Leonardo Senatore (Stanford)

$\lambda\phi^4$ in de Sitter

with V. Gorbenko

1911.00022

Introduction

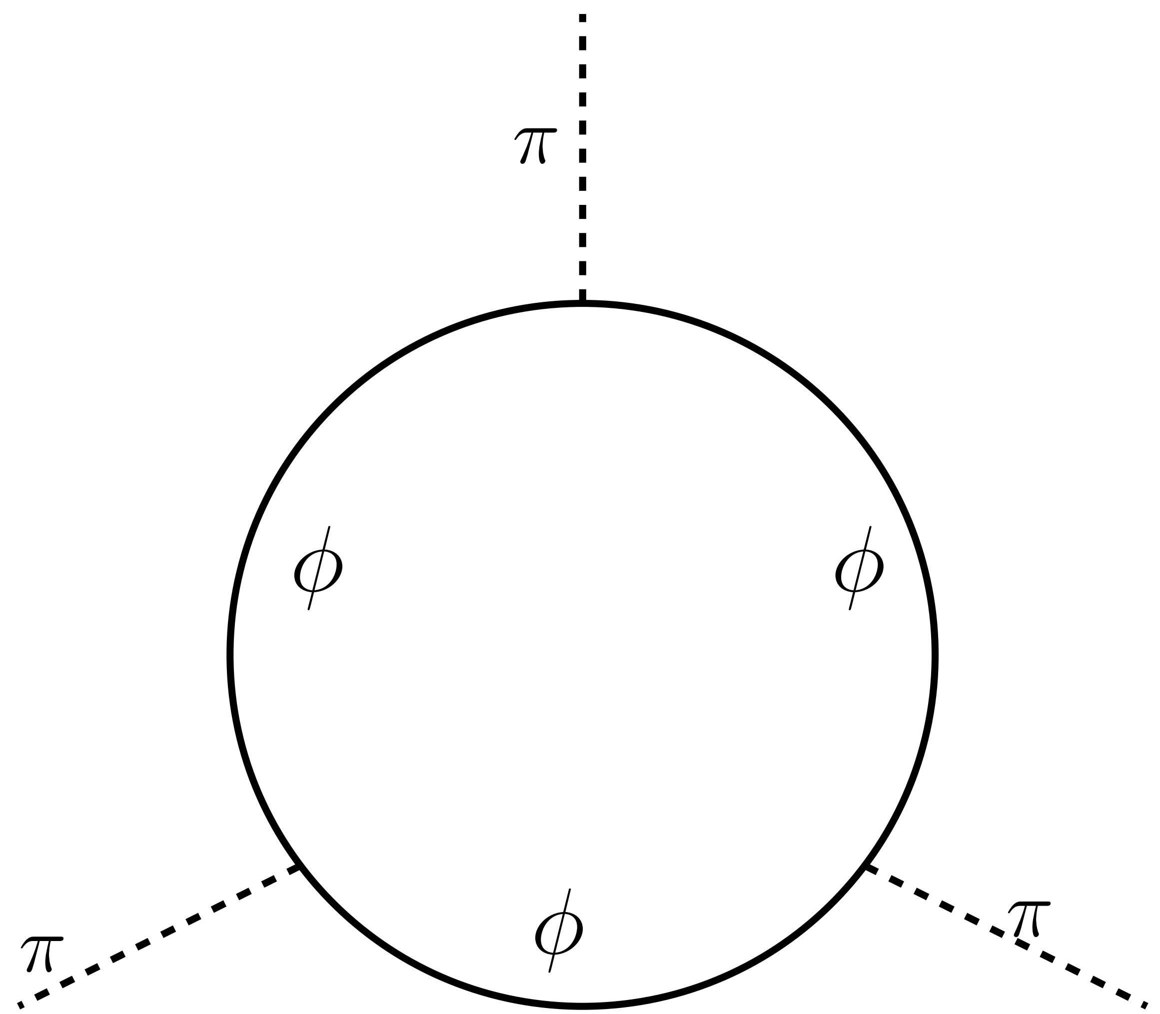
- Massless $\lambda\,\phi^4$ is IR-divergent in dS:
- Why do we care?
    - 1) This is somewhat embarrassing

Introduction

– Massless $\lambda\,\phi^4$ is IR-divergent in dS:

– Why do we care?

  – 2) It could have phenomenological consequences, for example to Black Holes from inflation, or to non-Gaussianities



Introduction

– Massless $\lambda\,\phi^4$ is IR-divergent in dS:

– Why do we care?

– 3) Since inflation is most probably the theory of the early universe, we should be able to understand its radiative corrections

- For single-field inflation, we have a satisfactory and complete understanding

$$\langle \zeta_k^2 \rangle \quad \supset \quad \log\left(\frac{H}{\mu}\right) \;, \quad Ht \;, \quad \log(kL)$$

Logarithmic Running
After S. Weinberg **2008** had begun the exploration,
finding $\log\,(k/\mu)$

- but not for non-derivatively coupled multifield

see KITP video of 2015 String Program

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Out-of-Horizon time-dependence:
absent, to all loops

- but not for non-derivatively coupled multifield

see KITP video of 2015 String Program

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$$\langle\zeta_k^2\rangle \quad \supset \quad \log\left(\frac{H}{\mu}\right) \ , \quad Ht \ , \quad \log(kL)$$

Logarithmic dependence on IR-cutoff of the universe: absent once define observable quantities.

- but not for non-derivatively coupled multifield

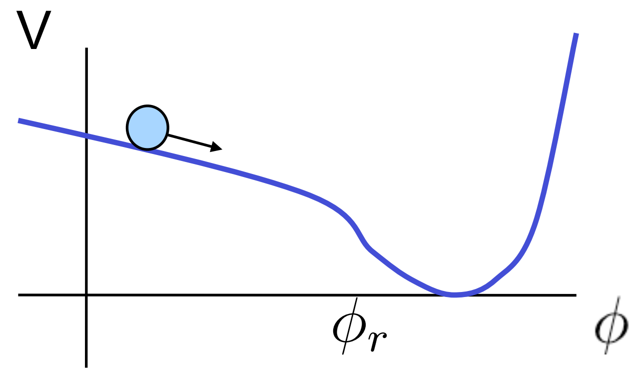
see KITP video of 2015 String Program

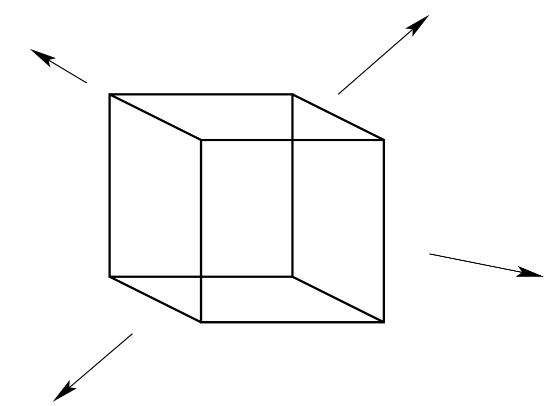
Introduction

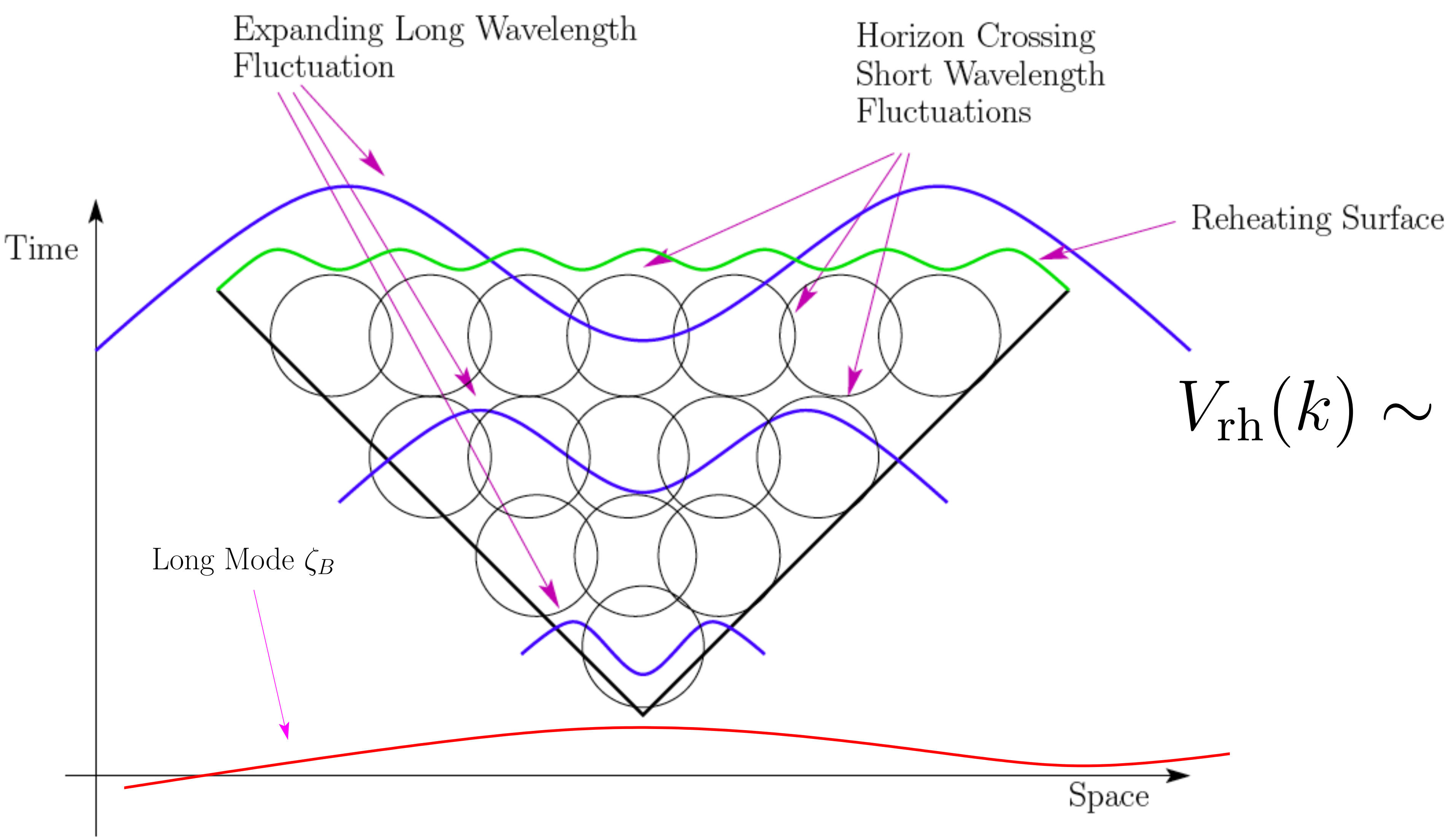
– Massless $\lambda\,\phi^4$ is IR-divergent in dS:

– Why do we care?

  – 4) Quantum-enhanced expansion and eternal inflation







$$V_{\rm rh}(k) \sim \left(1 + \langle \zeta^2 \rangle\right) \frac{e^{3N_e(k)}}{H^3}$$

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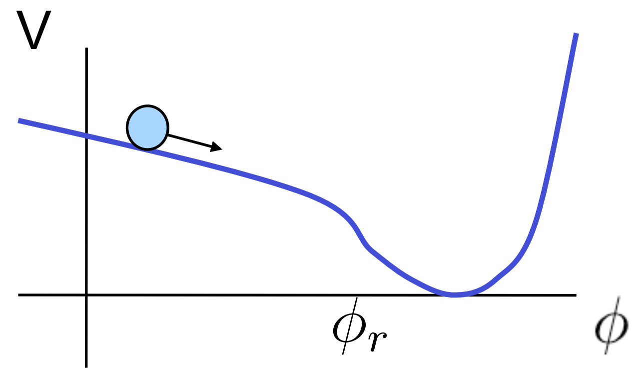

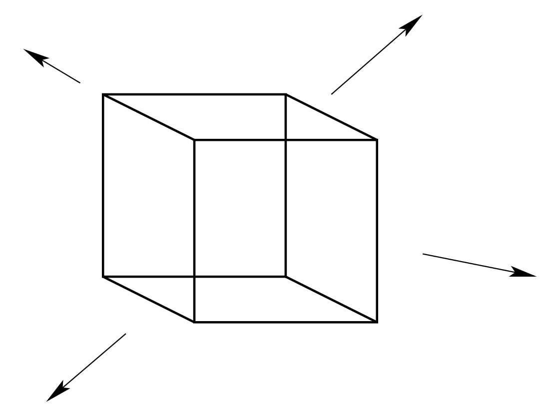

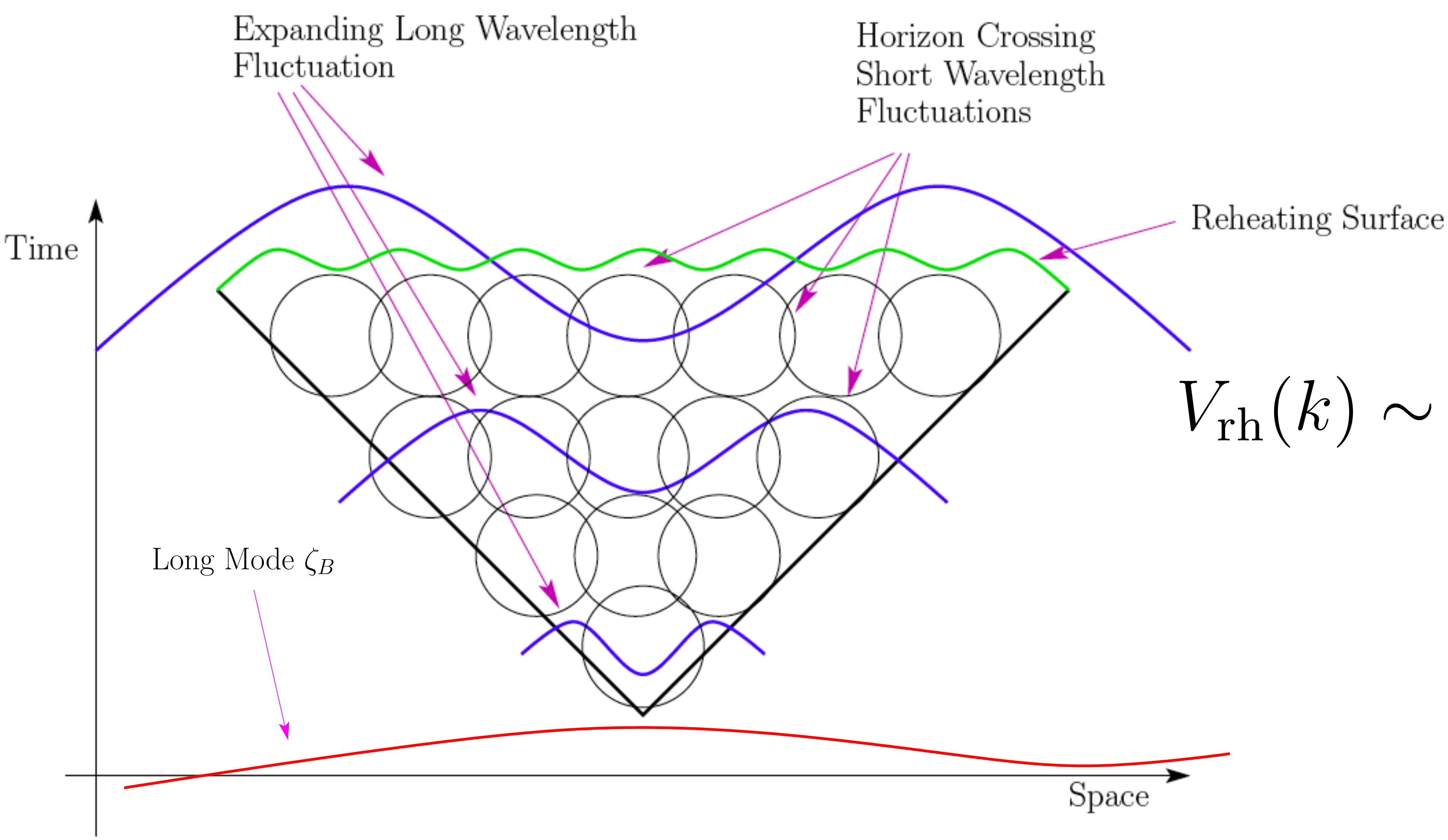

$$V_{\rm rh}(k) \sim \left(1 + \langle \zeta^2 \rangle\right) \frac{e^{3N_e(k)}}{H^3}$$

