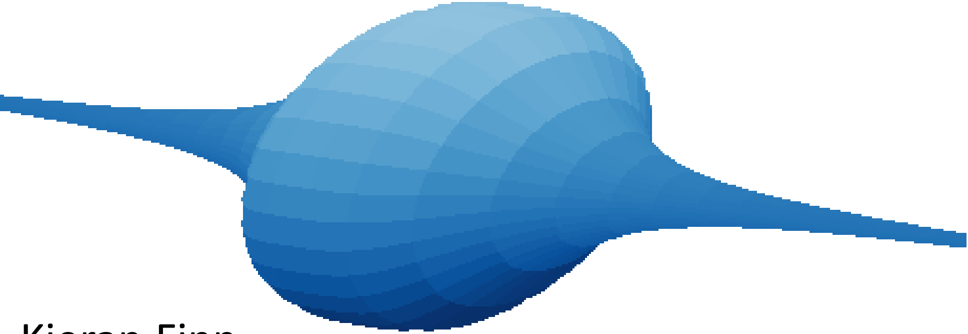
Geometric Approaches to QFT

The Eisenhart Lift



Kieran Finn

with Apostolos Pilaftsis and Sotirios Karamitsos

Monday 15th February 2021

arXiv:1806.02431 (PRD 2018)

arXiv:2012.15288 (PRD 2021)

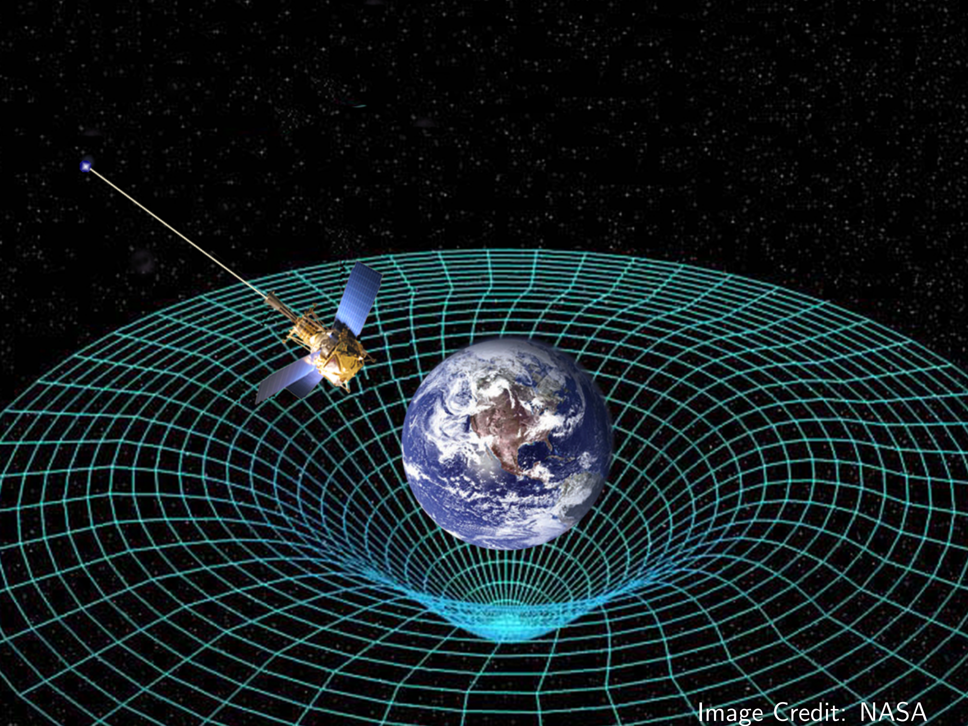


Image Credit: NASA

The Classical Eisenhart Lift

L. Eisenhart 1928

Particle Subject to Conservative Force

- $L = \frac{1}{2} m \, g_{ij} \dot{x}^i \dot{x}^j - V(\mathbf{x})$
- $m\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = -V^{,i}$

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Free Particle on Higher Dimensional Manifold

- Coordinates: $X^I = \{x^i, y\}$
- Metric: $G_{IJ} = \begin{pmatrix} g_{ij} & 0 \\ 0 & \frac{M^2}{mV} \end{pmatrix}$

