{\lambda^{n+m\Delta-4}} \int d^4x \, \frac{\partial^n \Phi^m(x)}{\Lambda^{n+m\Delta-4}} \hat{g}_{nm}$$

Their contribution scales to zero if $\Delta_{class} = [\partial^n \Phi^m] = n + \Delta m > 4$ as $\lambda \to \infty$

this limit corresponding to consider configurations of lower & lower momenta, i.e. $E \to 0$

Those op. are what we have been calling irrelevant operators.

Conversely, their contribution becomes more important relative to the K.T. in the same limit for operators with $\Delta_{cl} < 4$: the relevant perturbations. Marginal perturbations with $\Delta_{class} = 0$, such as ϕ^4 for a scalar theory, are those that are always kind of important (or kind of not important if $\hat{g} \ll 1$).

- **Important Remark:**

rather than $E \to 0$ one can look at $\Lambda \to \infty$ in $\frac{\delta \mathrm{Obs}}{\mathrm{Obs}}$

- irrelevant operators are $\boxed{\text{dominated by the smallest scale}}$

In the theory of photons only, for example, the irrelevant $(F_{\mu\nu}F^{\mu\nu})^2$ operator receive contributions by all particles in the UV, electron, muons, ..., but the dominant one comes from the lightest particle

$$\frac{g^2}{16\pi^2}(F_{\mu\nu}F^{\mu\nu})^2\left\{\frac{1}{m_e^4}+\frac{1}{m_\mu^4}+\cdots\right\}$$

UV-physics decouples
polyn. fast

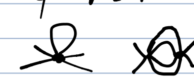
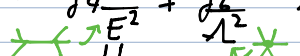
smallest scale gives the largest contribution

The effect of the muon is down by $o\left(\frac{m_e}{m_\mu}\right)^4$. There is a sort of erasing memory of the UV-physics as the lightest gives the leading effect.

- relevant operators are instead dominated by the largest scale

(this is the origin of the hierarchy problem)

- Remark

One may be confused by the fact that an operator like $\frac{\phi^6}{\Lambda^2}\hat{g}_6$ which would deem as irrelevant, it seemingly gives the leading 6-body amplitude: $M(123 \to 456) = \frac{\hat{g}_6}{\Lambda^2}$ which is constant, it does not scale to zero as $E \to 0$. But this is so simply because one forgot to include ϕ^4 term (which, even if absent at one scale, gets generated at lower scales )
so that actually $M(123 \to 456) = \frac{\hat{g}_4^2}{E^2} + \frac{\hat{g}_6}{\Lambda^2}$ so in fact $\frac{\delta M_{\hat{g}_6}(123\to456)}{M_{\hat{g}_4}} \propto \left(\frac{E}{\Lambda}\right)^2$ as expected. If we were  to add a mass, i.e. a relevant def. we would $E^2 \to E^2 - m^2$ in the numerator $\Rightarrow \frac{\delta M}{M}\Big|_m = o\left(\frac{m}{E}\right)^2$ also as expected.

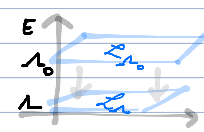
— Running Couplings: a first introduction

In order to deal with the loop-corrections from, a priori, large momenta, we integrate out not only heavy particles, but also energy-shells above a certain value Λ_0, meaning that experimentally we are not going to produce those high-modes anyway (even for the light particles in the IR). We put heavy modes & heavy particle on the same footing:

(1) $$Z[J] = \int [D\hat{\phi}]_{k<\Lambda_0} \exp\left\{ i S_{\Lambda_0}[\phi] + i\int J\phi \right\}$$

where $J(x)$ is such that $\hat{J}(k) = \theta(\Lambda_0 - k) f(k)$ has support only for $k<\Lambda_0$.
We can ask now what happens if we were to integrate-out even more modes, down to a new scale $\Lambda < \Lambda_0$, because actually we consider sources $\hat{J} = f(k)\theta(\Lambda - k)$ that probe only momenta $k<\Lambda<\Lambda_0$

