Topics in Effective Field Theory

PhD-course, Rome 2020 by B. Bellazzini

All the features we have seen in the previous example in QM are in fact general aspects of EFT;

- infinitely many operators vs locality in time → upgraded to locality in 4D space-time via Lorentz-symmetry
- expansion in derivatives & decoupling
- structure controlled by exact & broken symm.
- dimensional analysis vs explicit matching

rather than ∂_t^2 one has $\Box = \partial_\mu \partial^\mu = \partial_t^2 - \vec{\nabla}^2$

Moreover, the 4D Lorentz symmetry will allows to formulate tight constraints on EFT by requiring Causality of the underlying UV-theories (we'll see this in future lectures).

Today we go through some other 4D examples:

* Example: $\boxed{\text{Integrate out heavy vector}}$

We study regime $E \ll M_A$

(1) $$\mathcal{L}_{UV} = \underbrace{\bar{\psi} i \not{\partial} \psi}_{\mathcal{L}_{free}} + g J_\mu A^\mu - \frac{1}{4} F_{\mu\nu}^2 + \frac{M_A^2}{2} A_\mu^2 \qquad J_\mu \equiv \bar{\psi} \gamma_\mu \psi \text{ current}$$

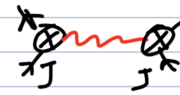
→ Integrating A_μ: $$Z[j] = \int [d\psi d\bar{\psi}] \exp\left[i \int d^4x\, \mathcal{L}_{free}\right] \cdot e^{i W[g J(\psi)]}$$

(2) again Gaussian integral: $$\int [dA] \exp\left\{ i \int \frac{1}{2} A_\mu \underbrace{\left[-\partial^\mu \partial^\nu + \eta^{\mu\nu} (\Box + M_A^2) \right]}_{\Delta_{\mu\nu}^{-1}} A_\nu + g J^\mu A_\mu \right\}$$

$\left(\Delta^{-1}_{\mu\nu}\Delta_{\nu\rho}(x) = \delta^4(x)\eta_{\mu\rho} \to \Delta_{\mu\nu} = \int \frac{d^4k}{(2\pi)^4} e^{-ikx}\frac{-\eta_{\mu\nu}+k_\mu k_\nu/M_A^2}{k^2-M_A^2+i\epsilon}\right)$

(3) $$e^{iW[gJ]} = \mathcal{N} e^{-i\int d^4x'\, d^4x\, g^2 J_\mu(x)\Delta_{\mu\nu}(x-x')J_\nu(x')}$$ ← non-local in spacetime

$$= \mathcal{N} e^{-i\int d^4k\, g^2 \hat J_\mu(k)\hat\Delta_{\mu\nu}(k)\hat J_\nu(-k)}$$ ← in momentum space

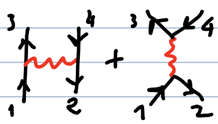
(4) $\exp\{i$  $\} = +i\int g^2 \hat J_\mu(k)\left\{\frac{1}{M_A^2}\left(1+\frac{k^2}{M_A^2}+\dots\right)\left(-\eta_{\mu\nu}+\frac{k_\mu k_\nu}{M_A^2}\right)\right\}\hat J_\nu(k)$

(the only connected diagram.) k's give rise to derivatives

(5) $$= i\int d^4x\left\{-\frac{g^2}{M_A^2}J_\mu(x)J^\mu(x) + \frac{g^2}{M_A^4}J_\mu(\Box\eta_{\mu\nu}-\partial_\mu\partial_\nu)J^\nu+\dots\right\}$$

which at low energy can be expanded in terms of local operators

The leading operator that contributes to the observables is thus $J_\mu J^\mu$ with coefficient $\propto \frac{g^2}{M_A^2}$. We could have guessed it using dimensional analysis + symmetries, as we will see later. For now, let's calculate an observable such as the 2-to-2 scattering S-matrix