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- $L = \frac{1}{2} m\, g_{ij} \dot{x}^i \dot{x}^j + \frac{1}{2} \frac{M^2}{V(\mathbf{x})} \dot{y}^2$
- $m\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = -\frac{1}{2} \left(\frac{M\dot{y}}{V(\mathbf{x})} \right)^2 V^{,i}$
- $\frac{d}{dt} \left(\frac{M\dot{y}}{V(\mathbf{x})} \right) = 0 \quad \Rightarrow \quad \frac{M\dot{y}}{V(\mathbf{x})} = A$

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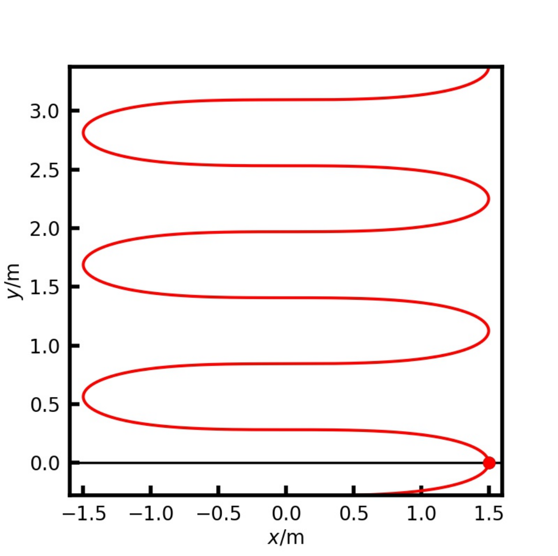
Example: Simple Harmonic Motion



Metric:

$$G_{IJ} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{k}{m^5 x^2} \end{pmatrix}$$

The Eisenhart Lift in Quantum Mechanics

KF, S. Karamitsos and A. Pilaftsis 2020 (Arxiv:2012.15288)

Original System

- $H = \frac{g^{ij} p_i p_j}{2m} + V(\mathbf{x})$
- $\dot{x}^i = \frac{g^{ij} p_j}{m}$
- $\dot{p}_i = -mV_{,i}$

Lifted System

- $H = \frac{g^{ij} p_i p_j}{2m} + V(\mathbf{x}) \frac{p_y^2}{2M^2}$
- $\dot{x}^i = \frac{g^{ij} p_j}{m}$
- $\dot{p}_i = -mV_{,i} \frac{p_y^2}{2M^2}$
- $\frac{M\dot{y}}{V(\mathbf{x})} = \frac{p_y}{M} = A$
- $\dot{p}_y = 0$

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Poisson Brackets

- $x^i, \; y, \; p_i, \; p_y$
- $\{x^i, p_i\} = \{y, p_y\} = 1$
- $\{x^i, p_y\} = \{y, p_i\} = 0$
- $\{x^i, x^j\} = \{y, y\} = 0$
- $\{x^i, y\} = 0$
- $\{p_i, p_j\} = \{p_y, p_y\} = 0$
- $\{p_i, p_y\} = 0$

Commutators

- $\hat{x}^i, \; \hat{y}, \; \hat{p}_i, \; \hat{p}_y$
- $[\hat{x}^i, \hat{p}_i] = [\hat{y}, \hat{p}_y] = i$
- $[\hat{x}^i, \hat{p}_y] = [\hat{y}, \hat{p}_i] = 0$
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Original Hamiltonian

$$\widehat{H} = \frac{1}{2m}\hat{p}_x^2 + V(\hat{x})$$

Original Schrödinger Equation

$$-\frac{1}{2m}\psi''(x) + V(x)\psi(x) = E\psi(x)$$

Lifted Hamiltonian

$$\widehat{H} = \frac{1}{2m}\hat{p}_x^2 + \frac{V(\hat{x})}{2M^2}\hat{p}_y^2$$

Lifted Schrödinger Equation

$$-\frac{1}{2m}\Psi_{,xx}(\boldsymbol{x}, y) - \frac{V(x)}{2M^2}\Psi_{,yy}(\boldsymbol{x}, y) = E\Psi(\boldsymbol{x}, y)$$

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Lifted Schrödinger Equation

$$-\frac{1}{2m}\Psi_{,xx}(\boldsymbol{x}, y) - \frac{V(x)}{2M^2}\Psi_{,yy}(\boldsymbol{x}, y) = E\Psi(\boldsymbol{x}, y)$$