(2) $$Z[J] = \int [D\hat{\phi}]_{k<\Lambda} \underbrace{\int [D\hat{\phi}]_{\Lambda<k<\Lambda_0} \exp\left\{ i S_{\Lambda_0}[\phi] + i\int J\phi \right\}}$$



$$= \int [D\hat{\phi}]_{k<\Lambda} \exp\left\{ i S_{\Lambda}[\phi] + i\int J\phi \right\}$$

This equation is telling us that, in order for the physical observables ($Z[J]$ and its derivatives that give correlat. functions) to be the same, given that we just split the integral in two steps (the 1° from Λ_0 to Λ, then from Λ to 0), that is $\Lambda \frac{dZ[J]}{d\Lambda} = 0$, the action S_Λ to be used in the second step is in general different than the original S_{Λ_0}:

(3) $$\boxed{S_{\Lambda_0} \longrightarrow S_\Lambda}$$

In other words, the couplings flow

(4) $\boxed{g_{\Lambda_0} \longrightarrow g_\Lambda}$ in order to keep fixed physical quantities at low energy.

This flow of couplings is called the Wilsonian Renormalization group, or RG-flow in short. As the flow connects theories at different scales it is convenient to work with dimensionless coupling constant $g_i(\Lambda)$ associated with operators, of scaling dimension Δ_i, that perturbe the free theory (or any other scale-invariant theory we started with)

(5) $S_\Lambda = S_0 + \sum_i \frac{g_i(\Lambda)}{\Lambda^{\Delta_i - 4}} \mathcal{O}_i(x)$

As the theory runs from one scale Λ_0 to $\Lambda < \Lambda_0$, the dimensionless couplings adjust such that low-energy observables are kept untouched: the new couplings that achieve that must be a function of the dimensionless quantity at our disposal i.e. $g_i(\Lambda_0)$ (the starting coupling) and Λ/Λ_0 (the relative change in scale)

(6) $g_i(\Lambda) = f(g_j(\Lambda_0), \Lambda/\Lambda_0)$

Looking at infinitesimal change $\Lambda = \Lambda_0 + \delta\Lambda$ means taking derivatives w.r.t. Λ at $\Lambda = \Lambda_0$, i.e. $\Lambda \frac{d}{d\Lambda} g_i(\Lambda)\Big|_{\Lambda=\Lambda_0} = \frac{\partial f}{\partial z}(g_j(\Lambda_0), z)\Big|_{z=1}$ because the $g_i(\Lambda_0)$ is Λ-independent (in fact, it represents the initial condition, or matching condition at $\Lambda = \Lambda_0$.

The infinitesimal change in the couplings gives:

(7) $$\boxed{\frac{d}{d\ln\Lambda} g_i(\Lambda) = \beta_i(g_j(\Lambda)) \quad \leftarrow \beta\text{-function}}$$

where $d/d\ln\Lambda = \Lambda\, d/d\Lambda$ and $\beta_i = \frac{\partial f}{\partial z}\Big|_{z=1}$. A theory with $\beta_i = 0$ does not change couplings as Λ changes and it's thus Λ-independent, that is scale-invariant.

- Remarks:
  - solving (7) with initial (matching) condition $g_i(\Lambda = \Lambda_0) = g_i|_0$ defines an EFT for any value of Λ, i.e. for energy below Λ
  - there are no other dimensionless ratios as everything is absorbed into the couplings, e.g. $m_\Lambda^2 \phi^2 = m_\Lambda^2/\Lambda^2 \, \Lambda^2 \phi^2 = g_2(\Lambda) \Lambda^2 \phi^2$.

