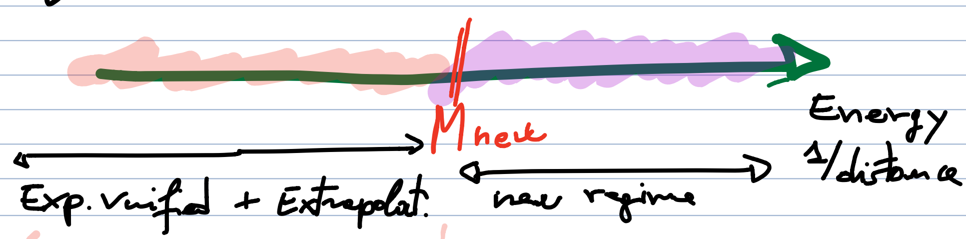
Topics in Effective Field Theory

PhD-course, Rome 2020 by B. Bellazzini

EFT-Manifesto

A given theory, say a QFT like e.g. QCD or QED, has a range of energies where it has been experimentally verified, within a certain precision δ, and one can immagine that the theory can be extrapolated to even higher energies, that is shorter distances. Renormalizable QFT such as QCD can be extrapolated without any inconsistency up to arbitrarily high energies.



new and old theory must agree within a precision δ

However, one should keep in mind
that new phenomena can be experimentally encountered at shorter distances, say a new particle of mass M_{new} or a new force of range $1/M_{NEW}$,
and that should come in a way that is consitant with

the experimental success at lower energies. In other words the low-energy theory should admit a deformation such that the variation in the observables $\mathcal{O}$ predicted by the new -updated theory in light of a discovery- is below the experimental precision: $\delta\mathcal{O}/\mathcal{O} \ll \delta_{exp}$.

EFT is the systematic way to parametrize all possible deformations (consistent w/ QM + few symmetries, e.g. Poincaré) in the IR theory (or its extrapolation up to $E < M_{NEW}$) in terms of the IR degrees of freedom only, i.e. those accessible experimentally. The Wilsonian version of the Renormalization Group put on the same footing of M_{NEW} all 4-momenta p heavier than some threshold Λ, $|p| \gg \Lambda$, and focus directly on modes $|p| \ll \Lambda$ relevant for observables at $E \ll \Lambda$. Let's see how this is done in practice via simple example in QM

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* Example: 2 coupled harmonic oscillators w/ $\omega \ll \Omega$

(1) $$\mathcal{L} = \frac{1}{2}\dot{q}^2 - \frac{\omega^2}{2}q^2 + \frac{1}{2}\dot{Q}^2 - \frac{\Omega^2}{2}Q^2 + \lambda q^2 Q$$

In this example one spring is much stiffer than the other and hardly will be excited off its ground state unless one injects energies $E \gtrsim \Omega$. We are instead interested in the regime $\omega \lesssim E \ll \Omega$. Let's consider the functional generator

(2) $Z[j] = \int [dq][dQ]\, \exp\left(i\, S[q,Q] + i\int dt\, j q\right)$

We have inserted a current for q only because we are going to produce experimentally only quanta for q (as long as $E \ll \Omega$). Now, this integral is gaussian w.r.t. Q and can be done exactly, with λq^2 the "external current" for Q in the path-integ.

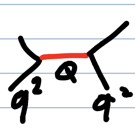
(3) $\int [dQ]\, \exp\left(i\int dt\, \frac{1}{2} Q(-\partial_t^2 - \Omega^2)Q + JQ\right) = N \exp\left(-\frac{i}{2}\int dt\, dt'\, J(t)\Delta(t,t')J(t')\right)$

where $\Delta(t)$ is the green function $(-\partial_t^2 - \Omega^2 + i\epsilon)\Delta(t-t') = \delta(t-t')$

$$\begin{cases} N = \text{irrelevant const.} \\ J = \lambda q^2 \end{cases}$$

(to make path-int. well defined via Wick rotation to Euclidean $t = i t_E$ such that for $t \to \infty$ projects on the vac.)