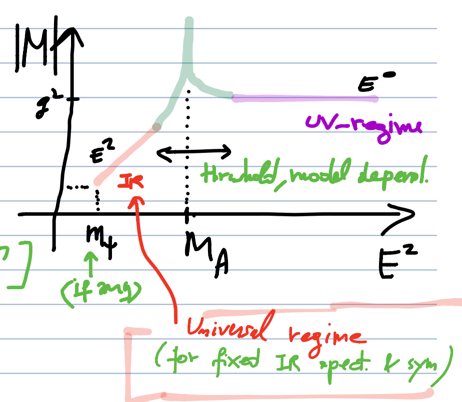
(6) $\mathcal{M}(\psi_1\bar\psi_2 \to \psi_3\bar\psi_4) \overset{\text{Wick th.}}{=} \frac{(ig)^2}{2!}\cdot 2\Big\{$  $\Big\}$

(7) $$= -g^2 (\bar u(3)\gamma^\mu u(1))\, \frac{i(-\eta_{\mu\nu}+k_\mu k_\nu/M_A^2)}{t-M_A^2+i\epsilon}\, \bar v(2)\gamma_\nu v(4) - g^2\, \bar v(2)\gamma^\mu u(1)\, \frac{i(-\eta_{\mu\nu}+k_\mu k_\nu/M_A^2)}{s-M_A^2+i\epsilon}\, \bar u(3)\gamma^\nu v(4)$$

(8) $$\sim g^2\frac{E^2}{E^2-M_A^2} = \begin{cases} O(g^2) & E\gg M_A^2 & \text{UV}\\ O\left(\frac{g^2E^2}{M_A^2}\right) & E\ll M_A^2 & \text{IR}\\ \text{resonance} & E\simeq M_A & \text{threshold}\end{cases}$$

$(u\sim\sqrt{E})$

[regularized by finite width $i\epsilon \to i m_A\Gamma$]



What about the EFT calculation?

If we use the leading low-energy operator $J^\mu J_\mu$ in the EFT we should get the same answer:

(9) $$\mathcal{M}_{EFT}(\psi_1\bar\psi_2 \to \psi_3\bar\psi_4) \underset{Eq(23)}{=} -i g^2 \{\langle 3\bar 4 | \bar\psi\gamma^\mu\psi\, \bar\psi\gamma_\mu\psi | 1\bar 2\rangle\} =$$

(10) $$= -i\frac{g^2}{M_A^2}\{\bar v_2\gamma_\mu u_1\, \bar u_3\gamma^\mu v_4 - \bar u_3\gamma^\mu u_1\, \bar v_2\gamma_\mu v_4\} \sim \frac{g^2 E^2}{M_A^2}$$

Which reproduces (24) (hence 25) when $s, t \ll M_A^2$ (also for the spin struct.)

Important:
The behavior at low-energy is actually completely fixed, it is universal and we could have gotten it without knowing the UV-theory: It's this universality that makes EFT very useful & powerful. Let's see it at work:

We assume:

- The IR spectrum (what an IR observer sees)

  In the case at hand a massless spin-1/2 particle described by a lefthanded field $\psi = \psi_L = (1/2, 0)$

  The theory contains the ψ kinetic term, $\bar\psi i \not\partial \psi = \psi^\dagger i\bar\sigma^\mu \partial_\mu \psi$
  the most important operator for a particle ← pretty much what defines having a particle in the sp.