- **Example:** ϕ^4 theory at 1-loop

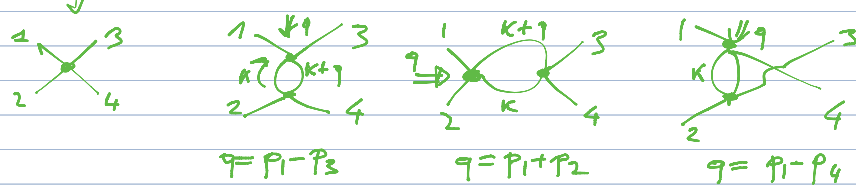
(a coupling, not the scale fact.)

$$(8) \qquad S_{\Lambda_0} = \int d^4x \left[\frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \right]_{\Lambda_0}$$

defined at $E=\Lambda_0$ with $\begin{cases} \lambda(\Lambda_0) \ll 16\pi^2 \\ m^2 \ll \Lambda^2 \end{cases}$

A physical observable sensitive to λ is the 2 → 2 scattering amplitude

$$(9) \quad i\mathcal{M}(12\to 34) = -i\lambda - \frac{\lambda^2}{2!} \int_{|K|<\Lambda_0} \frac{d^4K}{(2\pi)^4} \frac{i}{[(K+q)^2 - m^2 + i\epsilon]} \frac{i}{[K^2 - m^2 + i\epsilon]} + t\text{- and } u\text{-channel}$$



The tree-level contribution λ from ϕ^4 gets contributions from $p_i = 0$. The terms with momenta renormalize operators with derivatives (such as $(\partial\phi)^2\phi^2$, $(\partial\phi)^4$... which are irrelevant and not our focus). Since we are after the change of $\lambda(\Lambda_0) \to \lambda(\Lambda)$, the coeff. of ϕ^4 (for a canonically normalized KT) we can set $q=0$, and Wick-rotate $K^0 = iK^0_E$

I will drop the subscribe E hereafter

$$(10) \quad i\mathcal{M}(12\to34) = -i\left\{ \lambda(\Lambda_0) - \frac{3}{2}\lambda^2(\Lambda_0) \int_{|K|<\Lambda_0} \frac{d^4K}{(2\pi)^4} \frac{1}{(K^2+m^2)^2} \right\} (+ \text{ terms with } p_i)$$

where the factors of 3 comes from the 3-channels.

Now, we can match it to the answer from the theory at $E=\Lambda$

$$(11) \quad i\mathcal{M}(12\to34) = -i\left\{ \lambda(\Lambda) - \frac{3}{2}\lambda^2(\Lambda) \underbrace{\int_{|K|<\Lambda}}_{\text{different range}} \frac{d^4K}{(2\pi)^4} \frac{1}{(K^2+m^2)^2} \right\}$$

$$(12) \Rightarrow \boxed{\lambda(\Lambda) = \lambda(\Lambda_0) - \frac{3}{2}\lambda^2(\Lambda_0) \int_{\Lambda<|K|<\Lambda_0} \frac{d^4K}{(2\pi)^4} \frac{1}{(K^2+m^2)^2}}$$

small energy-shell if $\Lambda_0 = \Lambda + \delta\Lambda$

(where we used that $\frac{\lambda^2(\Lambda)}{16\pi^2} \ll \lambda(\Lambda)$ and that in the 1-loop corrections I can use the tree-level matching $\frac{\lambda^2(\Lambda_0)}{16\pi^2} = \frac{\lambda^2(\Lambda)}{16\pi^2}$.)

In other words,

$$\lambda(\Lambda) = \lambda(\Lambda_0) + \frac{3}{2}\frac{\lambda^2(\Lambda_0)}{16\pi^2}\int_{\Lambda^2}^{\Lambda_0^2} dk^2 \frac{k^2}{(k^2+m^2)^2} = \lambda(\Lambda_0) - \frac{3}{2}\frac{\lambda^2(\Lambda_0)}{16\pi^2}\ln\frac{\Lambda_0^2}{\Lambda^2} + o\left(\frac{m^2}{\Lambda^2}\right) \tag{13}$$

logarithmic running

that is

$$\boxed{\begin{cases} \lambda(\Lambda) = \lambda(\Lambda_0) + \dfrac{3\lambda^2(\Lambda_0)}{16\pi^2}\ln(\Lambda/\Lambda_0) \\ \beta(\lambda) = \dfrac{3\lambda^2}{16\pi^2} \end{cases}} \tag{14}$$

$$\left(\beta(\lambda(\Lambda_0)) = \frac{\partial \lambda(\Lambda)}{\partial \ln \Lambda}\Big|_{\Lambda=\Lambda_0} \right)$$