(4) $Z[j] = N\int [dq]\, \exp\left[i\int dt \left(\frac{\dot q^2}{2} - \frac{\omega^2}{2} q^2 + jq\right) - \frac{i\lambda^2}{2}\int dt\, dt'\, q^2(t)\Delta(t-t')q^2(t')\right]$



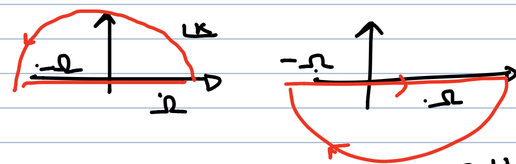
$= \left(\frac{i\lambda}{2!}\right)^2 q^2(-k) \frac{i}{k^2 - \Omega^2 + i\epsilon} q^2(k)$ The only connected Feynman diagram

Some important Remarks on the effective action for q:

(A) self-interactions q has now anharmonic q^4 interactions

(B) Non-Locality   these interactions are non-local in time but only for $|t-t'| \le 1/\Omega$, indeed

(5) $$\Delta(t) = \int \frac{dk}{2\pi} \frac{e^{ikt}}{k^2-\Omega^2+i\epsilon} = \theta(t)\, \frac{i}{-2\Omega} e^{i(\Omega+i\epsilon)t} + \frac{\theta(-t)(-i)\, e^{-i(\Omega-i\epsilon)t}}{2\Omega} = -e^{-i(\Omega-i\epsilon)|t|}\frac{i}{2\Omega}$$



In Euclidean signature, $\Omega \to -i\Omega$ above, $\Delta(t_E) \sim \frac{e^{-\Omega|t_E|}}{2\Omega} \xrightarrow[|t_E| \gg 1/\Omega]{} 0$

The same is visible in Minkowski spacetime in a distributional sense because for $|t| \gg 1/\Omega$ the Expression (5) oscillates very wildly and it averages to zero. For $\Omega \to \infty$ only the $t-t'=0$ region contributes non-trivially: it must be approximated by local contributions $\delta(t-t') + \frac{a}{\Omega^2}\delta^{(2)}(t-t') + \frac{b\,\delta^{(2m)}}{\Omega^{2m}}(t-t')$. Let's see it explicitly

(6) $$\text{new } q\text{-int.} = \frac{-i\lambda^2}{2}\int dt\, dt'\, q^2(t)\, \Delta(t-t')\, q^2(t') = \frac{-i\lambda^2}{2} \int_{\tau = t'-t} dt\, d\tau\; q^2(t)\, \underbrace{q^2(t-\tau)}_{q^2(t) - \tau \frac{dq^2}{dt}(t) + \frac{\tau^2}{2}\frac{d^2q^2}{dt^2}(t)+\dots}\, \overbrace{\Delta(-\tau)}^{-\frac{i}{2\Omega}e^{-i(\Omega-i\epsilon)|\tau|}}$$

$$= \frac{-i\lambda^2}{2}\int dt \left\{ q^2(t)\, q^2(t) \cdot 2 \underbrace{\int_0^\infty \frac{e^{-i(\Omega-i\epsilon)\tau}\, d\tau}{i2\Omega}}_{-\frac{1}{\Omega^2}} + q^2(t)\frac{d^2q^2}{dt^2}(t)\cdot 2 \int_0^\infty d\tau\, \frac{\tau^2}{2} \frac{\exp[-i(\Omega-i\epsilon)\tau]}{2i\Omega} + \dots \right\}$$

$$-\frac{1}{2i\Omega}\frac{1}{i}\frac{d}{d\Omega^2}\underbrace{\int_0 \exp[-i(\Omega-i\epsilon)\tau]}_{\frac{1}{i\Omega}}$$

$$= \frac{-i\lambda^2}{2}\int dt \left\{ -\frac{q^4(t)}{\Omega^2} + q^2(t)\frac{d^2q^2(t)}{dt^2}\frac{1}{\Omega^4} + \dots \right\}$$

it can be written as a series of infinitely many local terms

(7) $$\mathcal{L}^{\text{eff}}[q] \underset{(4)+(6)}{=} \frac{\dot q^2}{2} - \omega^2\frac{q^2}{2} + \frac{\lambda^2}{2\Omega^2}q^4 - \frac{\lambda^2}{2}\frac{1}{\Omega^4}q^2\frac{d^2q^2(t)}{dt^2} + \dots \quad \leftarrow \text{self-interact.}$$

The same could have been obtained more easily in energy-space $E = k$

(8) $$\int \frac{e^{ikt}}{k^2-\Omega^2-i\epsilon}\frac{dk}{2\pi} = \int \frac{dk}{2\pi} e^{ikt}\left(-\frac{1}{\Omega^2}\right)\left\{1 + \frac{k^2}{\Omega^2} + \dots\right\} = -\frac{1}{\Omega^2}\left[\delta(t) - \frac{\delta^{(2)}(t)}{\Omega^2} + \dots\right]$$

The non-locality is mild and harmless as long as $\Delta t \gg \frac{1}{\Omega}$ i.e. $E \ll \Omega$, which means that the series converges and the results are well approximated by a truncation to finite order terms [the order being determined by the precision sought]

moreover, if the experimentalist had precision $\Delta t \lesssim 1/\Omega$ so that it could resolve the non-locality it would mean that it could produce quanta of $Q(t)$, it would claim the "discovery" of Q