reheating volume
at classical level

Introduction

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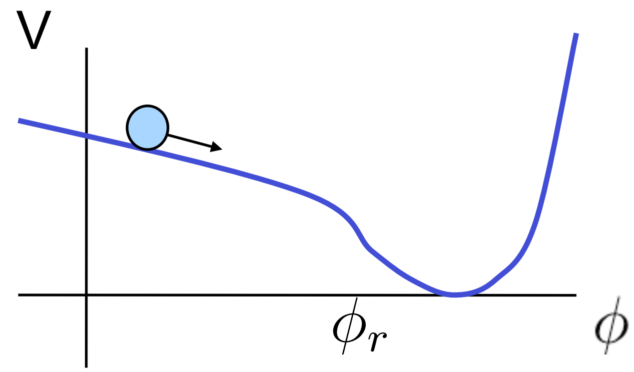
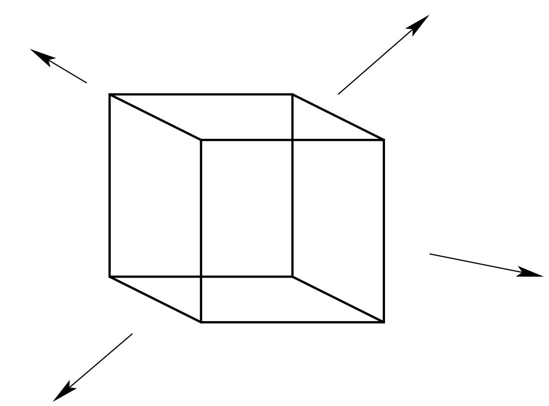
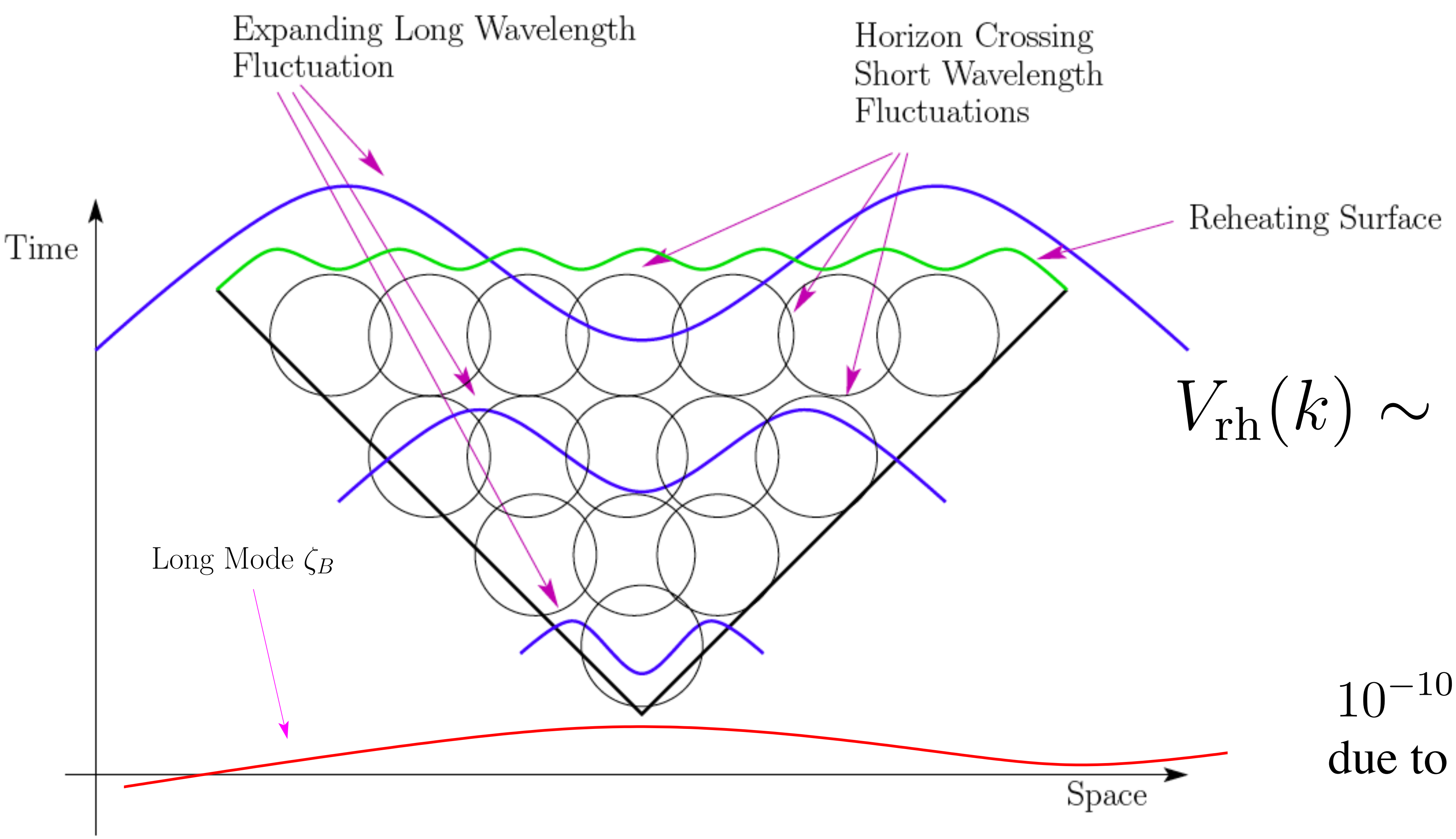






$$V_{\rm rh}(k) \sim \left(1 + \langle \zeta^2 \rangle\right) \frac{e^{3N_e(k)}}{H^3}$$

10^{-10} quantum enhancement
due to shorter modes

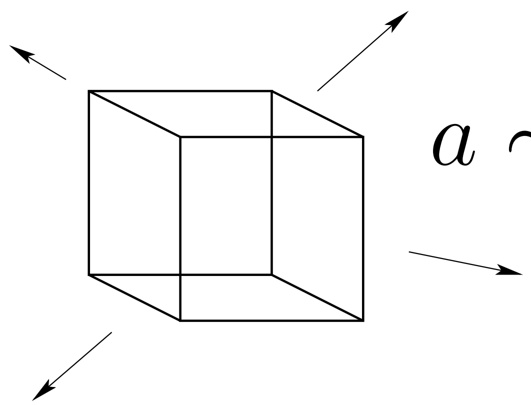
Introduction

– Massless $\lambda\,\phi^4$ is IR-divergent in dS:

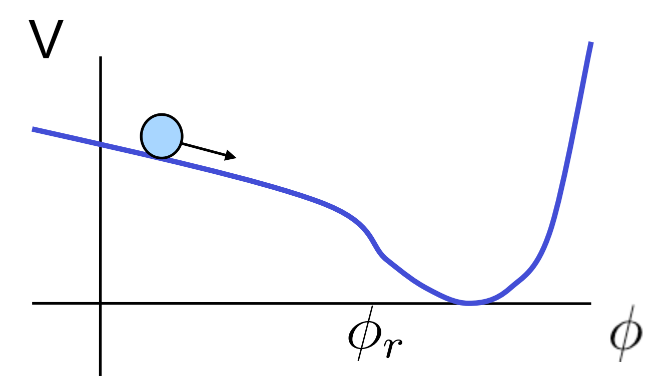
– Why do we care?

  – 4) Slow-Roll Eternal Inflation

    • This is a different application: let us elaborate



$$a \sim e^{Ht}$$



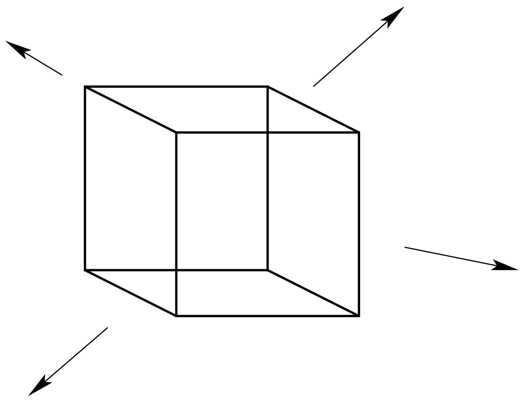
$$\zeta \sim H\frac{\delta\phi}{\dot\phi}$$

$$V_{\rm rh}(k) \sim \left(1 + \langle\zeta^2\rangle\right)\frac{e^{3N_e(k)}}{H^3}$$

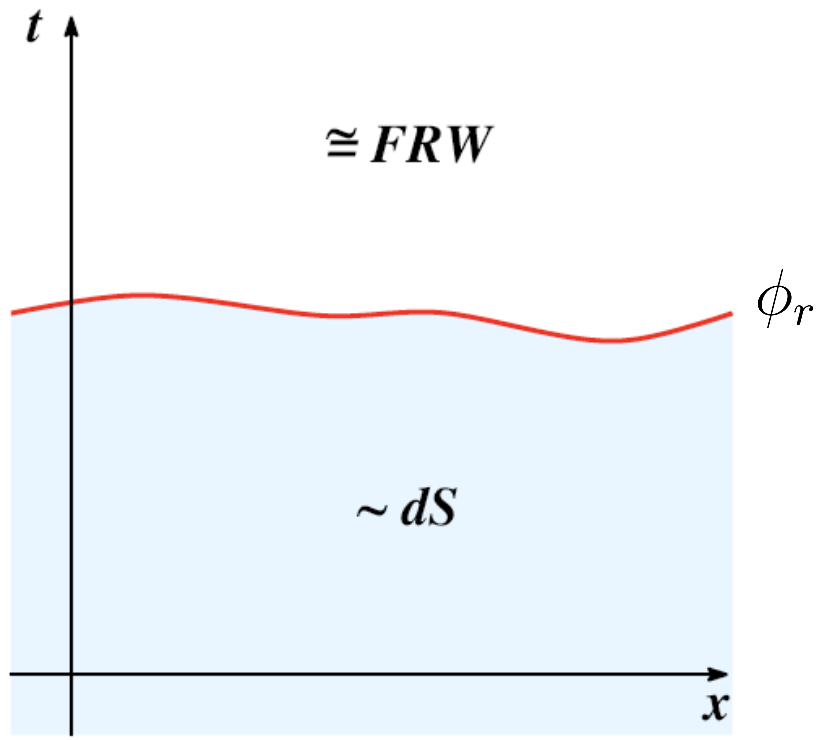
$$\sim 1$$

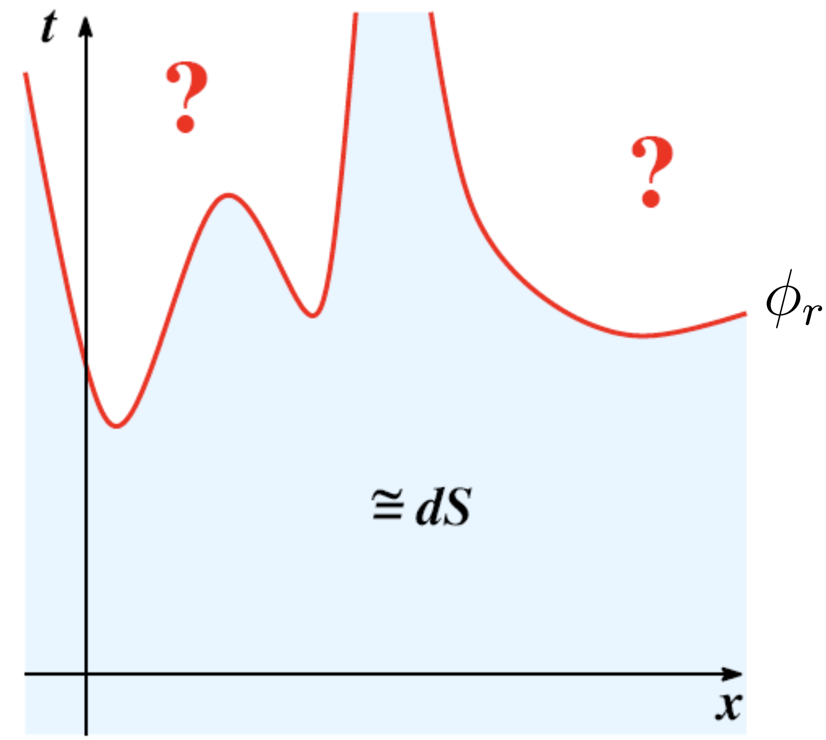
Slow-Roll Eternal Inflation

- The spacetime becomes stochastic

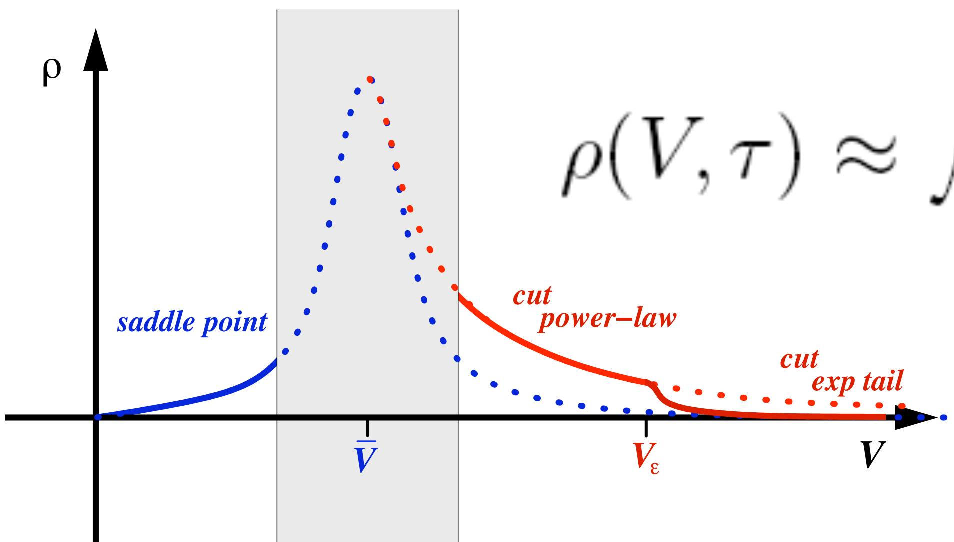


$$\Omega = \frac{\dot{\phi}^2}{H^4}$$





- Much more radical quantum effect on spacetime than the Black-hole evaporation
- Discovered in the 80's, we provided a first rigorous quantitative understanding



$$\rho(V,\tau) \approx \mathcal{N} e^{-\Omega\left[\frac{3N}{2}\left(1+\sqrt{1-\frac{1}{\Omega}}\right)-3N_c\right]^2}$$

With Creminelli, Dubovsky, Nicolis and Zaldarriaga **JHEP 2008**
With Dubovsky and Villadoro **JHEP2009; JHEP2012**

$$e^{3N_e} \to e^{\mathbf{6}N_e}$$

In particular: $P\left[V_{\rm finite} > e^{S_{\rm dS}/2}\right] \lesssim e^{-S_{\rm dS}}$

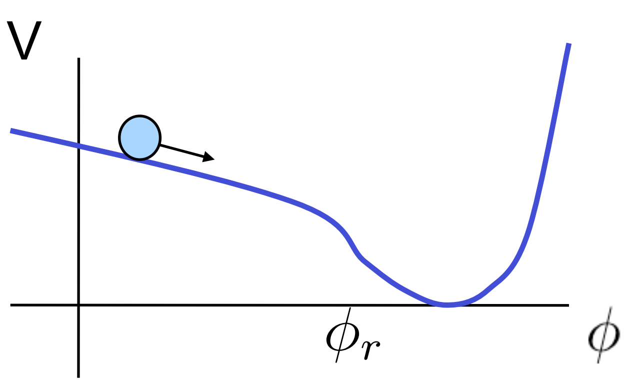
Slow-Roll Eternal Inflation

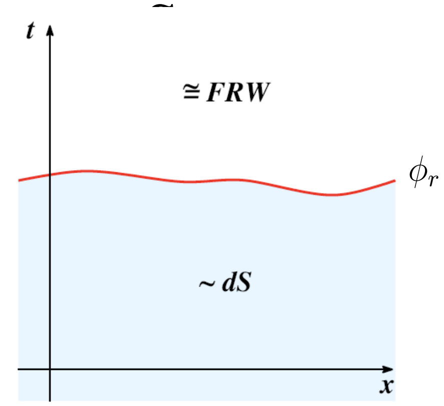
- Initial discovery and our quantitative understanding were based on a so-called Stochastic equation for inflationary fluctuations proposed and developed by Starobinsky in the 80's.