- IR-symmetries (testable as well against data, see below)

  We already assumed Lorentz ⇒ a Lorentz scalar needs an even # of fermion legs

(11) • 2 fields: $(1/2, 0)\otimes(1/2, 0) = (1,0) + (0,0)$

(12) $$\Rightarrow \psi^2 = \psi_\alpha\psi^\alpha = \psi_\alpha\epsilon^{\alpha\beta}\psi_\beta = \psi^T i\sigma^2\psi$$

Majorana mass (0,0)

(13) $(0,1/2)\otimes(1/2,0)=(1/2,1/2)\to\psi^\dagger\bar\sigma^\mu\psi$
$\psi^\dagger$

becomes a scalar if contracted with 1 derivative $\partial_\mu=(1/2,1/2)$
Kinetic energy (already included)

(14) Another symm. we can ask is $\boxed{\text{chiral symmetry } \psi\to e^{i\alpha}\psi}$
(it's a U(1) on a single chirality)

In fact this is an symmetry of the massless theory
The Majorana mass would break it: so the missing mass strongly suggest to upgrade it to an actual symmetry of the other terms and see what it buys for us: it says that # of ψ & $\psi^\dagger$ must be the same. For the 4 legs it means

(15) • 4 fields:
+ chiral sym.

$$\begin{cases}\psi^2\psi^{\dagger 2}=\psi_\alpha\psi^\alpha\,\psi^\dagger_{\dot\alpha}\psi^{\dagger\dot\alpha} & \text{or } (\psi\sigma^{\mu\nu}\psi\cdot\psi^\dagger\bar\sigma_{\mu\nu}\psi=0)\\ \psi^\dagger\bar\sigma^\mu\psi\;\psi^\dagger\bar\sigma_\mu\psi\end{cases}$$

In fact they are the same op. via Fierz identity
$\bar\sigma^\mu_{\alpha\beta}\,\bar\sigma_{\mu\gamma\delta}\propto\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}\ldots$

$$\boxed{\text{So, chiral sym + Lorentz + no-derivatives}\Longrightarrow\text{Unique Operator}}$$

unique, very predicting

(16) $$\mathcal{L}^{EFT}=\bar\Psi i\partial\!\!\!/\Psi+\frac{1}{f^2}\underbrace{(\bar\Psi\gamma^\mu\Psi)^2}_{J_\mu J^\mu}+O(\partial^2\psi^{\dagger 2}\psi^2)$$

completely fixed up to a constant

$$\mathcal{M}(\Psi\bar\Psi\to\Psi\bar\Psi)=\frac{1}{f^2}\left\{\bar v_2\gamma_\mu u_1\;\bar u_3\gamma^\mu v_4-\bar u_3\gamma^\mu u_1\;\bar v_2\gamma_\mu v_4\right\}\sim\frac{1}{f^2}E^2$$

It nicely reproduces the energy behavior & spin-structure of the amplitude in the full theory when evaluated to small energy. We use no info of UV theory except symmetries. Different UV-th with same symm. and IR sp.

deliver the same operator in the IR.

The only unknown $1/f^2$, can be fixed by one measurement, say by $M(\psi\psi\to\psi\psi)$ at a given energy, then all other measurements are fixed & the theory is predictive, as long as terms with more derivatives can be neglected (this is the case when $E\to 0$, the IR)

A low-energy measurement is teaching us something about the UV-theory: the following combinations of UV-param:

(17) $$\boxed{1/f^2 = \frac{g^2}{M_A^2}}$$ which is a specific example of Matching between the parameter in the two theories UV & IR, where both work well (in the IR)

Matching

The assumptions in the IR can be checked & exp. verified or refuted. For example, if we drop Chiral Symmetry, we can add other operators:

(18) $$m(\psi^2+\psi^{+2}),\ \frac{1}{f_1^2}(\psi^4+\psi^{+4})$$

(the $(\psi\bar{\sigma}^{\mu\nu}\psi)^2$ is actually not indep. via Fierz again)

So that the $U(1)$-conservation law would be broken: $M(\psi\,\psi\to\bar{\psi}\,\bar{\psi})\neq 0$

Notice that the violation is by two units because there is no operator built out of 3ψ and $1\psi^+$, by Lorentz symmetry

$$(1/2,0)^3\otimes(0,1/2) = (\#, 1/2) \not\ni (0,0)$$

What if we do a low-energy exp. and see no sign of $J^\mu J_\mu$? We would conclude that it is zero or very small ($f\to\infty$)

$\Rightarrow$ UV-theory either has a tiny g^2/M_A^2 or it's a different one with sizeable couplings and not so large masses, but with a symmetry that would forbid $J^\mu J_\mu$ in the first place.