This brings us to the next item

(c) Truncation:

we don't need to know all terms for a fixed precision

Let's calculate e.g. the shift in the energy levels, to first order in λ^2: $\boxed{\Delta E_m^{(1)} = \langle E_m^{(0)} | H^{(1)} | E_m^{(0)} \rangle}$ from QM

$$(9) \begin{cases} \mathcal{L}^{(0)} = \frac{1}{2}\dot{q}^2 - \frac{\omega^2}{2} q^2 \Rightarrow H^0 = \frac{p^2}{2} + \frac{\omega^2}{2} q^2 \Rightarrow E_m^{(0)} = m \cdot (\omega + 1/2) \\ \mathcal{L}^{(1)} = \frac{\lambda^2}{2\Omega^2} q^4 - \frac{\lambda^2}{2\Omega^4} q^2 \frac{d^2 q^2}{dt^2} + \dots \Rightarrow H^{(1)} \text{ is complicated by the fact that } \dot{q} \text{ depends on } p \text{ and } q \text{ too} \end{cases}$$

integration by parts $\hookrightarrow \frac{2\lambda^2}{\Omega^4} q^2 \dot{q}^2$

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = \dot{q}\left(1 + \frac{4\lambda^2}{\Omega^4} q^2\right) \quad \text{depends on } q$$

But this complication is easily avoided by doing a change of variables in the path-integral (usually called a field redefinition in 4D):

$$(10) \begin{cases} q(t) \to q(t) + f(q(t)) \\ \dot{q}(t) \to \dot{q}(t)\left[1 + \frac{\partial f}{\partial q}\right] \end{cases}$$

$$\Rightarrow \frac{\dot{q}^2}{2} \to \frac{\dot{q}^2}{2} + \frac{\dot{q}^2}{2}\left[1 + \frac{\partial f}{\partial q}\right]^2$$ so we can use it to cancel the $\frac{2\lambda^2 q^2 \dot{q}^2}{\Omega^4}$

choosing $\frac{\partial f}{\partial q} = -\frac{2\lambda^2}{\Omega^4} q^2 + o(\lambda^4)$, that is $f(q) = -\frac{2\lambda^2}{\Omega^4}\frac{q^3}{3} + o(\lambda^4)$

we get the following

$$\mathcal{L}^{(1)} = \frac{\lambda^2}{2\Omega^2} q^4 - \overbrace{\frac{\omega^2}{2} q^2\left(1-\frac{2\lambda^2}{\Omega^4}\frac{q^2}{3}\right)^2}^{\text{from } \omega^2 q^2/2} + o(\lambda^4) + (*)$$

[(*) There is also a contribution from the Jacobian $\frac{\delta q}{\delta q'} = \left(1+\frac{2\lambda^2}{\Omega^4}q^2\right)\delta(t-t')$ which gives to order λ^2 just a redefinition of ω, $\omega^2 \to \omega^2 - \frac{4\lambda^2\Lambda}{\Omega^4\pi} = \omega'^2$, where we cut $\int^\Lambda \frac{dk}{2\pi}$. (In dim-reg this shift is zero). Agreeing that we call ω what it's actually ω', the physical frequency, is nothing but adding a counter-term and renormalizing ω, the cool fact that we are in QM]

The net result is

the more $\frac{\partial_t^2}{\Omega^2}$ incl. the more $\frac{\omega^2}{\Omega^2}$

$$(11)\quad \mathcal{L}^{(1)} = \underbrace{\frac{\lambda^2}{2\Omega^2}q^4}_{\text{from } q^4} + \underbrace{\frac{2\lambda^2}{3\Omega^4}\omega^2 q^4}_{\text{from } q^2\partial_t^2 q^2} \longrightarrow H^{(1)} = \left(\frac{-\lambda^2}{2\Omega^2}\right)\left[1 + \left(\frac{\omega^2}{\Omega^2}\right)\frac{4}{3}\right] q^4$$

$$(12)\quad \to \Delta E_m^{(1)} = \langle E_m^{(0)}| H^{(1)} |E_m^{(0)}\rangle = \left(\frac{-\lambda^2}{2\Omega^2}\right)\left(1+\frac{\omega^2}{\Omega^2}\frac{4}{3}\right) \left\| q^2 |E_m^{(0)}\rangle\right\|^2 \quad \text{where} \quad q = \underbrace{\frac{a+a^\dagger}{\sqrt{2\omega}}}_{\text{raising/lowering op.}}$$