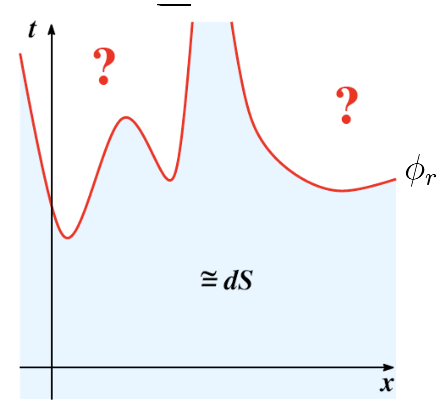
- Probability distribution of the quantum fluctuations satisfies a Fokker-Planck equation.

Starobinsky, **1980's**

$$\frac{\partial}{\partial t}P(\phi(\vec{x})) = H^2\frac{\partial^2}{\partial\phi(\vec{x})^2}P(\phi(\vec{x})) + \frac{\partial}{\partial\phi(\vec{x})}\left(V'(\phi(\vec{x}))P(\phi(\vec{x}))\right)$$







- This same equation is able to solve non-perturbatively $\lambda\phi^4$ in dS

- Lacking: a satisfactory derivation, an understanding of if this is a toy model or the leading expansion in `*something*', and, if it is the second, if it can be made a precise approach.

Summary of Introduction

- Both
  - Massless $\lambda\,\phi^4$ in dS
  - Slow-Roll eternal inflation
- are non-perturbative phenomena
- The Stochastic equation $\frac{\partial}{\partial t}P(\phi(\vec{x})) = H^2 \frac{\partial^2}{\partial\phi(\vec{x})^2}P(\phi(\vec{x})) + \frac{\partial}{\partial\phi(\vec{x})}\left(V'(\phi(\vec{x}))P(\phi(\vec{x}))\right)$
  - might provide a way to solve for them
  - lacks a systematic derivation and proof of accuracy

- We will prove that that equation is the leading-order truncation of a generalized equation, from which we can derive arbitrary accurate results.
  - Proving the existence slow-roll eternal inflation
  - Solving $\lambda\,\phi^4$ in rigid de Sitter

Let us start

Secular-divergencies

– Consider a free massive field in dS

$$\langle \phi_k^2 \rangle \sim e^{-3Ht} \left[H^{(1)}_{\nu=\sqrt{9/4-m^2/H^2}} \left(\frac{k}{aH} \right) \right]^2 \qquad a \sim e^{Ht}$$

– Assuming the mass is small:

$$\langle \phi_k^2 \rangle \to \frac{H^2}{k^3} \left(1 + \frac{m^2}{H^2} \log \left(\frac{k}{a(t)H} \right) \right)$$

– This perturbative expansion breaks at late times $\quad \frac{k}{a(t)H} \sim e^{-\frac{H^2}{m^2}}$

- This is the answer we would have gotten if we had treated the mass as a perturbation:

$$\ddot{\phi} + 3H\dot{\phi} + \frac{k^2}{a^2}\phi = m^2\phi \quad \Rightarrow \quad \phi^{(1)} \sim \int dt' \, \frac{1}{3H} m^2 \phi^{(0)}(t') \sim \phi^{(0)} \, \frac{m^2}{H^2} H \, t$$

Secular-divergencies

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Secular-divergencies

– Now, consider massless $\lambda\,\phi^4$

– Solving perturbatively

$$\ddot{\phi}_k + 3H\dot{\phi}_k + \frac{k^2}{a^2}\phi_k = \lambda[\phi^3]_k \quad \Rightarrow \quad \phi_k^{(1)} \sim \int dt' \, \frac{1}{3H}\lambda \left[\phi^{(0)3}\right]_k$$

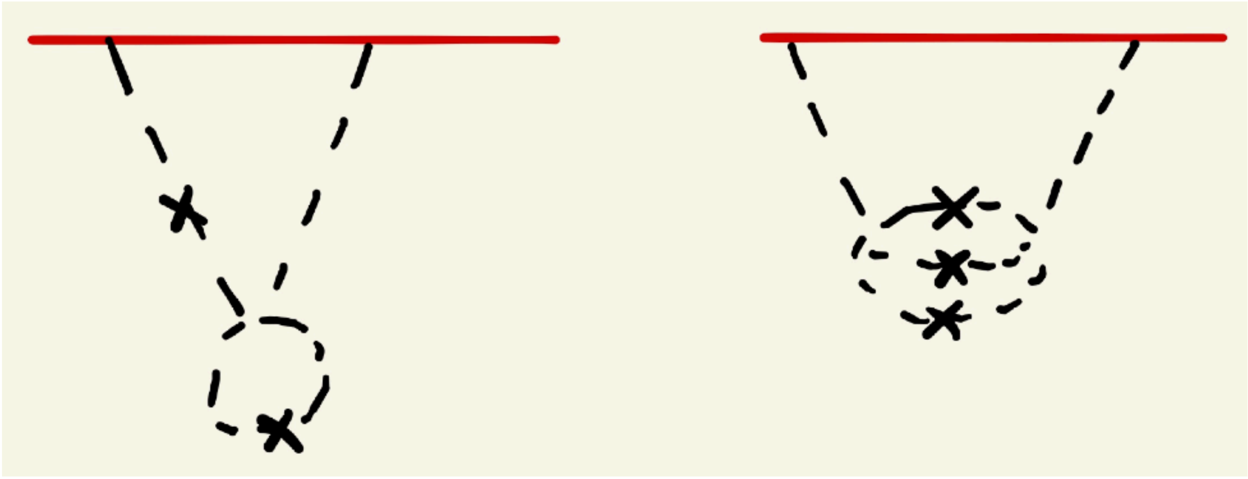
– At loop one level:

$$\langle \phi_k^2 \rangle \supset \langle \phi_k^{(0)} \phi_k^{(1)} \rangle \sim \lambda \int dt \; \langle (\phi^{(0)})^2 \rangle \langle \phi_k^{(0)} \phi_k^{(0)} \rangle \sim \lambda t$$

– secular divergent

- Not just some simple mean field term:

$$\langle \phi_k^2 \rangle \supset \langle \left(\phi_k^{(1)} \right)^2 \rangle \sim \lambda^2 \left(\int dt \; d^3q \right)^2 \langle \phi_q^{(0)} \phi_q^{(0)} \phi_q^{(0)} \; \phi_q^{(0)} \phi_q^{(0)} \phi_q^{(0)} \rangle \sim \lambda^2 t^2$$



Secular-divergencies

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– secular divergent

- Not just some simple mean field term:

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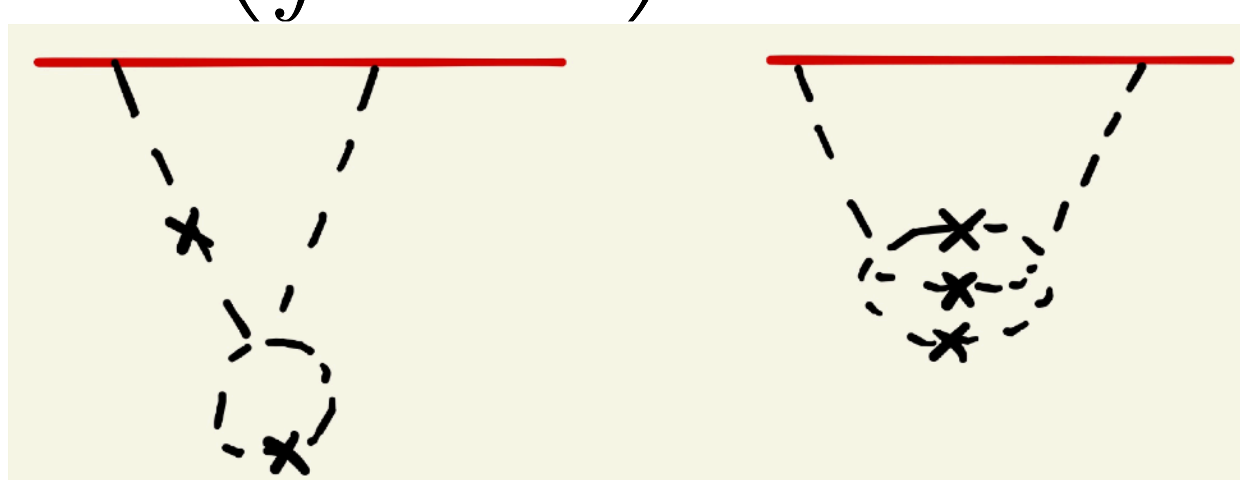
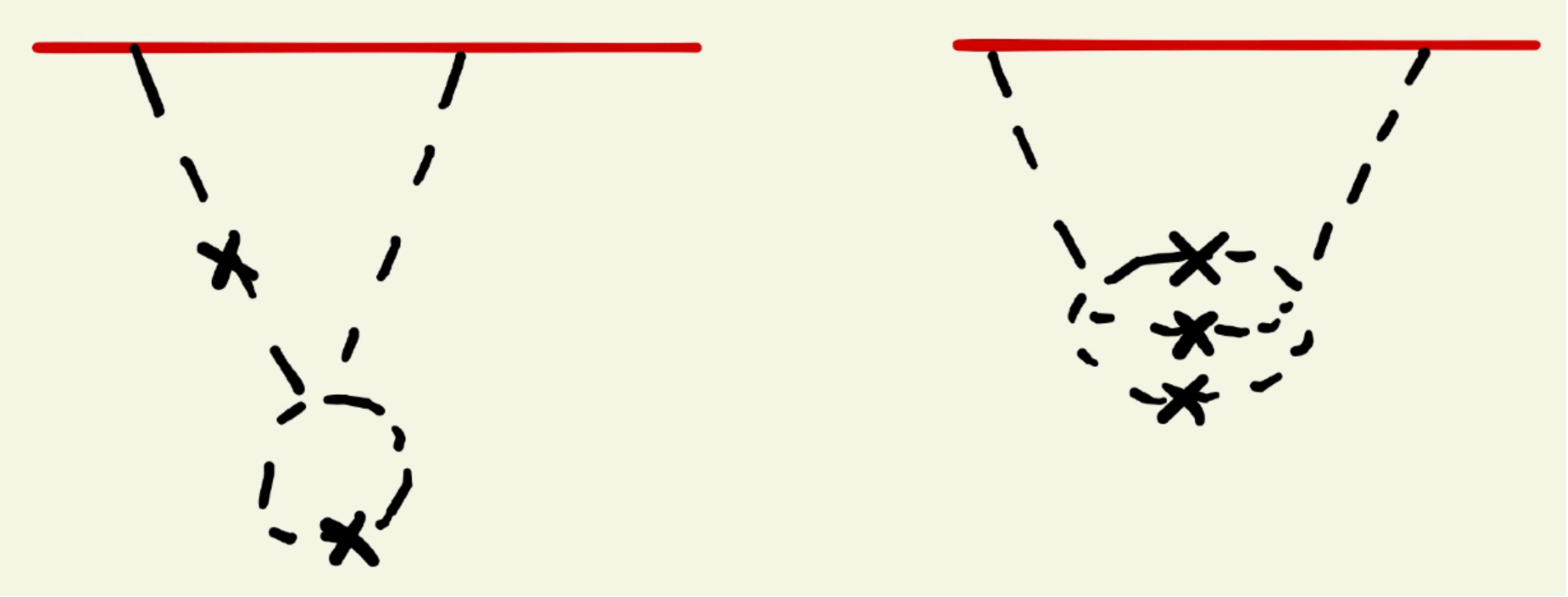


Secular-divergencies

–Rigorous calculation (see for example Burgess et al. **0912**)

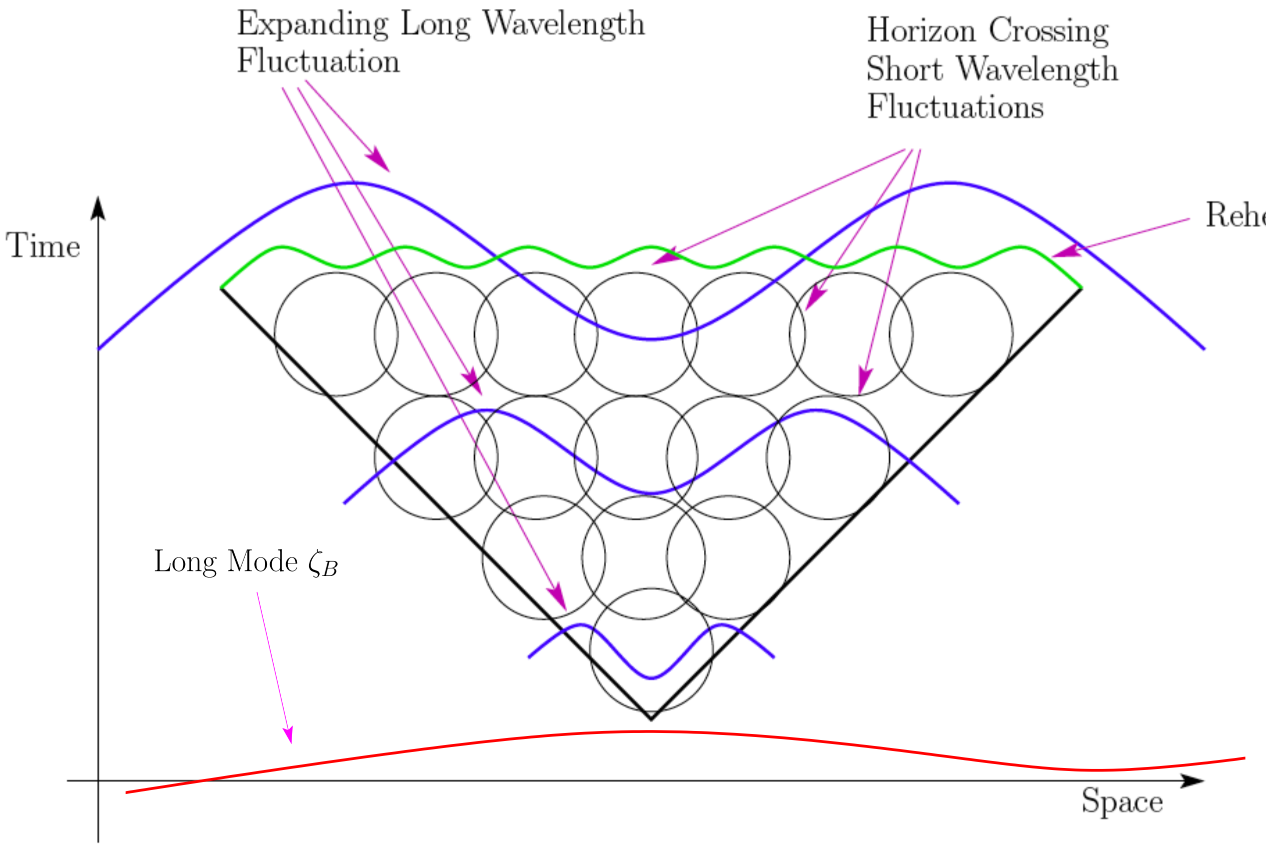
$$\langle \mathcal{O}(t) \rangle = \left\langle \mathrm{in} \middle| \left[\overline{T} \exp\left(i \int_{t_{\mathrm{in}}}^{t} \mathrm{d}t' \, \mathcal{H}(t') \right) \right] \mathcal{O}(t) \left[T \exp\left(-i \int_{t_{\mathrm{in}}}^{t} \mathrm{d}t' \, \mathcal{H}(t') \right) \right] \middle| \mathrm{in} \right\rangle ,$$