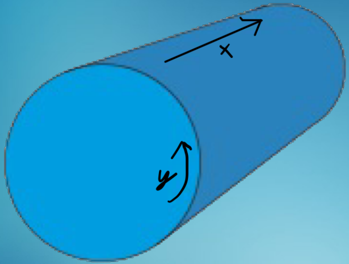
Separation of Variables

$$\Psi(x, y) = \psi(x)\chi(y)$$

$$\chi''(y) = -P^2\chi(y)$$

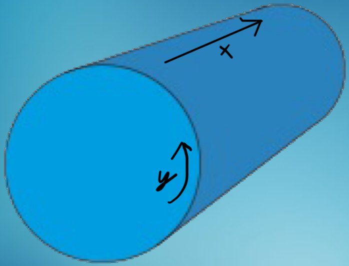
$$-\frac{1}{2m}\psi''(x) + \frac{P^2}{2M^2}V(x)\psi(x) = E\psi(x)$$

The Eisenhart Lift in Quantum Mechanics



- Compactify fictitious dimension y so that $0 \leq y < \ell$
- $P = \frac{2\pi k}{\ell}, \qquad k \in \mathbb{Z}$
- $-\frac{1}{2m}\psi''(x) + \frac{2\pi^2 k^2}{M^2 \ell^2} V(x)\psi(x) = E\psi(x)$

The Eisenhart Lift in Quantum Mechanics



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- $-\frac{1}{2m}\psi''(x) + k^2 V(x)\psi(x) = E\psi(x)$

Example: Simple Harmonic Oscillator

- Hamiltonian: $\widehat{H} = \frac{\hat{p}_x^2}{2m} + m\omega^2 \hat{x}^2 \frac{\hat{p}_y^2}{2M^2}$
- Energy Eigenstates: $\Psi(x, y) = \langle \boldsymbol{x} | k, n \rangle = \exp\left(\frac{2\pi iky}{\ell}\right) \psi_n\left(\frac{x}{k}\right)$
- Energy Eigenvalues: $k\omega\left(n + \frac{1}{2}\right)$
- Orthonormality: $\langle k', n' | k, n \rangle = \delta_{k'k}\, \delta_{n'n}$

The Eisenhart Lift for Classical Field Theory

KF, S. Karamitsos and A. Pilaftsis 2018 (arXiv:1806.02431)

Original Theory

- $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$
- $\Box\phi = -V'(\phi)$

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Lifted Theory

- $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\frac{M^4}{V(\phi)}\partial_\mu B^\mu\partial_\nu B^\nu$
- $\Box\phi = -\frac{1}{2}\left(\frac{M^2\partial_\nu B^\nu}{V(\phi)}\right)^2 V'(\phi)$
- $\partial_\mu\left(\frac{M^2\partial_\nu B^\nu}{V(\phi)}\right) = 0 \quad \Rightarrow \quad \frac{M^2\partial_\nu B^\nu}{V(\phi)} = A$

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Hamiltonian Formulation

Lifted Hamiltonian

$$\mathcal{H}_{\text{lift}} = \frac{\pi_\phi^2 + (\nabla\phi)^2}{2} + \frac{V(\phi)}{M^4}\pi_0^2 - \pi_0\partial_i B^i + \pi_i \dot{B}^i$$

Hamilton's Equations

$$\dot{\phi} = \pi_\phi \qquad \dot{\pi}_\phi = \nabla^2\phi - \frac{1}{2M^4}\pi_0^2 V'(\phi)$$

$$\pi_0 = M^4 \frac{\partial_\mu B^\mu}{V(\phi)} = M^2 A \qquad \partial_\mu \pi_0 = 0$$

$$\pi_i = 0$$

The Eisenhart Lift for Quantum Field Theory

Lifted Hamiltonian

$$\widehat{\mathcal{H}}_{\text{lift}} = \frac{\hat{\pi}_\phi^2 + (\nabla\hat{\phi})^2}{2} + \frac{m^2\hat{\phi}^2}{2M^4}\hat{\pi}_0^2 - \hat{\pi}_0\partial_i\widehat{B}^i + \hat{\pi}_i\dot{\widehat{B}}^i$$