[[QM-reminder: $H^{(0)} = \frac{p^2}{2} + \frac{\omega^2 q^2}{2} = \omega\left(\overbrace{-i\frac{p}{\sqrt{2\omega}} + \sqrt{\frac{\omega}{2}}q}^{a^\dagger}\right)\left(\overbrace{\frac{ip}{\sqrt{2\omega}} + \sqrt{\frac{\omega}{2}}q}^{a}\right) - \frac{i\omega}{2}[q,p]$

$= \omega(a^\dagger a + 1/2)$

$$|n^{(0)}\rangle = \frac{(a^\dagger)^m}{\sqrt{m!}}|0\rangle \qquad \begin{cases} (a+a^\dagger)\frac{1}{\sqrt{2\omega}} = q \\ \frac{(a-a^\dagger)}{i}\sqrt{\frac{\omega}{2}} = p \end{cases} \qquad (*)\ \begin{aligned} a|n^{(0)}\rangle &= \sqrt{n}\,|n-1\rangle \\ a^\dagger|n^{(0)}\rangle &= \sqrt{n+1}\,|n+1\rangle \end{aligned}$$

$$q^2|n^{(0)}\rangle = \frac{(a+a^\dagger)^2}{2\omega}|n^{(0)}\rangle \Rightarrow \langle n^{(0)}|q^4|n^{(0)}\rangle = \left\|q^2|n^{(0)}\rangle\right\|^2 = \frac{3}{4\omega^2}\left[(n+1)^2+n^2\right]$$]]

So that the corrections from $o(\lambda^2)$ are

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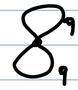
(13) $$\boxed{\Delta E_m^{(1)} = \frac{-\lambda^2}{2\Omega^2}\left(\frac{3}{4\omega^3}\right)\left[(m+1)^2+m^2\right]\left\{1+\left(\frac{\omega}{\Omega}\right)^2\frac{4}{3}+\cdots\right\}}$$ expansion in $\left(\frac{\omega}{\Omega}\right)^2 \ll 1$

- Only the first few terms are needed to match a certain experimental precision
- notice that $\Delta E_m^{(1)}$ grows as m^2 whereas $E_m^{(0)} = \omega(m+\frac{1}{2})$ so that this expansion is useful only for low-quanta, which was our goal anyway: the theory becomes strongly coupled when $\boxed{\frac{\lambda^2}{2\Omega^2}\frac{3}{4\omega^2}\bar{m} = \omega \Rightarrow \Lambda_{strong} = \frac{2\Omega^2 4\omega^4}{\lambda^2 3}}$
- The correction to the ground energy due to q^4 can be calculated with a bubble diagram too:

(14)  $$3\frac{i\lambda}{2\Omega^2}\Bigg(\underbrace{\int\frac{dk}{2\pi}\frac{i}{k^2-\omega^2+i\epsilon}}_{1/2\omega}\Bigg)^2 = 3\frac{i\lambda^2}{2\Omega^2}\left(\frac{1}{2\omega}\right)^2$$

In agreement with (13) for $m=0$ (and $\omega \ll \Omega$ so that we could neglect $q^2\dot{q}^2$

(D) Symmetries

The original theory in Eq. (1) has discrete symmetries

$$P: \begin{cases} q(t) \to -q(t) \\ Q(t) \to Q(t) \\ j(t) \to -j(t) \end{cases} \qquad T: \begin{cases} q(t) \to q(-t) \\ Q(t) \to Q(-t) \\ j(t) \to j(-t) \end{cases}$$

This explains why the effective action depends only on even-powers of both q's & time-derivatives: $q^2, q\ddot{q}, q^2\dot{q}^2$ $\cancel{q\dot{q}}, \cancel{\ddot{q}\dot{q}}, \cancel{q^3}, \dots$

there is for another useful "spurious symmetry" that explains why only even-powers of λ appears:

$P' = \begin{cases} q(t) \to q(t) \\ Q(t) \to -Q(t) \\ j(t) \to j(t) \end{cases}$ is not a symmetry, broken by $\lambda q^2 Q$ because this vertex flips signs

It is, however, restored if we think of λ as a so called, spurion field upon which we did not integrate over yet: the resulting action is invariant under

$\mathcal{P}' : \begin{cases} q(t) \to q(t) \\ Q(t) \to -Q(t) \\ j(t) \to j(t) \\ \lambda \to -\lambda \end{cases}$ $\Rightarrow$ Since Q does not appear in the IR the only singlet under $\mathcal{P}'$ is λ^2

Of course this theory is also invariant under a continuous symmetry, translations in time $t \to t + \text{const}$ which gives rise to energy conservation.