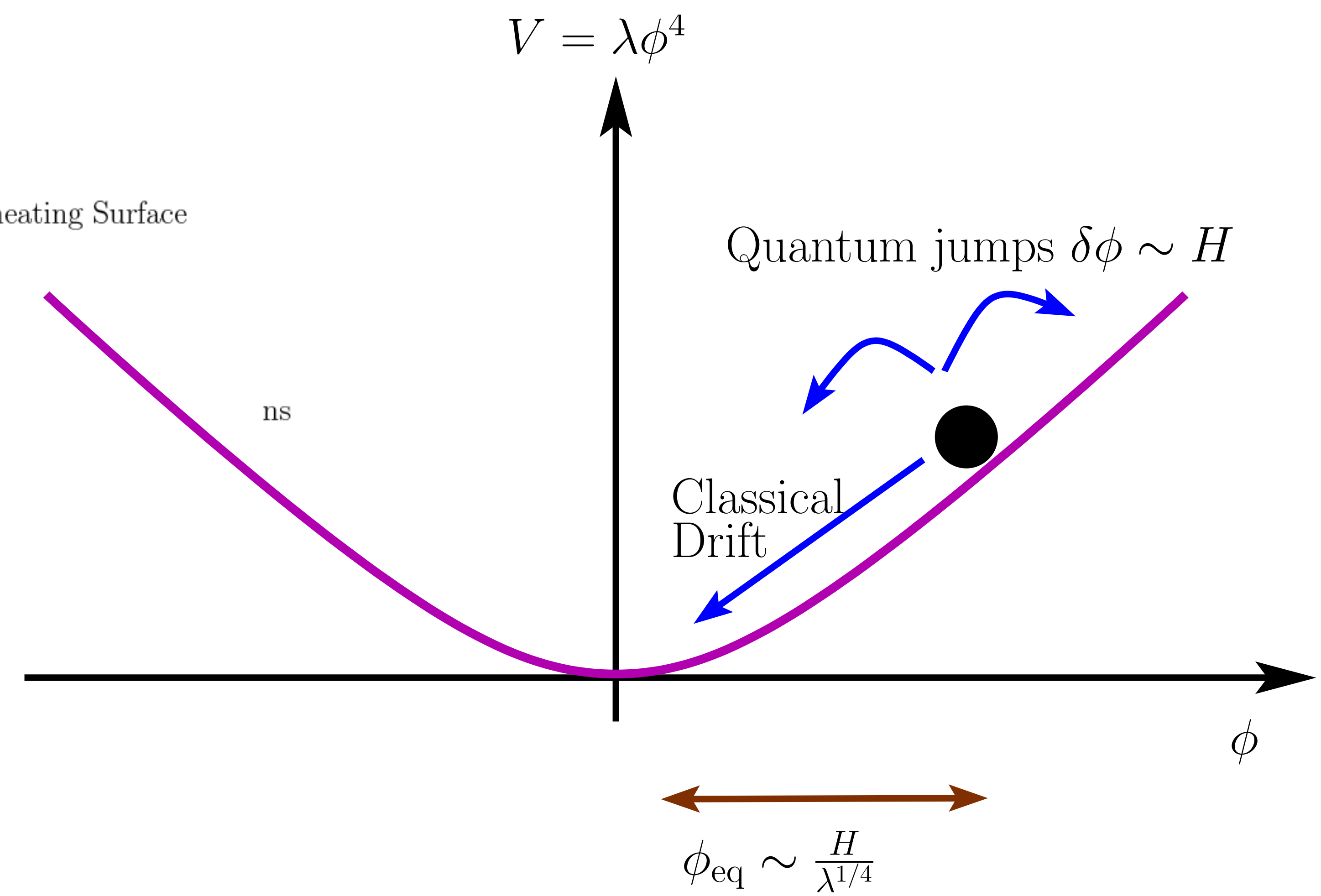
$$\Rightarrow \quad \langle \phi_k^2 \rangle \sim \lambda \log\left(\frac{k}{a(t) H} \right) \qquad\qquad a \sim e^{Ht}$$



Intuition

– What is going on?





– there is an $\mathcal{O}(1)$ change to perturbative evolution: $\Rightarrow$ solution is non perturbative

– We expect a diffusion-upwards, stopped by a drift-downwards, reaching a sort of equilibrium distribution with

$$\text{Energy} \sim H \quad \Rightarrow \quad V(\phi) = \lambda\phi^4 \sim H^4 \quad \Rightarrow \phi \sim \frac{H}{\lambda^{1/4}}$$

– How to obtain a rigorous calculation? with arbitrary precision?

A Generalized Stochastic Approach

– We are going to define a rigorous formalism to solve the problem, that, at zeroth order in all the expansion parameters we will identify and introduce, reduces to the remarkable Stochastic approach of Starobinsky.

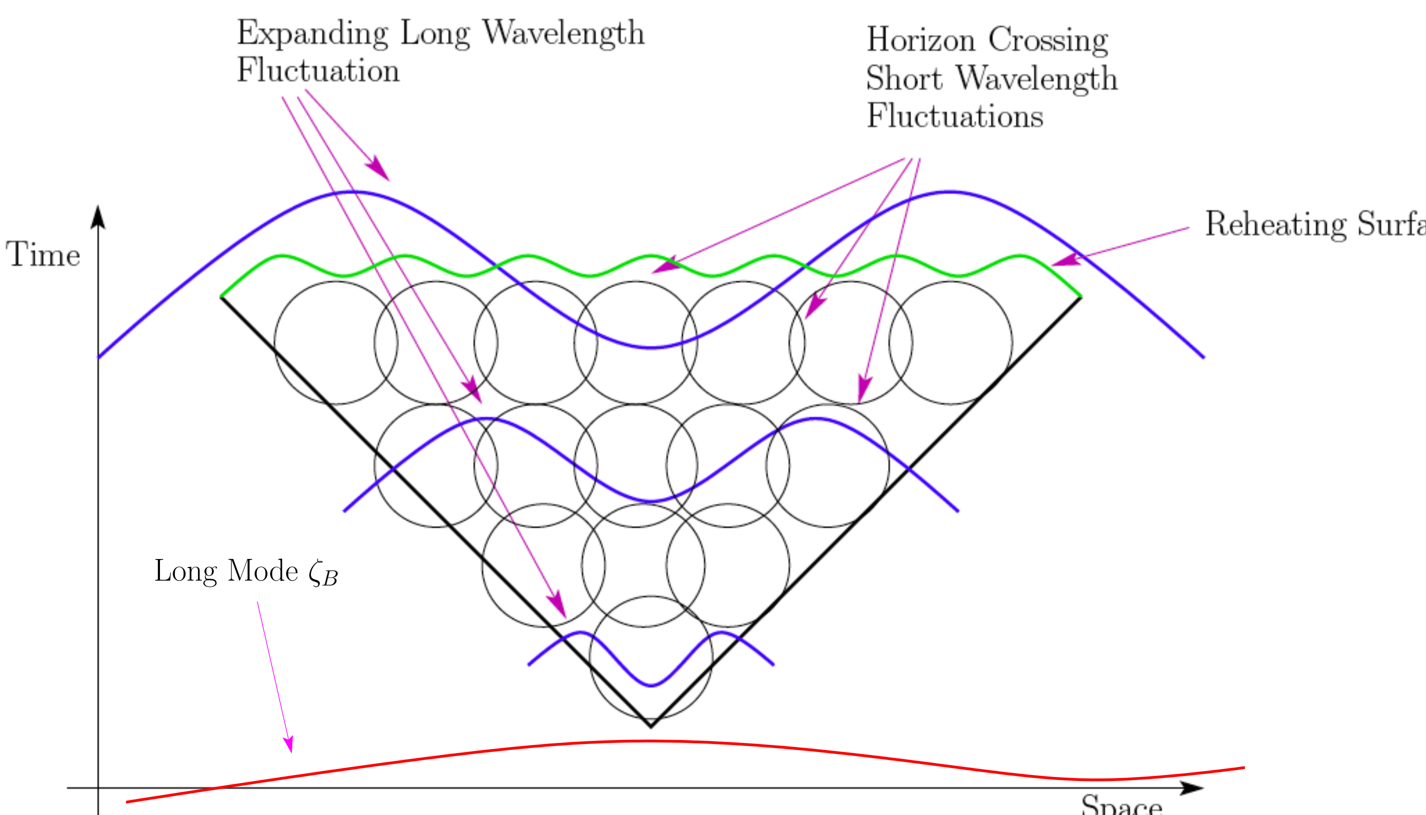
- Two crucial simplifications, around which we will expand, that allow us to solve a non-perturbative quantum problem in rigid curved spacetime.

A Generalized Stochastic Approach

– Two crucial expansion parameters:

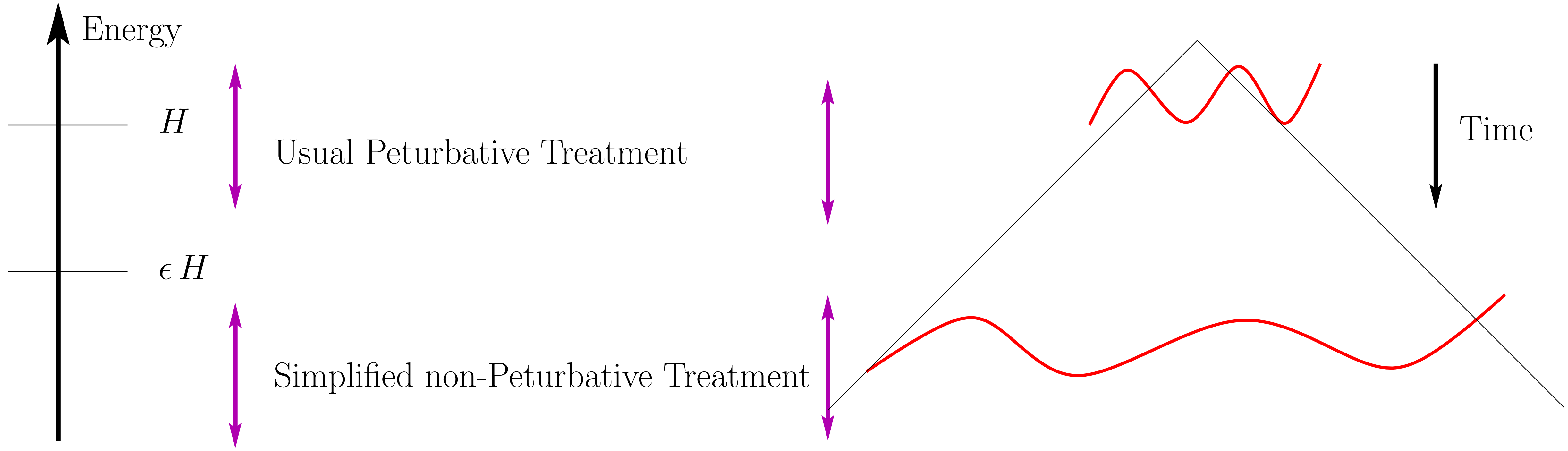
– (1): outside of horizon, gradients are negligible. We can expand around an exactly local-in-space evolution.

– (2): perturbativity of coupling constant: $\lambda \ll 1 \quad \Rightarrow \quad \sqrt{\lambda} \ll 1$



A Generalized Stochastic Approach

– Main idea:



– separate long and short modes at an *artificial* fixed physical scale:

- in terms of wavenumbers it is a time-dependent scale $k = \Lambda(t) = \epsilon a(t) H \ , \quad \epsilon \ll 1$
- modes move from \`short' to \`long'

– for short modes: use usual quantum perturbation theory: $\lambda t \sim \lambda \log \epsilon \ll 1$

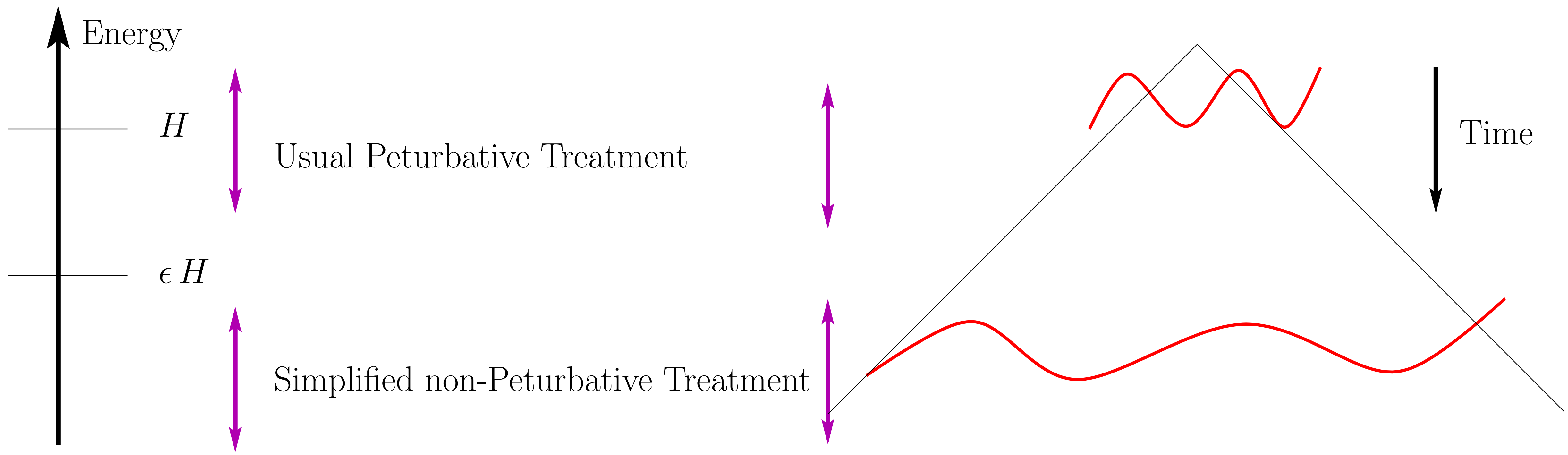
$$\langle \mathcal{O}(t) \rangle = \left\langle \mathrm{in} \left| \left[\overline{T} \exp\left(i \int_{t_{\mathrm{in}}}^{t} \mathrm{d}t' \, \mathcal{H}(t') \right) \right] \mathcal{O}(t) \left[T \exp\left(-i \int_{t_{\mathrm{in}}}^{t} \mathrm{d}t' \, \mathcal{H}(t') \right) \right] \right| \mathrm{in} \right\rangle ,$$

– for long modes, use the helps of the two expansion (1,2) above: expansion in

$$\epsilon \ll 1 \ , \quad \lambda \ll 1 \ ,$$

A Generalized Stochastic Approach

– Main idea:



– With time, modes pass from `short' to `long' regime.

– Can choose: $\sqrt{\lambda}\log\epsilon \ll 1 \quad \Rightarrow \quad e^{-\frac{1}{\sqrt{\lambda}}} \ll \epsilon \ll 1$

– $\Longrightarrow$ gradient and quantum corrections can be made much smaller than $\sqrt{\lambda}$ corrections

What we wish to compute

– We wish to compute $\langle\phi(x_1)\ldots\phi(x_n)\rangle$

– Given by:

$$\langle\phi(x_1)\ldots\phi(x_n)\rangle = \int \mathcal{D}\phi\ \Psi^\star[\phi]\Psi[\phi]\ \phi(x_1)\ldots\phi(x_n) = \int \mathcal{D}\phi\ P[\phi]\ \phi(x_1)\ldots\phi(x_n)$$

– with $P[\phi] = \Psi^\star[\phi]\Psi[\phi]$

– Formally, we could do the integral over the intermediate points:

$$\langle\phi(x_1)\ldots\phi(x_n)\rangle = \int d\phi_1\ldots d\phi_n\ P(\phi_1,\ldots,\phi_n)\ \phi_1\ldots\phi_n$$

– where

$$P(\phi_1,\ldots,\phi_n) = \int \mathcal{D}\phi\ \delta^{(1)}(\phi_1-\phi(x_1))\ldots\delta^{(1)}(\phi_n-\phi(x_n))\ P[\phi]$$

– hard to compute $P[\phi]$, and then to make such functional integral.

– Our strategy: find an equation that is satisfied by $P(\phi_1,\ldots,\phi_n)$

Solving for the field density

– Let us start with $P[\phi] = \Psi^\star[\phi]\Psi[\phi]$, which satisfies the following equation:

$$\frac{\partial P[\phi,t]}{\partial t} = \frac{\delta}{\delta\phi(\vec{x})}\left(\Psi[\phi,t]^*\frac{\delta}{\delta\phi(\vec{x})}\Psi[\phi,t] - \Psi[\phi,t]\frac{\delta}{\delta\phi(\vec{x})}\Psi^*[\phi,t]\right)$$

– which is functional and not even closed.