Is this possible?

Yes, non-linearly realized SUSY is one such example where ψ is the Goldstino (the "Goldstone-Fermion" of susy-break) and the first operator allowed is

(19) Non-linear SUSY: $\dfrac{\psi^{\dagger 2}\Box\psi^2}{f^4} \Rightarrow M(\psi\bar\psi\to\psi\bar\psi)\sim\dfrac{E^4}{f^4}$.

Another example is a fermionic shift symmetry: $\psi\to\psi+\epsilon$ — similar to susy but internal, no spacetime involved

so that the invariant operators are built out of $\partial\psi$, which is invariant ← Kinetic term, only up to a total der. $\bar\psi i\gamma\psi\to\partial_\mu(\bar\epsilon i\gamma^\mu\psi)$

(20) $\Rightarrow \mathcal{L} > \dfrac{(\partial\psi)^{\dagger 2}(\partial\psi)^2}{f^6} \Rightarrow \boxed{M=\dfrac{E^6}{f^6}}$.

Two remarks:

- Measuring well low-energy amplitudes (or observables) can teach us about structural properties of the UV, such as the symmetries
- it seems however that we can always find a sym. that would forbid terms in the amplitudes: in fact we will see that other constraints apply to EFT (that come from Causality) and in fact the E^4-term must always be present in the low-energy amplit. ($\Rightarrow \psi\to\psi+\epsilon$ is never an exact sym.)

Example: theory with Photon only

(no charged particles around, $E \ll m_e$)

- IR-spectrum: $s=1$ $m_\gamma=0$ $\Rightarrow$ everything can be written down in terms of $F_{\mu\nu}$ and $\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}=\tilde{F}_{\mu\nu}$

[no currents, so among the irrep $(1,0)$; $(0,1)$; $\cancel{(1/2,1/2)}$] Keep only the gauge inv.

- Symmetries: Let's assume Parity $\Rightarrow$ no $\varepsilon_{\mu\nu\rho\sigma}$ around, only $F_{\mu\nu}$ is needed

(21) $$\rightarrow \mathcal{L}_{EFT}(E\ll m_e) = -\frac{1}{4g^2}F_{\mu\nu}^2 + \frac{1}{\Lambda_1^4}(F_{\mu\nu}F^{\mu\nu})^2 + \frac{1}{\Lambda_2^4}(F_{\mu\nu}F^{\nu\rho})^2 + \dots$$

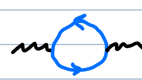
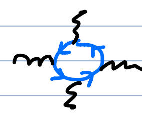
[notice: $\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\rho\sigma\alpha\beta} \sim \delta^\alpha_\mu \delta^\beta_\nu$ and other identities like this $(F\tilde{F})^2 \sim a(FF)^2 + bF^4$]

(22) $$\Rightarrow M(\gamma\gamma\rightarrow\gamma\gamma)(E\ll m_e) \propto \frac{g^4E^4}{\Lambda^4}$$

(only $g^4/\Lambda_i^4 = \frac{1}{f_i^4}$ enters in the IR)

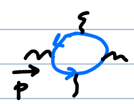
- In this example we did not specified the UV-theory. It could be, for ex. a QED-like th. $\mathcal{L}_{UV}^{-1} = -\frac{1}{4g^2_{UV}(\Lambda)}F_{\mu\nu}^2 + J_\mu A_\mu + \bar{\psi}(i\partial\!\!\!/ - M)\psi$

only log-decoup

(23)   $$\Rightarrow \begin{cases} \frac{1}{g^2} = \frac{1}{g_{UV}^2(\Lambda)} + \frac{\#}{16\pi^2}\ln\left(\frac{\Lambda}{M}\right) \\ \frac{1}{\Lambda_i^4} = \# \frac{1}{16\pi^2}\frac{1}{M^4} \end{cases}$$

power-law decoupl.