(E) Decoupling

* For $\Omega \to \infty$ (at fixed λ) the corrections to $\mathcal{L}_{\text{harmonic}}(q)$ vanish, because the Q becomes impossible to excite w.r.t its ground state

* In the limit $\lambda \to 0$ (at fixed Ω) the q and Q decouple from each other and the problem factorizes it into two independent harmonic oscillators: no effect of Q affects q.

* In the limit $\lambda \to \infty$, $\Omega \to \infty$ keeping λ^2/Ω^2 - fixed the $\mathcal{L}_{eff.}$ is non-trivial because the coupling strength gets large compensating the mass gap [something similar happens in chiral

gauge theories where $m \propto$ Yukawas, at fixed VEV; it happens also for purely Dirac-Neutrinos]

* The only non-decoupling effect as $\Omega \to \infty$ (fixed λ) is the change into the vacuum-energy

(15) $$N = \left[2\pi / \det(-\partial_t^2 - \Omega^2 + i\epsilon)\right]^{1/2}.$$

in Eq.(4). This determinant is nothing but a complicated way to determine the zero-point energy contribution due to Q, i.e. $\Delta E_0 = \frac{\Omega}{2}$:

(16) $$N = \sqrt{2\pi}\, \exp\left[-\frac{1}{2} \mathrm{Tr} \ln(-\partial_t^2 - \Omega^2 + i\epsilon)\right] = \sqrt{2\pi}\, \exp\left[-\frac{1}{2} \int \frac{dk}{2\pi} \ln(+k^2 - \Omega^2 + i\epsilon)\right]$$

sum eigenval.

This is formally divergent but the Ω-dependent term is finite

$$\frac{\partial}{\partial \Omega} \int \frac{dk}{2\pi} \ln(\ldots) = \int \frac{dk}{2\pi} \frac{-2\Omega}{k^2 - \Omega^2 + i\epsilon} = -2\Omega \, \frac{2\pi i}{2\pi} \, \frac{1}{-2\Omega} = i$$

residue

(17) $$\Rightarrow N = \text{const} \cdot \exp\left[-i \frac{\Omega}{2}\right]$$

This corresponds to a vacuum energy $\Delta E_0 = \frac{\Omega}{2} \gg \frac{\omega}{2}$ and does not decouple. Nevertheless it does not affect anything else as long $M_{Planck} \to \infty$ too (this is the origin of the cc problem when M_{Planck} is finite)

(F) Dimensional Analysis

Most of the qualitative results follow by Dim. Anal + Decoupling + symm.
The mass-dimensions are ($\hbar = 1$, $c = 1$)

(18) $$[\mathcal{L}] = 1, \quad [\partial_t] = 1 \Rightarrow [\varphi] = [Q] = -\frac{1}{2}, \quad [\omega] = [\Omega] = 1, \quad [\lambda] = 5/2$$

So the corrections to the energy levels $[\Delta E_n] = 1$ must be of the type

$$(19)\quad \Delta E_m^{(1)} \sim \lambda^2 \cdot \frac{1}{\Omega^2} \cdot \omega^{1-2[\lambda]+2[\Omega]=-2} = \lambda^2 \frac{1}{\Omega^2}\frac{1}{\omega^2}$$

matching (13)

largest order compatible w/ $\mathcal{L}'$ in (14) and decoupling $\lambda \to 0$

largest order compatible with decoupling $\Omega \to \infty$

We get the correct scaling of the higher-derivative terms too, at leading order in λ^2, since then only ω^2/Ω^2 is the dimensionless expansion param.

$$(20)\quad \Delta E_m^{(1)} \sim \lambda^2 \frac{1}{\Omega^2}\frac{1}{\omega^2}\left\{1 + \cancel{\alpha_1}\, \frac{\omega^2}{\Lambda^2} + \dots\right\}.$$

All the features we have seen in the previous example in QM are in fact general aspects of EFT:

- infinitely many operators vs locality in time $\longrightarrow$ upgraded to locality in 4D space-time via Lorentz-symmetry
- expansion in derivatives & decoupling
- structure controlled by exact & broken symm.
- dimensional analysis vs explicit matching

rather than ∂_t^2 one has $\Box = \partial_\mu \partial^\mu = \partial_t^2 - \vec{\nabla}^2$

We will see these over and over again in the next lectures. Notice that "leak" of heavy modes into low-energy observables was due to QM, as it is illustrated by restoring $\hbar$-factors (or more easily look at the loops in (16) & (17)), it's not special to QFT in higher D only. What's special about QM+Relativity is the restriction due to causality. We will see in future lectures what causality imply on EFT.