– But we actually can compute the wavefunction in dS, in some sense.

  – This will allow us to manipulate this equation.

Solving for the wavefunction

– Relevant literature has already emphasized how to compute the wavefunction in dS. For example Arkani-Hamed et al. ..., **2017, 2018, ...** For $\lambda\phi^4$: Anninos, Anous, Freedman, Kostantinidis, **2015**

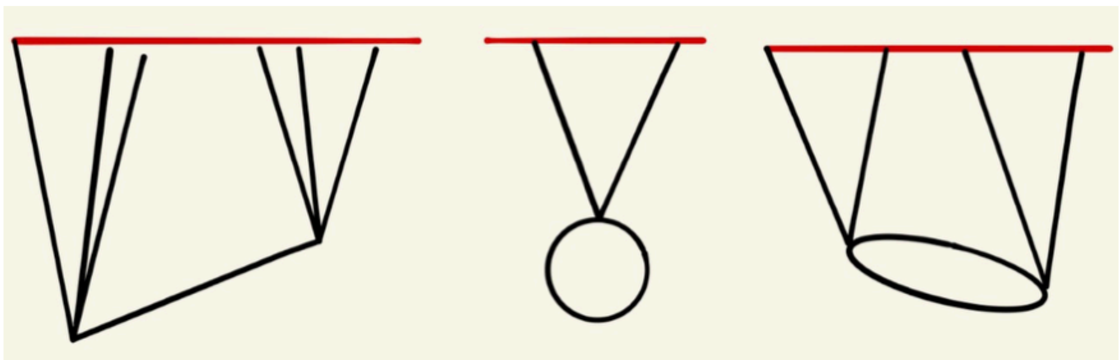
$$i\frac{\partial}{\partial t}\Psi[\phi,t]=\mathcal{H}\left[-i\frac{\delta}{\delta\phi},\phi,t\right]\Psi[\phi,t]\ ,\Rightarrow\quad \Psi_{BD}[\phi,\eta]=\int\mathcal{D}\varphi\ e^{iS[\varphi]}\ ,$$

– Perturbative structure is extremely different than for correlation functions, because of the different boundary conditions the propagators have. Two propagators:

– Bulk-to-Bulk: $G(\eta_1,\eta_2,k;\eta)$

– Bulk-to-Boundary (the `transfer function'): $K(\eta_1,k;\eta)$

– Both propagators are regular for $k\rightarrow 0$, so there are no IR-divergencies (of course, they come back once one tries to compute correlation functions of ϕ)



Perturbativity of Wavefunction

$$\log\Psi_{BD}[\phi,\eta]\sim\sum_L\int\left\{\frac{\phi(\vec{k})^2}{H^2}\left[\frac{i\tilde{m}^2}{\eta^3}+\frac{ik^2}{\eta}+\sum k_i^3\left(\lambda\log\left(-k_j\eta\right)\right)^L\right]\right.$$
$$+\lambda\frac{\phi(\vec{k})^4}{H^4}\left[\frac{i}{\eta^3}+(1+i)\sum k_i^3\log\left(-k_i\eta\right)\left(\lambda\log\left(-k_j\eta\right)\right)^L\right]$$
$$+\lambda^2\frac{\phi(\vec{k})^6}{H^6}\left[\frac{i}{\eta^3}+(1+i)\sum k_i^3\left(\log\left(-k_i\eta\right)\right)^2\left(\lambda\log\left(-k_j\eta\right)\right)^L\right]+$$
$$\left.+\ldots+\lambda^{\frac{E-2}{2}}\frac{\phi(\vec{k})^E}{H^E}\left[\frac{i}{\eta^3}+(1+i)\sum k_i^3\left(\log\left(-k_i\eta\right)\right)^{\frac{E-2}{2}}\left(\lambda\log\left(-k_j\eta\right)\right)^L\right]+\ldots\right\}$$

– Inspection of diagrams allow to prove the form upstairs, and that, upon *assuming* that

$$\phi\sim\frac{1}{\lambda^{1/4}}$$

– all terms are hierarchically organized in $\sqrt{\lambda}$

– We can perturbatively compute: $\dfrac{\delta}{\delta\phi(\vec{x})}\Psi[\phi]=\Pi[\phi(\vec{x})]\Psi[\phi]$

– But we cannot compute correlation functions with it

Solving for the field density

– Back to $P[\phi] = \Psi^{\star}[\phi]\Psi[\phi]$, which satisfies:

$$\frac{\partial P[\phi,t]}{\partial t} = \frac{\delta}{\delta\phi(\vec{x})}\left(\Psi[\phi,t]^{*}\frac{\delta}{\delta\phi(\vec{x})}\Psi[\phi,t] - \Psi[\phi,t]\frac{\delta}{\delta\phi(\vec{x})}\Psi^{*}[\phi,t]\right)$$

– we just established that we can compute $\frac{\delta}{\delta\phi(\vec{x})}\Psi[\phi] = \Pi[\phi(\vec{x})]\Psi[\phi]$

– let us set up to compute $P(\phi_1,\ldots,\phi_n)$

– and then $\langle\phi\ldots\phi\rangle$

Our Strategy

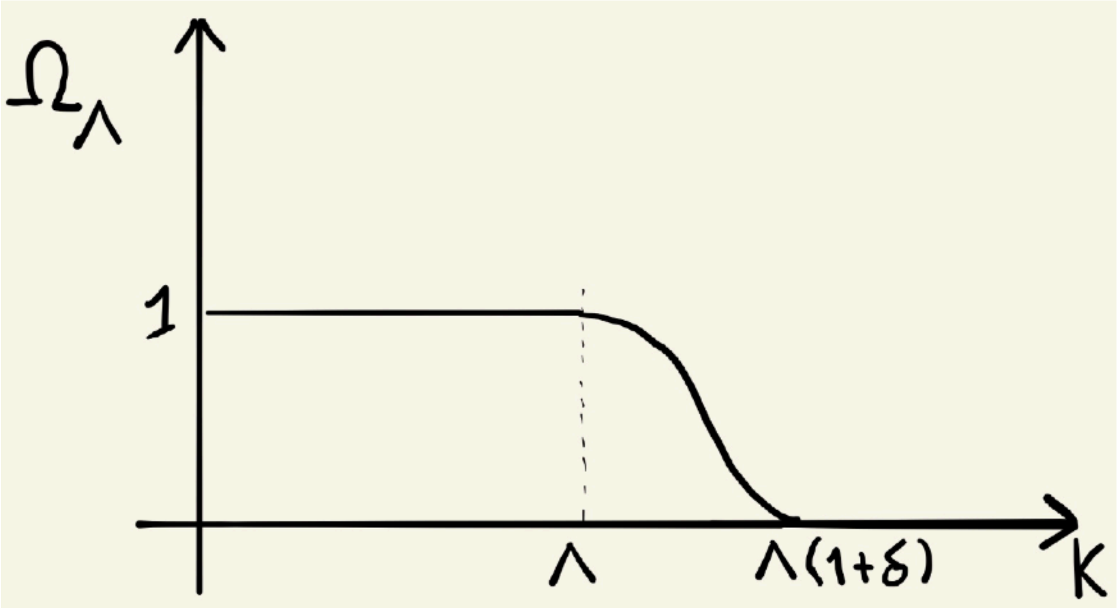
– Separate treatment for long and short modes: *short are perturbative, long are local*.

$$\phi(\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, \Omega_{\Lambda(t)}(k)\;\; e^{i\vec{k}\cdot\vec{x}}\phi(\vec{k}) + \int \frac{d^3k}{(2\pi)^3}\;\left(1-\Omega_{\Lambda(t)}(k)\right)\; e^{i\vec{k}\cdot\vec{x}}\phi(\vec{k}) \equiv \phi_\ell(\vec{x}) + \phi_s(\vec{x})\;,$$

– with

$$\Omega_{\Lambda(t)}(k) = \begin{cases} 1 & \text{for } k \leq \Lambda(t), \\ 0 & \text{for } \; k \geq (1+\delta)\Lambda(t)\,. \end{cases}$$

– smooth and wide enough $\Lambda(t) = \varepsilon a(t) H$



$$\frac{e^{-\frac{1}{\sqrt{\lambda}}}}{\epsilon} \ll \delta \ll \sqrt{\lambda},$$

– Effective long Density Function

$$P_\ell[\phi_\ell, t] = \int \mathcal{D}\phi\;\delta\left[\phi_\ell(\vec{x}) - \int^{\Lambda(t)} d^3k\; e^{i\vec{k}\cdot\vec{x}}\phi(\vec{k})\right]\; P[\phi, t]$$

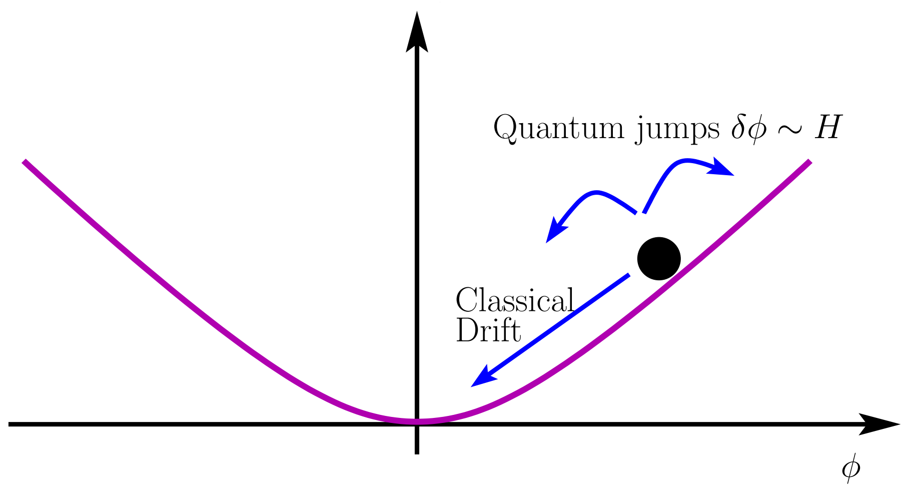
Effective Quasi-Probability for long modes

– Given:

$$P_\ell[\phi_\ell, t] = \int \mathcal{D}\phi \; \delta\left[\phi_\ell(\vec{x}) - \int^{\Lambda(t)} d^3k \; e^{i\vec{k}\cdot\vec{x}}\phi(\vec{k})\right] \; P[\phi, t]$$

– Find the effective time-evolution for the probability of the long modes

$$\frac{\partial P_\ell[\phi_\ell]}{\partial t} = \text{Drift} + \text{Diff}.$$



– Drift:

$$\text{Drift} = \int \mathcal{D}\phi \; \delta\left[\phi_\ell(\vec{x}) - \int^{\Lambda(t)} d^3k \; e^{i\vec{k}\cdot\vec{x}}\phi(\vec{k})\right] \; \frac{\partial P[\phi, t]}{\partial t}$$

$$= \int \mathcal{D}\phi \; \delta\left[\phi_\ell(\vec{x}) - \int^{\Lambda(t)} d^3k \; e^{i\vec{k}\cdot\vec{x}}\phi(\vec{k})\right] \; \frac{\delta}{\delta\phi(\vec{x})}\left(\Psi[\phi]^* \frac{\delta}{\delta\phi(\vec{x})}\Psi[\phi] - \Psi[\phi]\frac{\delta}{\delta\phi}\Psi^*[\phi]\right)$$

Effective Quasi-Probability for long modes

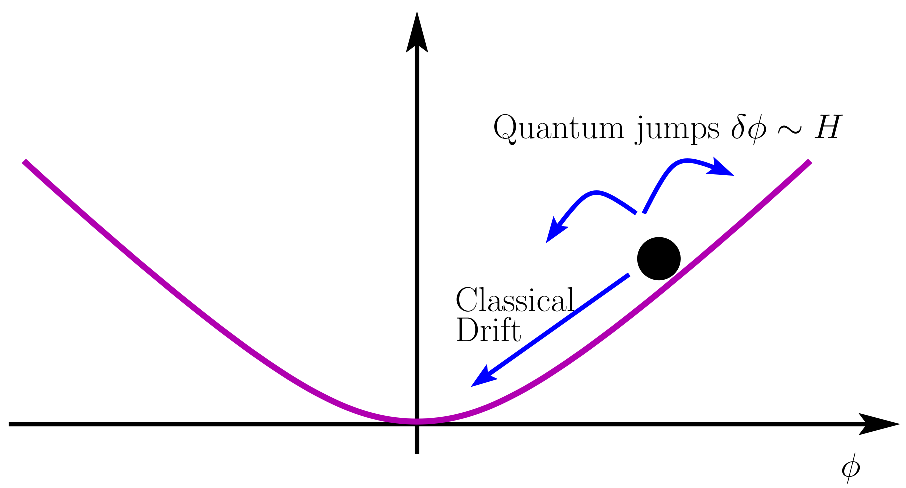
–Given:

$$P_\ell[\phi_\ell, t] = \int \mathcal{D}\phi\, \delta\left[\phi_\ell(\vec{x}) - \int^{\Lambda(t)} d^3k\; e^{i\vec{k}\cdot\vec{x}}\phi(\vec{k})\right] P[\phi, t]$$

Keep long modes fixed

– Find the effective time-evolution for the probability of the long modes

$$\frac{\partial P_\ell[\phi_\ell]}{\partial t} = \text{Drift} + \text{Diff}.$$



–Drift:

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$$= \int \mathcal{D}\phi\, \delta\left[\phi_\ell(\vec{x}) - \int^{\Lambda(t)} d^3k\; e^{i\vec{k}\cdot\vec{x}}\phi(\vec{k})\right] \frac{\delta}{\delta\phi(\vec{x})}\left(\Psi[\phi]^* \frac{\delta}{\delta\phi(\vec{x})}\Psi[\phi] - \Psi[\phi]\frac{\delta}{\delta\phi}\Psi^*[\phi]\right)$$

Effective Quasi-Probability for long modes

– Given:

$$P_\ell[\phi_\ell, t] = \int \mathcal{D}\phi\ \delta\left[\phi_\ell(\vec{x}) - \int^{\Lambda(t)} d^3k\ e^{i\vec{k}\cdot\vec{x}}\phi(\vec{k})\right]\ P[\phi, t]$$

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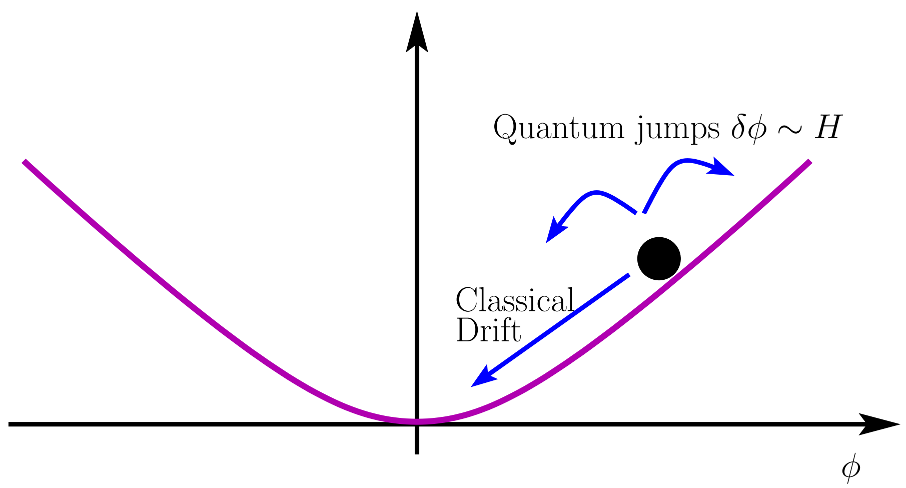

– Drift:

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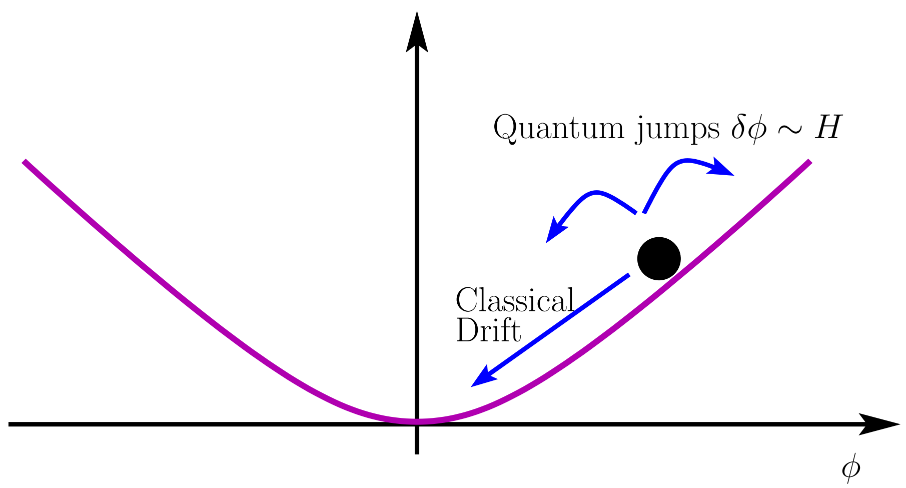
Effective Quasi-Probability for long modes

– Given:

$$P_\ell[\phi_\ell, t] = \int \mathcal{D}\phi \; \delta\left[\phi_\ell(\vec{x}) - \int^{\Lambda(t)} d^3k \; e^{i\vec{k}\cdot\vec{x}} \phi(\vec{k})\right] \; P[\phi, t]$$

– Find the effective time-evolution for the probability of the long modes

$$\frac{\partial P_\ell[\phi_\ell]}{\partial t} = \text{Drift} + \text{Diff}.$$



– Drift:

$$\text{Drift} = \int \mathcal{D}\phi \; \delta\left[\phi_\ell(\vec{x}) - \int^{\Lambda(t)} d^3k \; e^{i\vec{k}\cdot\vec{x}} \phi(\vec{k})\right] \; \frac{\partial P[\phi, t]}{\partial t}$$

$$= \int \mathcal{D}\phi \; \delta\left[\phi_\ell(\vec{x}) - \int^{\Lambda(t)} d^3k \; e^{i\vec{k}\cdot\vec{x}} \phi(\vec{k})\right] \; \frac{\delta}{\delta\phi(\vec{x})}\left(\Psi[\phi]^* \frac{\delta}{\delta\phi(\vec{x})}\Psi[\phi] - \Psi[\phi]\frac{\delta}{\delta\phi}\Psi^*[\phi]\right)$$

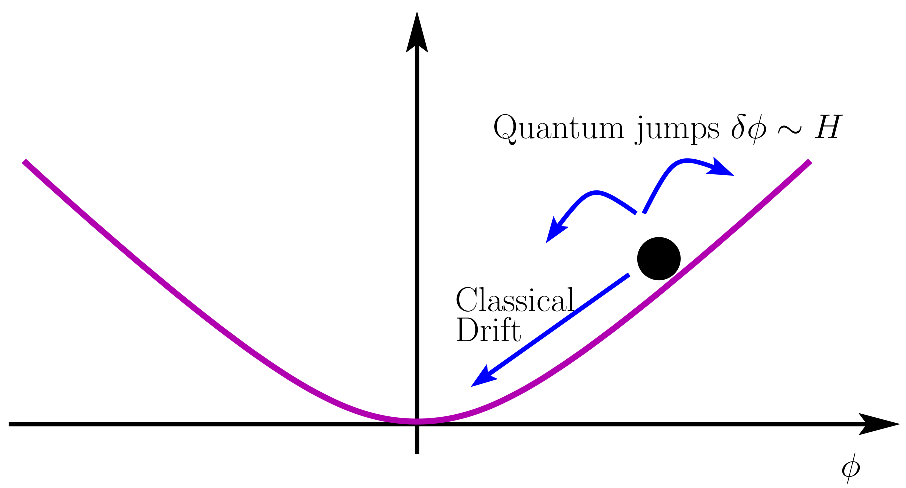
Effective Quasi-Probability for long modes

– Given:

$$P_\ell[\phi_\ell, t] = \int \mathcal{D}\phi\ \delta\left[\phi_\ell(\vec{x}) - \int^{\Lambda(t)} d^3k\ e^{i\vec{k}\cdot\vec{x}}\phi(\vec{k})\right]\ P[\phi, t]$$

– Find the effective time-evolution for the probability of the long modes

$$\frac{\partial P_\ell[\phi_\ell]}{\partial t} = \text{Drift} + \text{Diff}.$$



– Drift:

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Effective Probability for long modes

– manipulate:

$$\text{Drift} = -\int \mathcal{D}\phi\ \delta\left[\phi_\ell(\vec{x}) - \int^{\Lambda(t)} d^3k\ e^{i\vec{k}\cdot\vec{x}}\phi(\vec{k})\right]\ \frac{\delta}{\delta\phi(\vec{x})}\left(\Psi[\phi]^*\frac{\delta}{\delta\phi(\vec{x})}\Psi[\phi] - \Psi[\phi]\frac{\delta}{\delta\phi}\Psi^*[\phi]\right) =$$

$$= \frac{\delta}{\delta\phi_\ell(\vec{x})}\int \mathcal{D}\phi\ \delta\left[\phi_\ell(\vec{x}) - \int^{\Lambda(t)} d^3k\ e^{i\vec{k}\cdot\vec{x}}\phi(\vec{k})\right]\ \left(\Psi[\phi]^*\frac{\delta}{\delta\phi(\vec{x}')}\Psi[\phi] - \Psi[\phi]\frac{\delta}{\delta\phi(\vec{x}')}\Psi^*[\phi]\right) =$$

$$= \frac{\delta}{\delta\phi_\ell(\vec{x})}\int \mathcal{D}\phi\ \delta\left[\phi_\ell(\vec{x}) - \int^{\Lambda(t)} d^3k\ e^{i\vec{k}\cdot\vec{x}}\phi(\vec{k})\right]\ \left[\text{Re}\left(\frac{\Pi[\phi(\vec{x})]}{a^3}\right)\right]_\Lambda\ P[\phi]$$

– where $\Pi[\phi(\vec{x})]\Psi[\phi] = -i\frac{\delta}{\delta\phi(\vec{x})}\Psi[\phi]$

– Last path integral: expectation value of $\Pi(\phi)$ with fixed long-background.

$$\text{Drift} = \frac{\delta}{\delta\phi_\ell(\vec{x})}\left(\left\langle\left[\text{Re}\left(\frac{\Pi(\phi(\vec{x}))}{a^3}\right)\right]_\Lambda\right\rangle_{\phi_\ell} P_\ell[\phi_\ell]\right)$$

Effective Probability for long modes

– manipulate:

$$\text{Drift} = -\int \mathcal{D}\phi\ \delta\left[\phi_\ell(\vec{x}) - \int^{\Lambda(t)} d^3k\ e^{i\vec{k}\cdot\vec{x}}\phi(\vec{k})\right] \frac{\delta}{\delta\phi(\vec{x})}\left(\Psi[\phi]^*\frac{\delta}{\delta\phi(\vec{x})}\Psi[\phi] - \Psi[\phi]\frac{\delta}{\delta\phi}\Psi^*[\phi]\right) =$$

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$$= \frac{\delta}{\delta\phi_\ell(\vec{x})}\int \mathcal{D}\phi\ \delta\left[\phi_\ell(\vec{x}) - \int^{\Lambda(t)} d^3k\ e^{i\vec{k}\cdot\vec{x}}\phi(\vec{k})\right] \left[\mathrm{Re}\left(\frac{\Pi[\phi(\vec{x})]}{a^3}\right)\right]_\Lambda P[\phi]$$

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Effective Probability for long modes

– manipulate:

$$\text{Drift} = -\int \mathcal{D}\phi\ \delta\left[\phi_\ell(\vec{x}) - \int^{\Lambda(t)} d^3k\ e^{i\vec{k}\cdot\vec{x}}\phi(\vec{k})\right]\ \frac{\delta}{\delta\phi(\vec{x})}\left(\Psi[\phi]^*\frac{\delta}{\delta\phi(\vec{x})}\Psi[\phi] - \Psi[\phi]\frac{\delta}{\delta\phi}\Psi^*[\phi]\right) =$$

$$= \frac{\delta}{\delta\phi_\ell(\vec{x})}\int \mathcal{D}\phi\ \delta\left[\phi_\ell(\vec{x}) - \int^{\Lambda(t)} d^3k\ e^{i\vec{k}\cdot\vec{x}}\phi(\vec{k})\right]\ \left(\Psi[\phi]^*\frac{\delta}{\delta\phi(\vec{x}')}\Psi[\phi] - \Psi[\phi]\frac{\delta}{\delta\phi(\vec{x}')}\Psi^*[\phi]\right) =$$

$$= \frac{\delta}{\delta\phi_\ell(\vec{x})}\int \mathcal{D}\phi\ \delta\left[\phi_\ell(\vec{x}) - \int^{\Lambda(t)} d^3k\ e^{i\vec{k}\cdot\vec{x}}\phi(\vec{k})\right]\ \left[\mathrm{Re}\left(\frac{\Pi[\phi(\vec{x})]}{a^3}\right)\right]_\Lambda\ P[\phi]$$

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– Last path integral: expectation value of $\Pi(\phi)$ with fixed long-background.

$$\text{Drift} = \frac{\delta}{\delta\phi_\ell(\vec{x})}\left(\left\langle\left[\mathrm{Re}\left(\frac{\Pi(\phi(\vec{x}))}{a^3}\right)\right]_\Lambda\right\rangle_{\phi_\ell}\ P_\ell[\phi_\ell]\right)$$

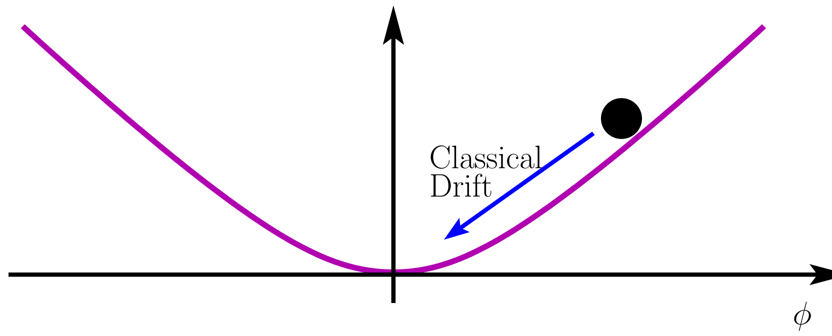
Effective Quasi-Probability for long modes

– Therefore:

$$\text{Drift} = \frac{\delta}{\delta\phi_\ell(\vec{x})}\left(\left\langle\left[\text{Re}\left(\frac{\Pi(\phi(\vec{x}))}{a^3}\right)\right]_\Lambda\right\rangle_{\phi_\ell} P_\ell[\phi_\ell]\right)$$

– Expectation value over the short modes: perturbative methods.

$$\langle\mathcal{O}(t)\rangle = \left\langle \text{in}\left|\left[\overline{T}\exp\left(i\int_{t_{\text{in}}}^{t}\mathrm{d}t'\,\mathcal{H}(t')\right)\right]\mathcal{O}(t)\left[T\exp\left(-i\int_{t_{\text{in}}}^{t}\mathrm{d}t'\,\mathcal{H}(t')\right)\right]\right|\text{in}\right\rangle,$$



– This is all well since we know the functional form of

$$\Pi[\phi(\vec{x})]\Psi[\phi] = -i\frac{\delta}{\delta\phi(\vec{x})}\Psi[\phi]$$

– Before writing that, let us do the diffusion

$$\frac{\partial P_\ell[\phi_\ell]}{\partial t} = \text{Drift} + \text{Diff}.$$

Effective Quasi-Probability for long modes

– Diffusion term: it arises because our cutoff is time-dependent

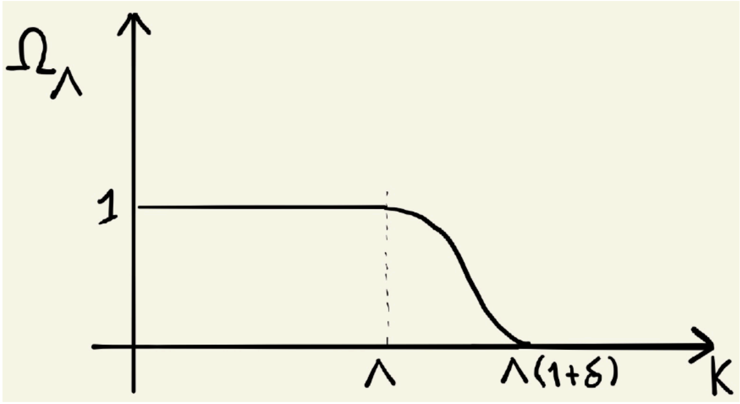
$$\mathrm{Diff} = \int \mathcal{D}\phi\, \frac{\partial}{\partial t}\delta\left[\phi_\ell(\vec{x}) - \int d^3k\, \Omega_{\Lambda(t)} e^{i\vec{k}\cdot\vec{x}}\phi(\vec{k})\right]\; P[\phi, t]$$

New modes enter the long theory

Keep long modes fixed

– Finite-thickness shell of modes entering the long theory:

$$\Delta\phi(\vec{x}) = \int_{\Lambda(t)}^{(1+\delta)\Lambda(t)} \frac{d^3k}{(2\pi)^3}\Omega_{\Lambda(t)}(k)\; e^{i\vec{k}\cdot\vec{x}}\phi(\vec{k})\ ,$$

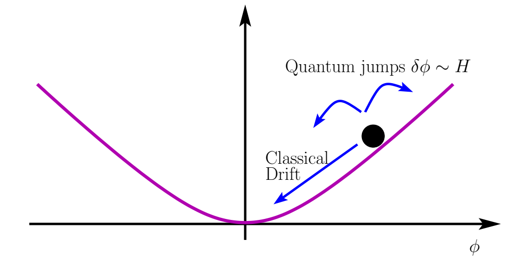


– Similar:

$$\mathrm{Diff.} = \frac{\delta}{\delta\phi_\ell(\vec{x})}\left(\left\langle\left(-\frac{\partial}{\partial t}\Delta\phi(\vec{x})\right)\right\rangle_{\phi_\ell} P_\ell[\phi_\ell, t]\right) + \frac{\delta^2}{\delta\phi_\ell(\vec{x})\delta\phi_{\ell,b}(\vec{x}')}\left(\left\langle\frac{\partial}{\partial t}\left(\Delta\phi(\vec{x})\Delta\phi(\vec{x}')\right)\right\rangle_{\phi_\ell} P_\ell[\phi_\ell, t]\right)$$

Effective Quasi-Probability for long modes

– To all orders in λ & ϵ and leading in δ , we obtain the following effective equation. It is Fokker-Planck-like, but it has differences



$$\frac{\partial P_\ell[\phi_\ell]}{\partial t} = \text{Drift} + \text{Diff.}$$

$$\text{Drift} = \frac{\delta}{\delta\phi_\ell(\vec{x})}\left(\left\langle\left[\text{Re}\left(\frac{\Pi(\phi(\vec{x}))}{a^3}\right)\right]_\Lambda\right\rangle_{\phi_\ell} P_\ell[\phi_\ell]\right)$$

$$\text{Diff.} = \frac{\delta}{\delta\phi_\ell(\vec{x})}\left(\left\langle\left(-\frac{\partial}{\partial t}\Delta\phi(\vec{x})\right)\right\rangle_{\phi_\ell} P_\ell[\phi_\ell, t]\right)$$

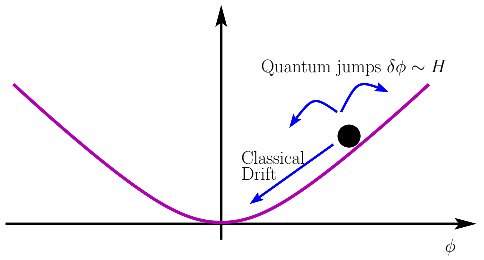
$$+\frac{\delta^2}{\delta\phi_\ell(\vec{x})\delta\phi_{\ell,b}(\vec{x}')}\left(\left\langle\frac{\partial}{\partial t}\left(\Delta\phi(\vec{x})\Delta\phi(\vec{x}')\right)\right\rangle_{\phi_\ell} P_\ell[\phi_\ell, t]\right)$$

– tadpole-diffusion term, and in principle higher derivative terms.