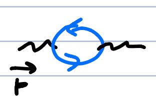
It's interesting to see how these estimates come about:

 $\overset{?}{\sim} A^4 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(p \text{ or } k \text{ or } M)^4}$ but it needs to be gauge invariant so

(24) $$\text{actually} \sim (p^{[\mu}A^{\nu]})^4 \underbrace{\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k \text{ or } M)^8}} \sim p^{[\mu}A^{\nu]}\frac{1}{16\pi^2}\frac{1}{M^4}$$

we set $p=0$ as interested in F^4, not $(\partial F)^4$

The integral is IR-dominated (IR-div. for $M\to 0$) so pulled out as many M to make it convergent.

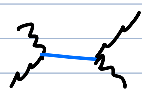
(25)  $$\sim A^2 \int \frac{d^4k}{(2\pi)^4}\frac{1}{(p \text{ or } k \text{ or } M)^2} \Rightarrow (p^{[\mu}A^{\nu]})^2 \int \frac{d^4k}{(2\pi)^4}\frac{1}{(k \text{ or } M)^4} \sim \frac{\#\ln\Lambda}{16\pi^2\, M}$$

L2/8

The F^4 term decouples very fast as $M \to \infty$, while F^2-corrections are only logarithmic in M as $M \to \infty$. Nevertheless, to scattering only F^4 contributes (up to rescaling canonically F^2) so it represents the leading correction to scattering in the IR

an alternative UV-theory would be a higgs-like theory (or dilaton-like)

(26) $$\mathcal{L}^2_{UV} = -\frac{1}{4g^2_{UV}} F_{\mu\nu}^2 + \frac{1}{f g^2_{UV}} h F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial h)^2 - \frac{M^2}{2} h^2$$

(27) $\Rightarrow$  $$\frac{1}{g^4_{UV} f^2} F_{\mu\nu}^2 \left(\frac{1}{M^2} - \frac{\Box}{M^4} + \dots\right) F_{\mu\nu}^2 \Rightarrow \begin{cases} \frac{1}{g^2} = \frac{1}{g^2_{UV}} \\ \frac{g^4}{\Lambda_1^4} \sim \frac{1}{f^2 M^2} \end{cases}$$

Again, very fast decoupling, as $1/M^2$.

However, if we had chosen a different UV-theory, e.g.

(28) $$\mathcal{L}^3_{UV} = -\frac{1}{4g^2_{UV}} F_{\mu\nu}^2 + \frac{1}{f g^2_{UV}} h F_{\mu\nu}^2 + \frac{1}{2}(\partial h)^2 - \lambda (h^2 - v^2)^2$$

such that $M \propto \lambda v$ and $\langle h \rangle = v$, then the

(29) $$\frac{1}{g^2_{IR}} = \frac{1}{g^2_{UV}} - 4\frac{M}{\lambda f g^2_{UV}}$$

so that $M \to \infty$, $\lambda \to \infty$ at v-fixed does not decouple (unless one fine-tunes $1/g^2_{UV}$ against it)

We will come back to these type of sensitivities to the UV param. in future lectures.