- **Remarks**

  - The expression (13) is perfectly well defined, neither UV or IR divergences are present as the integration is in a small regular shell $\Lambda < |k| < \Lambda_0$.
    Taking the shell infinitesimal means integrating the $\beta(\lambda)$ from one Λ_0 to an arbitrary $\Lambda < \Lambda_0$:

    $$\begin{cases} \dfrac{d}{d\ln\Lambda}\lambda(\Lambda) = \dfrac{3\lambda^2}{16\pi^2} \\ \lambda(\Lambda_0) = \lambda_0 \end{cases} \Rightarrow -\frac{1}{\lambda} + \frac{1}{\lambda_0} = \frac{3}{16\pi^2}\ln\Lambda/\Lambda_0 \rightarrow \lambda = \frac{1}{\frac{3}{16\pi^2}\ln\tilde{\Lambda}_0/\Lambda}$$

    where we traded λ_0 for a dimensionful scale $\tilde{\Lambda}_0$, this is an example of dimensional transmutation.

  - the ϕ^4 operator, which is marginal at tree-level $\Delta_{tree} = 4$, is actually marginally irrelevant since the coupling decreases as Λ/Λ_0 is becoming smaller (remember that $E \leq \Lambda$).
    In the limit $\Lambda/\Lambda_0 \to 0$, either $\Lambda_0 \to \infty$ or $\Lambda \to 0$ (with $m \to 0$), the theory becomes free. (known as continuum limit)
    However, it takes an exponentially long "RG-time" to reach

the free theory $\lambda=0$ from $\lambda(\Lambda_0)=\lambda_0$, $\Lambda=\Lambda_0 \exp\left[-\frac{16\pi^2}{3\lambda_0^2}\right]$

Conversely, for a finite $\lambda(\Lambda)$ measured at $E=\Lambda$, the coupling must grow at higher energy, invalidating our perturbative calculation at exponentially large energies,

(15) $$\Lambda_{LANDAU} = \Lambda \exp\left\{\frac{16\pi^2}{3\lambda(\Lambda)^2}\right\}$$ known as Landau pole.

- Another remark is that the β-function obtained by requiring $M(12\to 34)$ at Λ_0 should match the one at Λ for any $\Lambda<\Lambda_0$, is equivalent to the RG-independence on Λ of a physical quantity like $M(12\to 34)$, that is

(16) $$\underset{\substack{\uparrow \\ \text{RG-independence}}}{0} = \Lambda \frac{d M(12\to 34)}{d\Lambda} \underset{\text{eq.(11)}}{=} \Lambda \frac{d}{d\Lambda}\lambda(\Lambda) - \frac{3}{2}\frac{1}{16\pi^2}\Lambda\frac{d}{d\Lambda}\left\{\int_0^{\Lambda^2} dk^2 \frac{k^2}{(k^2+m^2)^2}\cdot\lambda^2(\Lambda)\right\}$$

$$= \beta - \underbrace{\frac{3\lambda}{16\pi^2}\beta\cdot\int^{\Lambda^2} dk^2\frac{k^2}{(k^2+m^2)^2}}_{\text{2 loop correction}} - \frac{3}{2}\frac{\lambda^2}{16\pi^2}\underbrace{2\Lambda^2\frac{d}{d\Lambda^2}\int^{\Lambda^2} dk^2\frac{k^2}{(k^2+m^2)^2}}_{\frac{2\Lambda^4}{(\Lambda^2+m^2)^2}\simeq 2+O(m^2/\Lambda^2)} \simeq \beta - \frac{3\lambda^2}{16\pi^2}$$

$$\Rightarrow \beta = \frac{3\lambda^2}{16\pi^2} \quad \text{at 1-loop}$$

We will have much more to say about loops in the next lectures.