– Strategy: compute these expectation values for the short modes in perturbation theory with a given background for the long modes in expansion in $\lambda t \sim \lambda \log\epsilon \ll 1, \sqrt{\lambda} \ll 1$, and solve this functional Fokker-Planck-like equation containing only long modes.

Effective Quasi-Probability for long modes

– To all orders in λ & ϵ and leading in δ, we obtain the following effective equation. It is Fokker-Planck-like, but it has differences



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$$\text{Diff.} = \frac{\delta}{\delta\phi_\ell(\vec{x})}\left(\left\langle\left(-\frac{\partial}{\partial t}\Delta\phi(\vec{x})\right)\right\rangle_{\phi_\ell} P_\ell[\phi_\ell, t]\right)$$

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The Momentum

– We finally need $$\Pi[\phi(\vec{x})]\Psi[\phi] = -i\frac{\delta}{\delta\phi(\vec{x})}\Psi[\phi]$$

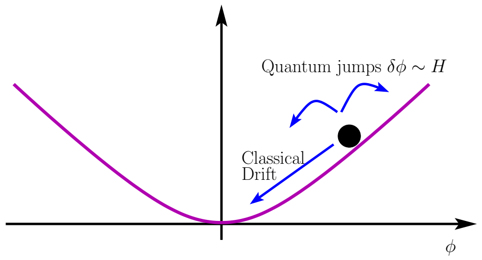
– Wavefunction reads

$$\Psi[\phi] \sim \mathrm{Exp}\left(-i\,a(t)^3\left(\frac{\lambda}{12H}\phi(\vec{x})^4 - \frac{\lambda^2}{54H^4}\phi(\vec{x})^6\right) + \mathcal{O}(\epsilon, \epsilon^3(\lambda\log(k\eta))^n)\right)$$

$$\Rightarrow \quad \frac{\Pi[\phi]}{a^3} = -\frac{\lambda}{3H}\phi(\vec{x})^3 + \mathcal{O}(\epsilon, \sqrt{\lambda}) = \mathrm{slow-roll\ solution} + \mathcal{O}(\epsilon)$$

Effective Quasi-Probability for long modes

– To all orders in λ & ϵ and leading in δ, we obtain the following effective equation. It is Fokker-Planck-like, but it has differences



$$\frac{\partial P_\ell[\phi_\ell]}{\partial t} = \text{Drift} + \text{Diff.}$$

$$\text{Drift} = \frac{\delta}{\delta\phi_\ell(\vec{x})}\left(\left\langle\left[\text{Re}\left(\frac{\Pi(\phi(\vec{x}))}{a^3}\right)\right]_\Lambda\right\rangle_{\phi_\ell} P_\ell[\phi_\ell]\right)$$

$$\text{Diff.} = \frac{\delta}{\delta\phi_\ell(\vec{x})}\left(\left\langle\left(-\frac{\partial}{\partial t}\Delta\phi(\vec{x})\right)\right\rangle_{\phi_\ell} P_\ell[\phi_\ell, t]\right)$$

$$+\frac{\delta^2}{\delta\phi_\ell(\vec{x})\delta\phi_{\ell,b}(\vec{x}')}\left(\left\langle\frac{\partial}{\partial t}\left(\Delta\phi(\vec{x})\Delta\phi(\vec{x}')\right)\right\rangle_{\phi_\ell} P_\ell[\phi_\ell, t]\right)$$

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One-location

Effective Probability

– Expand in number of locations, as evolution is quasi local, thanks to dS (this is the opposite of what we do for perturbative theories in Minkowski).

– One-location probability distribution:

$$P_{\ell,1}(\phi_\ell(\vec{x}_1) = \phi_1) = \int \mathcal{D}\phi_\ell \; \delta^{(1)}\left[\phi_1 - \phi_\ell(\vec{x}_1)\right] P_\ell[\phi_\ell]$$

Keep long fields at one point fixed

Effective Probability

– Expand in number of locations, as evolution is quasi local, thanks to dS (this is the opposite of what we do for perturbative theories in Minkowski).

– One-location probability distribution:

$$P_{\ell,1}(\phi_\ell(\vec{x}_1)=\phi_1)=\int \mathcal{D}\phi_\ell\ \delta^{(1)}\left[\phi_1-\phi_\ell(\vec{x}_1)\right] P_\ell[\phi_\ell]$$

– Resulting equation:

$$\frac{\partial P_{\ell,1}(\phi_1,t)}{\partial t}=\frac{1}{2}\frac{\partial^2}{\partial\phi_1^2}\left(\left\langle\frac{\partial}{\partial t}\Delta\phi(\vec{x}_1)^2\right\rangle_{\phi_1}P_{\ell,1}(\phi_1,t)\right)-\frac{\partial}{\partial\phi_1}\left(\left\langle\left[\frac{\Pi(\phi(\vec{x}_1))}{a^2}\right]_\Lambda\right\rangle_{\phi_1}P_{\ell,1}(\phi_1,t)\right)$$

– We dropped the term $\left\langle\left(-\frac{\partial}{\partial t}\Delta\phi(\vec{x})\right)\right\rangle_{\phi_\ell}$ because it will not contribute at the order at which we will compute.

– Expectation value of the short modes on the long depends only on the field at the same location

Effective Probability

– Expand in number of locations, as evolution is quasi local, thanks to dS (this is the opposite of what we do for perturbative theories in Minkowski).

– One-location probability distribution:

$$P_{\ell,1}(\phi_\ell(\vec{x}_1)=\phi_1)=\int \mathcal{D}\phi_\ell\ \delta^{(1)}\left[\phi_1-\phi_\ell(\vec{x}_1)\right] P_\ell[\phi_\ell]$$

Diffusion

Drift

– Resulting equation:

$$\frac{\partial P_{\ell,1}(\phi_1,t)}{\partial t}=\frac{1}{2}\frac{\partial^2}{\partial\phi_1^2}\left(\left\langle\frac{\partial}{\partial t}\Delta\phi(\vec{x}_1)^2\right\rangle_{\phi_1}P_{\ell,1}(\phi_1,t)\right)-\frac{\partial}{\partial\phi_1}\left(\left\langle\left[\frac{\Pi(\phi(\vec{x}_1))}{a^2}\right]_\Lambda\right\rangle_{\phi_1}P_{\ell,1}(\phi_1,t)\right)$$

– We dropped the term $\left\langle\left(-\frac{\partial}{\partial t}\Delta\phi(\vec{x})\right)\right\rangle_{\phi_\ell}$ because it will not contribute at the order at which we will compute.

– Expectation value of the short modes on the long depends only on the field at the same location

Solving at one-location: leading order

– Compute various ingredients, assuming counting $\phi_\ell \sim \frac{1}{\lambda^{1/4}}$

$$\langle \hat{\phi}(\vec{k},t)\hat{\phi}(-\vec{k},t)\rangle' = \frac{H^2}{2k^3}\left(1 + O\left(\sqrt{\lambda}\right)\right)$$

$$\left\langle \left[\frac{\Pi(\phi(\vec{x}_1))}{a^2}\right]_\Lambda \right\rangle_{\phi_1} = -\frac{\lambda}{3H}\phi_1^3$$

- Obtain

$$\frac{\partial P_{\ell,1}(\phi_1)}{\partial t} = \Gamma_{\phi_1} P_{\ell,1}(\phi_1, t)\left(1 + \mathcal{O}(\lambda^{1/2}, \delta, \epsilon^2)\right)$$

$$\Gamma_\phi = -\frac{\partial}{\partial \phi}\left(\frac{\lambda}{3H}\phi^3\right) + \frac{H^3}{8\pi^2}\frac{\partial^2}{\partial \phi^2}$$

Solving at one-location: leading order

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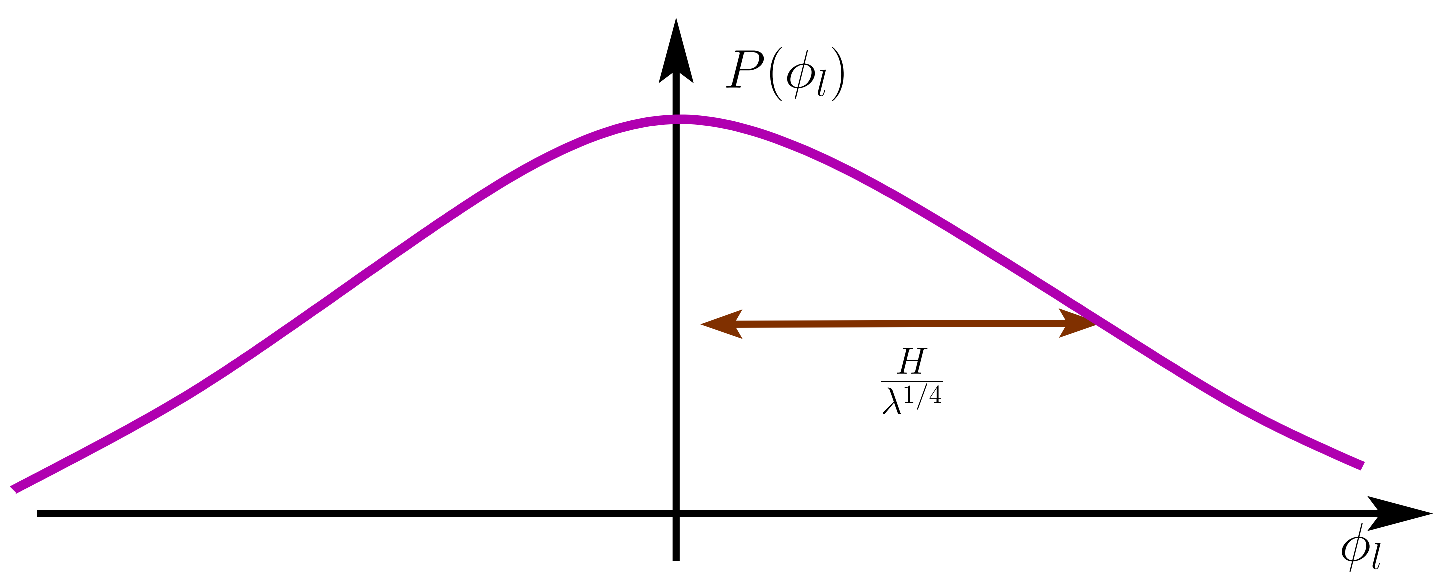
$$\Gamma_\phi = -\frac{\partial}{\partial\phi}\left(\frac{\lambda}{3H}\phi^3\right) + \frac{H^3}{8\pi^2}\frac{\partial^2}{\partial\phi^2}$$

– The famous Starobinsky equations, but now rigorously derived with control of approximation and we can include them.

– There is an equilibrium, *i.e.* static, solution:

$$P_1^{eq} \sim e^{-\frac{\lambda\,\phi^4}{H^4}}$$

- – $\phi_l \sim \dfrac{H}{\lambda^{1/4}}$: so our *counting is correct: self-consistency*
- – Static solution is the one corresponding to the BD vacuum
- – time-dependent solutions decay
    - suggesting stability and attractor



Solving at one-location: sub-leading order

– We start again from

$$\frac{\partial P_{\ell,1}(\phi_1,t)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial\phi_1^2}\left(\left\langle \frac{\partial}{\partial t}\Delta\phi(\vec{x}_1)^2\right\rangle_{\phi_1} P_{\ell,1}(\phi_1,t)\right) - \frac{\partial}{\partial\phi_1}\left(\left\langle\left[\frac{\Pi(\phi(\vec{x}_1))}{a^2}\right]_\Lambda\right\rangle_{\phi_1} P_{\ell,1}(\phi_1,t)\right)$$

– with counting $\quad \phi_1 \sim H\lambda^{-1/4}, \quad \phi_s \sim H,$

– and compute to next order the various expectation values

– drift term:

$$\left\langle\left[\frac{\Pi(\phi(\vec{x}_1))}{a^2}\right]_\Lambda\right\rangle_{\phi_1} = -\frac{\lambda}{3H}\phi_1^3 - \frac{\lambda^2}{9H^3}\phi_1^5 - \frac{\lambda}{H}\langle\phi_s(\vec{x})^2\rangle\phi_1 + m^2\phi_1$$

– *sixtic potential and a mass* from short modes on longs: $\left\langle\hat{\phi}_s(x_1^\mu)^2\right\rangle = -\frac{H^2}{4\pi^2}\log\epsilon + H^2 V_2\ ,$

– diffusion term: *mass* induced from long modes on shorts: $\delta m_s^2 = 3\lambda\phi_1^2$

$$\left\langle\hat{\phi}(\vec{k}_s,t)\hat{\phi}(-\vec{k}_s,t)\right\rangle'_{\phi_1} = \frac{H^2}{2k_s^3}\left(1+\log\epsilon\frac{2\delta m_s^2}{3H^2}\right) + O(\lambda)$$

Solving at one-location: sub-leading order

– Summary so far:

$$\frac{\partial P_{\ell,1}(\phi_1,t)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial \phi_1^2}\left(\left\langle \frac{\partial}{\partial t}\Delta\phi(\vec{x}_1)^2\right\rangle_{\phi_1} P_{\ell,1}(\phi_1,t)\right) - \frac{\partial}{\partial \phi_1}\left(\left\langle\left[\frac{\Pi(\phi(\vec{x}_1))}{a^2}\right]_\Lambda\right\rangle_{\phi_1} P_{\ell,1}(\phi_1,t)\right)$$

– where

$$\left\langle \frac{\partial}{\partial t}\Delta\phi(\vec{x}_1)^2\right\rangle_{\phi_1} = H^3\left(1+\lambda\frac{\phi_1^2}{H^2}\log(\epsilon)\right)\ ,$$

Effective mass

$$\left\langle\left[\frac{\Pi(\phi(\vec{x}_1))}{a^2}\right]_\Lambda\right\rangle_{\phi_1} = -\frac{\lambda}{H}\phi_1^3 - \frac{\lambda^2}{H^3}\phi_1^5 - \lambda H\log\epsilon\,\phi_1 + \bar{m}^2\phi_1$$

Solving at one-location: sub-leading order

– Summary so far:

$$\frac{\partial P_{\ell,1}(\phi_1,t)}{\partial t}=\frac{1}{2}\frac{\partial^2}{\partial\phi_1^2}\left(\left\langle\frac{\partial}{\partial t}\Delta\phi(\vec{x}_1)^2\right\rangle_{\phi_1}P_{\ell,1}(\phi_1,t)\right)-\frac{\partial}{\partial\phi_1}\left(\left\langle\left[\frac{\Pi(\phi(\vec{x}_1))}{a^2}\right]_\Lambda\right\rangle_{\phi_1}P_{\ell,1}(\phi_1,t)\right)$$

– where

$$\left\langle\frac{\partial}{\partial t}\Delta\phi(\vec{x}_1)^2\right\rangle_{\phi_1}=H^3\left(1+\lambda\frac{\phi_1^2}{H^2}\log(\epsilon)\right),$$

dependence on $\log\epsilon$ is unphysical

$$\left\langle\left[\frac{\Pi(\phi(\vec{x}_1))}{a^2}\right]_\Lambda\right\rangle_{\phi_1}=-\frac{\lambda}{H}\phi_1^3-\frac{\lambda^2}{H^3}\phi_1^5-\lambda H\log\epsilon\,\phi_1+\bar{m}^2\phi_1$$

Subleading order

– Solve the same equation, including subleading terms:

$$P_1^{eq}(\phi_1) = N e^{-\frac{2\pi^2\lambda\phi_1^4}{3H^4}} \left(1 - \lambda\frac{\phi_1^2}{H^2}\left(\log\left(\epsilon/2\right) - \psi\left(3/2\right) + \frac{4\pi^2}{3}\frac{\bar{m}^2}{\lambda H^2}\right) + \right.$$
$$\left. + \frac{8\pi^2}{9}\frac{\lambda^2\phi_1^6}{H^6}\left(\log\left(\epsilon/2\right) - \psi\left(3/2\right) - \frac{1}{6}\right) + O(\lambda)\right)$$

$$\Rightarrow \quad \langle\phi_l(\vec{x})^n\rangle = \int d\phi_l \; \bar{P}_{1,\,\mathrm{eq}}(\phi_l)\,\phi_l^n = \text{depends on } \log(\epsilon)$$ unphysical!