Different UV-theories can give rise to the same leading IR-operators

— Integrating out @ tree- and loop-level —

It's often useful to know that integrating out a field at tree-level corresponds to solve the classical equations of motion, in the bkg of the other fields, and plug the solution back-in the action. Indeed, consider the path integral

$$\int[d\phi]\int[dH]\,\exp\left\{i\,S[\phi,H]+i\int J\phi\right\} \tag{30}$$

the heavy field

expand in H around a $\bar{H}(x)$ and change variable $H-\bar{H}=H'$

$$S[\phi,H]=S[\phi,\bar{H}]+\int d^4x\,H'(x)\left.\frac{\delta S}{\delta H(x)}\right|_{H=\bar{H}}+\int d^4x\,d^4y\,\frac{1}{2}H'(x)\left.\frac{\delta^2 S}{\delta H(x)\,\delta H(y)}\right|_{H=\bar{H}}H'(y)+\dots \tag{31}$$

Now if we choose the $\bar{H}=\bar{H}(\phi)$ such that it solve the e.o.m

$$\left.\frac{\delta S(\phi,H)}{\delta H}\right|_{H=\bar{H}(\phi)}=0 \Rightarrow \text{the } (31)=S[\phi,\bar{H}(\phi)]+\int d^4x\,d^4y\,\frac{H'(x)}{2}\left.\frac{\delta^2 S}{\delta H\,\delta H}\right|_{\bar{H}(\phi)}H'(y)+\dots \tag{32}$$

But the $\frac{\delta^2 S}{\delta H^2}$ and following terms contribute @ 1-loop or more because they have at least 2 H'-legs and no external current. Therefore, at tree level the net effect is just

$$\boxed{S^{EFT}[\phi]\Big|_{\text{tree-level}}=S[\phi,\bar{H}(\phi)]} \quad \text{where} \quad \boxed{\left.\frac{\delta S}{\delta H}\right|_{\bar{H}(\phi)}=0} \tag{33}$$

Ex: $gH\phi^2$-vertex

$\Rightarrow \bar{H}(\phi)=\frac{1}{-\Box-M^2}\,g\phi^2$

$\Rightarrow \delta S^{EFT}=\frac{1}{2}g\phi^2\frac{1}{-\Box-M^2}g\phi^2$

compare with (27)

Example: take Eq.(26) $\mathcal{L}=\frac{1}{2}h(-\Box-M^2)h+\frac{1}{f g_{UV}^2}h\,F_{\mu\nu}F^{\mu\nu}-\frac{1}{4g_{UV}^2}F_{\mu\nu}F^{\mu\nu}$

$$\frac{\delta S}{\delta h}=0 \Rightarrow h=-\frac{1}{-\Box-M^2+i\epsilon}\,\frac{F^2}{f g_{UV}^2} \tag{34}$$

(33) $$\Rightarrow S_{\text{tree}}^{\text{EFF}}[A_\mu] = \frac{F^2}{f g_{UV}^2}\left(\frac{1}{-\Box - M^2 + i\epsilon}\right)\frac{F^2}{f g_{UV}^2}\left(-\frac{1}{2}\right) = +\frac{1}{2}\cdot\frac{1}{f^2 g_{UV}^4}\frac{1}{M^2} F_{\mu\nu}F^{\mu\nu}\left(1 - \frac{\Box}{M^2} + \dots\right)F_{\mu\nu}F^{\mu\nu}$$

in agreement with (27).

(If we take instead $\frac{1}{f g_{UV}^2} h F_{\mu\nu}^2 + \frac{1}{2}(\partial h)^2 - \lambda(h^2 - v^2)^2$ with $\lambda \to \infty$

$\frac{\delta S}{\delta h} = -\Box h - 2\lambda(h^2 - v^2)2h = 0$ for $\lambda \to \infty$ only if $h \to v$, and ∂ it's vev

because fluctuations cost infinite energy.)