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Creation and Annihilation Operators

$$\mathfrak{a}_{\mathbf{k}}, \mathfrak{a}_{\mathbf{k}}^\dagger = \int d^3\mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}} \left[\frac{\hat{\pi}_0(\mathbf{x})}{\sqrt{2}M^2}\sqrt{\frac{\omega_{\mathbf{k}}}{2}}\hat{\phi}(\mathbf{x}) \pm \frac{i}{\sqrt{2\omega_{\mathbf{k}}}}\hat{\pi}_\phi(\mathbf{x})\right]$$

Commutation Relations

- $[\mathfrak{a}_{\mathbf{k}}, \mathfrak{a}_{\mathbf{q}}^\dagger] = \frac{\hat{\pi}_0}{\sqrt{2}M^2}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{q})$
- $[\widehat{H}_{\text{lift}}, \mathfrak{a}_{\mathbf{k}}^\dagger] = \frac{\hat{\pi}_0}{\sqrt{2}M^2}\omega_{\mathbf{k}}\mathfrak{a}_{\mathbf{k}}^\dagger$
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Conserved Momentum

- $[\widehat{\mathcal{H}}_{\rm lift}, \hat{\pi}_0] = 0$
- $[\widehat{P}_i, \hat{\pi}_0] = 0$
- $\therefore\ \hat{\pi}_0$ conserved in time and space

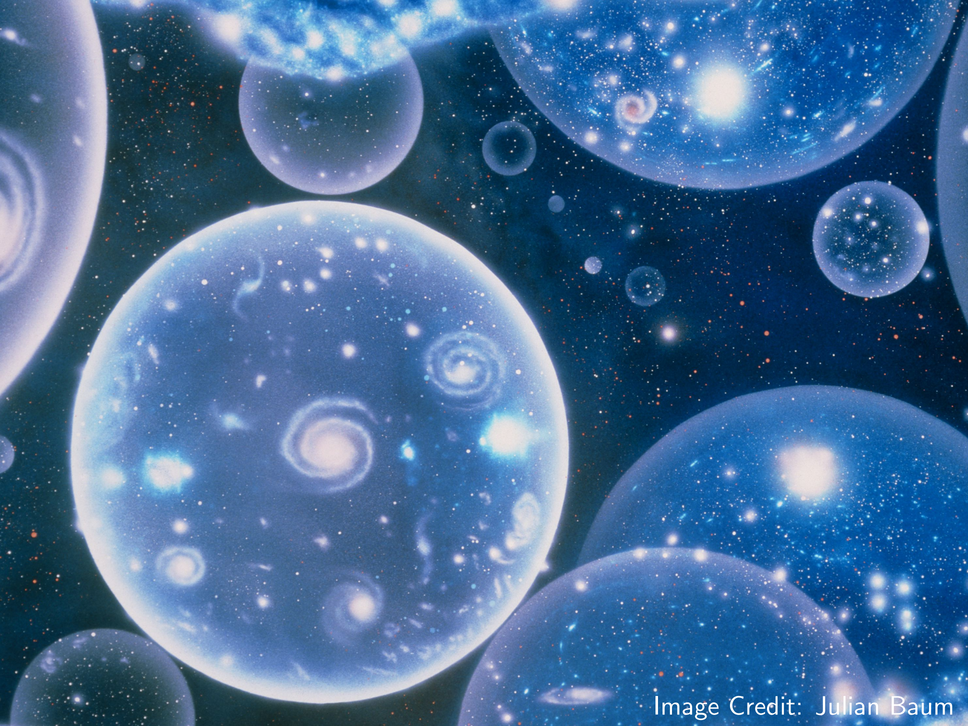

Image Credit: Julian Baum

Possible Applications

- Anthropic explanation of the cosmological constant problem
- Gauge hierarchy problem
- The initial conditions of inflation (see arXiv:1812.07095)

Image Credit: Julian Baum

Summary

- A conservative force can be described by the curvature of a higher dimension
- The potential term of a quantum field theory can be described as the kinetic term of a vector field
- Different values of the conjugate momentum π_0 lead to an ensemble of universes with different physics
- May have applications to cosmological constant, gauge hierarchy and measure problem