— 1-loop —

One can also go to 1-loop since $H' \frac{\delta^2 S}{\delta H \delta H} H'$ in (32) gives just a Gaussian integral:

(36) $$\int [\mathcal{D}H'] \exp\left\{ i \frac{H'}{2} \cdot \frac{\delta^2 S}{\delta H^2}\bigg|_{H = \bar{H}(\phi)} \cdot H' \right\} \underset{\text{if real boson}}{=} N\left(\det \frac{\delta^2 S}{\delta H^2}\bigg|_{H = \bar{H}(\phi)}\right)^{-1/2} = N \exp\left\{ -\tfrac{1}{2} \ln \det \left(\frac{\delta^2 S}{\delta H \delta H}\bigg|_{H = \bar{H}(\phi)}\right) \right\}$$

and analogous for fermions. (Recall also that $\log\det = \mathrm{Tr}\log$)

This functional integral is complicated (however doable, see Fedkoski's lect. not 33-34) but it becomes a simple one if one considers constant ϕ, like when looking for the effective potential.

Example: Take $\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}(\partial\Phi)^2 - \frac{M^2\Phi^2}{2} - \frac{g^2\Phi^2\varphi^2}{2}$

$$\Rightarrow \frac{\delta^2 S}{\delta\Phi(x)\delta\Phi(y)} = \left[(-\Box_x - M^2) - g^2\varphi^2(x)\right]\delta^4(x-y)$$

$$\Rightarrow (36) \underset{\text{const } \varphi}{=} \exp\left\{ -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \ln\left(k^2 - M^2(\phi) + i\epsilon\right) \right\} N = V_{CW}$$

$M^2(\phi) = M^2 + g^2\varphi^2$ (φ-const)

This is known as Coleman-Weinberg potential for φ generated ∂ Φ loop

(It is formally divergent but just need to add counterterms $A + B\phi^2 + C\phi^4$ since $\frac{\partial^3 V}{\partial M^3}$ is finite, then integrate it back to V_{CW})

— Aside: Why is the sky blue? —

To address this question we need to add to the EFT of photons the molecules of atmosphere upon which light from the sun scatter. These molecules are essentially pointlike w.r.t. the wavelength sent so they are represented as a field ϕ ($\phi^\dagger$) which annihilates (creates) the lowest energy mode (we are going to neglect excited & ionized states assuming $E \ll \Delta E$ — the splittings of levels $\sim \alpha_{em}\Lambda$ (where $\Lambda \simeq \alpha_{em} M \ll M$).

Moreover, it is highly non relativistic and no anti-molecules are around (neither needed, $E \ll M$); hence we are integrating out also the antiparticles. Effectively, it means that of the two branches of $E^2 = p^2 + M^2$ we keep only one: $E = \sqrt{p^2+M^2} = M + \frac{p^2}{2M} + \dots$ corresponding to invariance under the Galilean group: $\mathcal{L}_{kin}^{non-rel.} \simeq \phi^\dagger\left(i\partial_t - M + \frac{\vec{\nabla}^2}{2M}\right)\phi \Rightarrow [\phi] = 3/2$

The molecule is neutral, so it couples to $F_{\mu\nu}$ (and it has conserved # so as many ϕ as $\phi^\dagger$ in the vertex):

$$\mathcal{L}_{int} = \frac{\phi^\dagger \phi F_{\mu\nu}^2}{\Lambda^3}$$

where $\Lambda^{-1} = r_0$ the typical geometric size, that is the ~ Bohr radius $\Lambda^{-1} = 1/\alpha_{em} M$

$\Rightarrow \mathcal{M}(\phi\gamma \to \phi\gamma)$ scales with $1/\Lambda^3 \Rightarrow \sigma(\gamma\phi \to \gamma\phi) = \# \frac{E^4}{\Lambda^6} = r_0^6 E^4$

(since $[\sigma] = -2$) $\Rightarrow$ Sky is blue because blue frequency is scattered more than red.

Since we integrated out energy splitting $\Delta E \sim \alpha^{em}\Lambda$ the theory is valid as long as $E_\gamma/\Delta E = E_\gamma/M\alpha_{em}^2 \ll 1$.

(On Mars the sky is reddish because light scatter off dust with size $\sim 1/E$ which can't be considered pointlike (no spherical sym.) so that various levels contribute significant, not just ground state. (Moreover, there is less atmos here).