– Ok, as $\langle\phi_l(\vec{x})^n\rangle$ is UV sensitive

– What is physical is $\langle\phi(\vec{x})^n\rangle = \langle(\phi_s(\vec{x}) + \phi_l(\vec{x}))^n\rangle$

– Counting: $\phi_s \sim H \sim \phi_l \cdot \lambda^{1/4} \quad \Rightarrow \quad \frac{\phi_s^2}{\phi_l^2} \sim \sqrt{\lambda}$

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$$\mathcal{O}(\lambda \cdot \frac{1}{\lambda^{1/2}} = \lambda^{1/2}) \qquad \left. + \frac{8\pi^2}{9}\frac{\lambda^2\phi_1^6}{H^6}\left(\log\left(\epsilon/2\right) - \psi\left(3/2\right) - \frac{1}{6}\right) + O(\lambda)\right)$$

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Subleading order

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$$\mathcal{O}(\lambda^2 \cdot \frac{1}{\lambda^{3/2}} = \lambda^{1/2})$$

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Result for 1-location

– Using
$$P_1^{eq}(\phi_1) = N e^{-\frac{2\pi^2\lambda\phi_1^4}{3H^4}} \left(1 - \lambda \frac{\phi_1^2}{H^2}\left(\log\left(\epsilon/2\right) - \psi\left(3/2\right) + \frac{4\pi^2}{3}\frac{\bar{m}^2}{\lambda H^2}\right) + \right.$$
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– Obtain
$$\langle \phi(\vec{x})^n \rangle = \langle (\phi_s(\vec{x}) + \phi_l(\vec{x}))^n \rangle$$

$$\left\langle \hat{\phi}(x_1^\mu)^{2n} \right\rangle = \lambda^{-\frac{n}{2}} H^{2n} \left(\frac{3}{2}\right)^{n/2} \left[\frac{\pi^{-n}\Gamma\left(\frac{n}{2}+\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} + \right.$$
$$+\sqrt{\lambda}\,\frac{6^{-3/2}\pi^{-n-1}}{\Gamma\left(\frac{1}{4}\right)^2}\left(\left(3 - 24\,\pi^2\frac{\bar{m}^2}{\lambda H^2} - n\left(2 - 48\pi^2 V_2\right)\right)\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{n}{2}+\frac{3}{4}\right) - \right.$$
$$\left.\left. + \left(3 - 24\,\pi^2\frac{\bar{m}^2}{\lambda H^2}\right)\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{n}{2}+\frac{1}{4}\right)\right) + O(\lambda)\right].$$

– $\log(\epsilon)$ cancelled!

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– $\log(\epsilon)$ cancelled!

2-locations

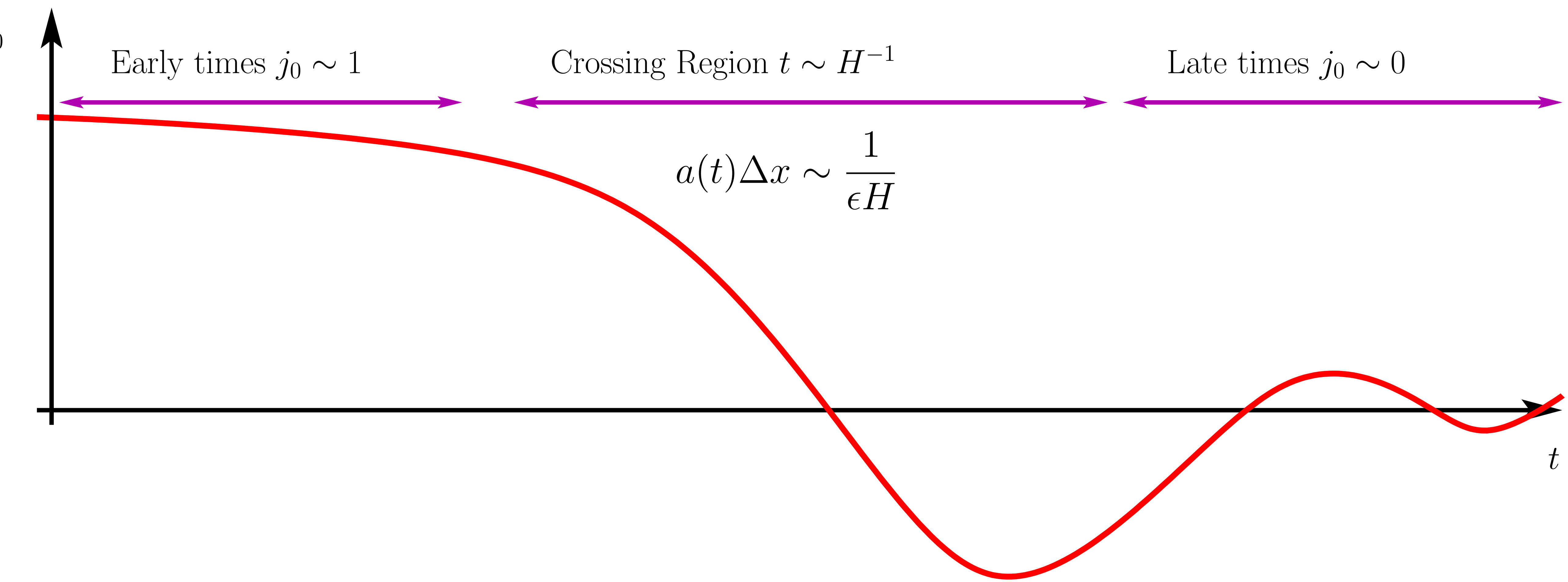
2-locations n-point function

– Analogous Fokker-Planck-like equation for the distribution at 2-points:

$$\frac{\partial}{\partial t}P_2(\phi_1,\phi_2,\Delta x,t) = \left(\Gamma_{\phi_1}+\Gamma_{\phi_2}\right)P_2 + j_0\left(\epsilon a(t)H\Delta x\right)\frac{\partial^2}{\partial\phi_1\partial\phi_2}P_2$$

where $\quad \dfrac{\partial}{\partial t}P_1(\phi_1,t) = \Gamma_{\phi_1}P_1 = \dfrac{\partial^2}{\partial\phi_1^2}P_1 + \dfrac{\partial}{\partial\phi_1}\left(V'(\phi_1)P_1\right)$

– Last term strongly depends on distance



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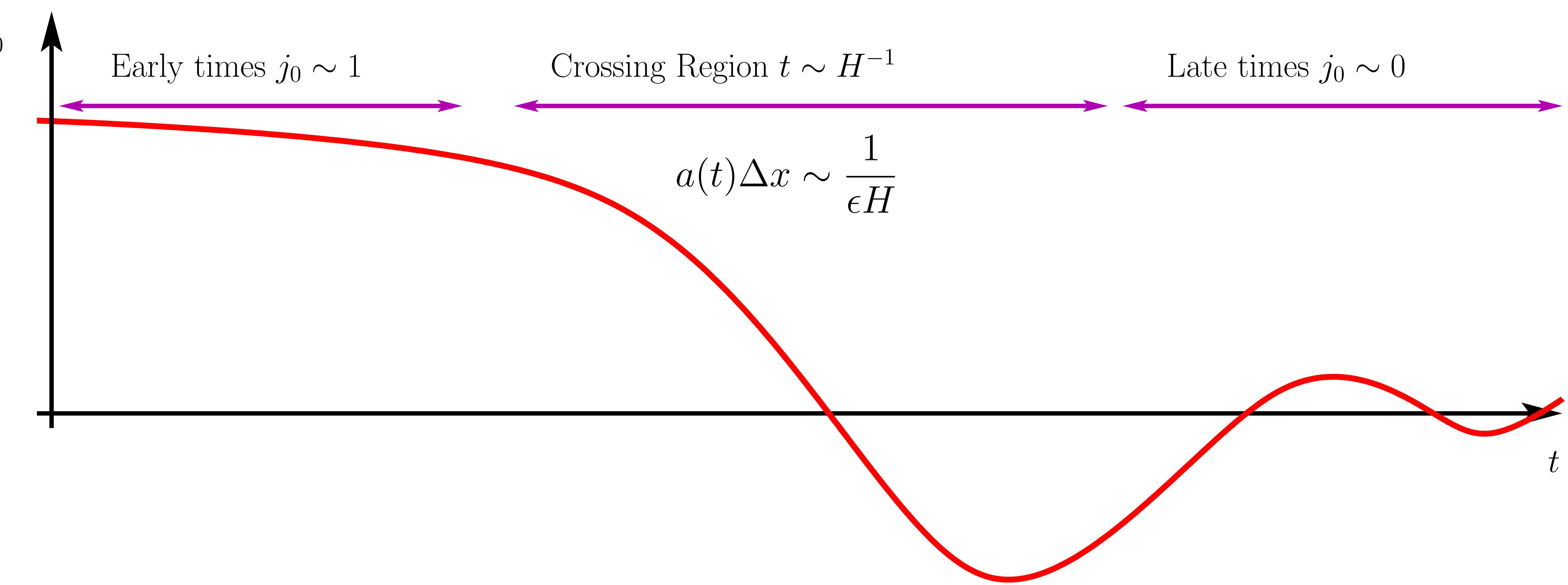

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$$\frac{\partial}{\partial t} P_2(\phi_1, \phi_2, \Delta x, t) = \left(\Gamma_{\phi_1} + \Gamma_{\phi_2}\right) P_2 + j_0\left(\epsilon a(t) H \Delta x\right) \frac{\partial^2}{\partial \phi_1 \partial \phi_2} P_2$$

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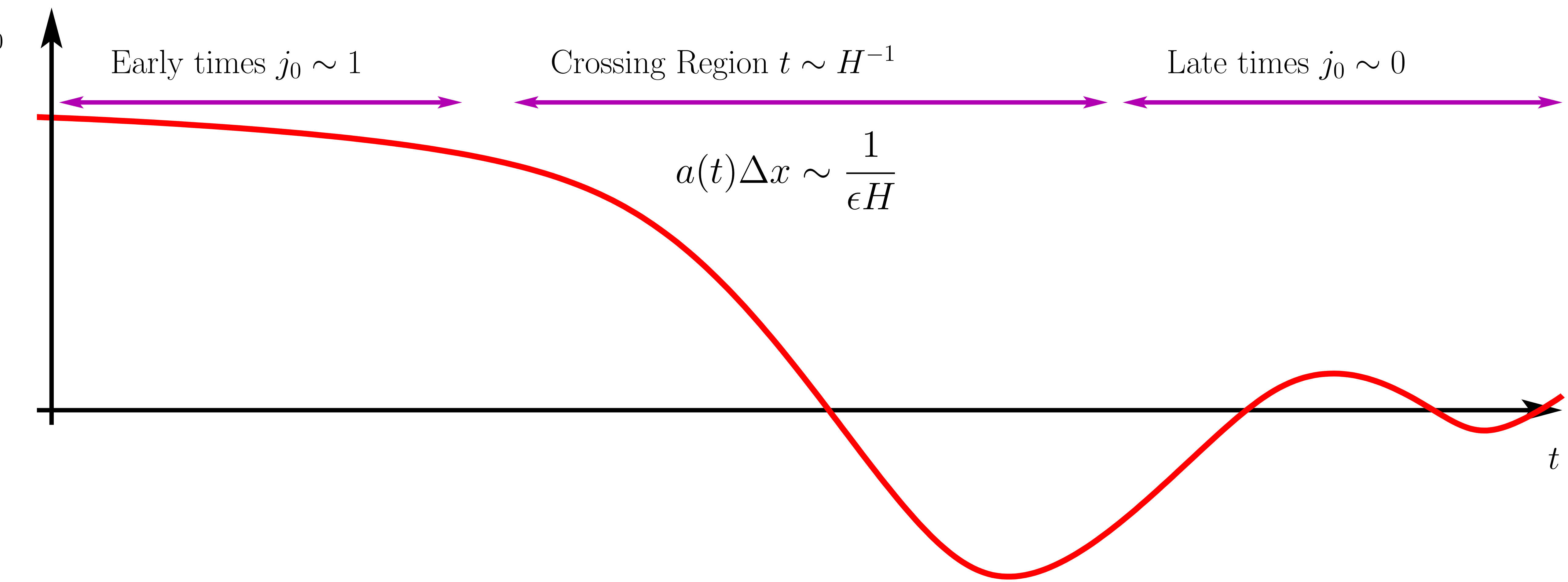

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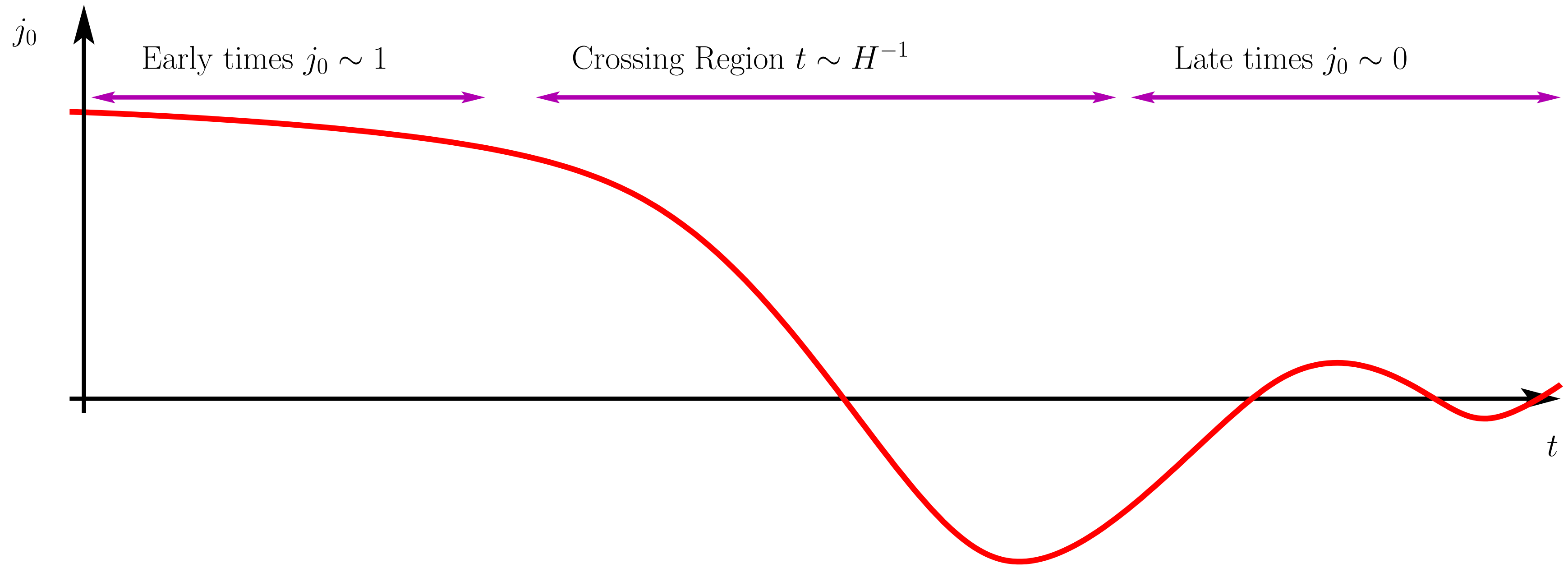

– At early times, solutions is $P_2(\phi_1, \phi_2, t_{\rm early}) \sim \delta^{(1)}(\phi_1 - \phi_2)\ P_{\rm eq,1}(\phi_1)$

– At late times is $P_2(\phi_1, \phi_2,\ t_{\rm late}) \sim P_{1,a}(\phi_1,\ t_{\rm late})\ P_{1,b}(\phi_2\, t_{\rm late})$

– Time scale of diff equation is $H^{-1}/\sqrt{\lambda}$, but crossing time $H^{-1} \ll H^{-1}/\sqrt{\lambda}$

– $\Rightarrow$ glue using `sudden perturbation theory', which corresponds to expansion in $\sqrt{\lambda}$

de Sitter invariance

– Analogous Fokker-Planck-like equation for the distribution at 2-points:

$$\frac{\partial}{\partial t} P_2(\phi_1, \phi_2, \Delta x, t) = \left(\Gamma_{\phi_1} + \Gamma_{\phi_2}\right) P_2 + j_0 \left(\epsilon a(t) H \Delta x\right) \frac{\partial^2}{\partial \phi_1 \partial \phi_2} P_2$$

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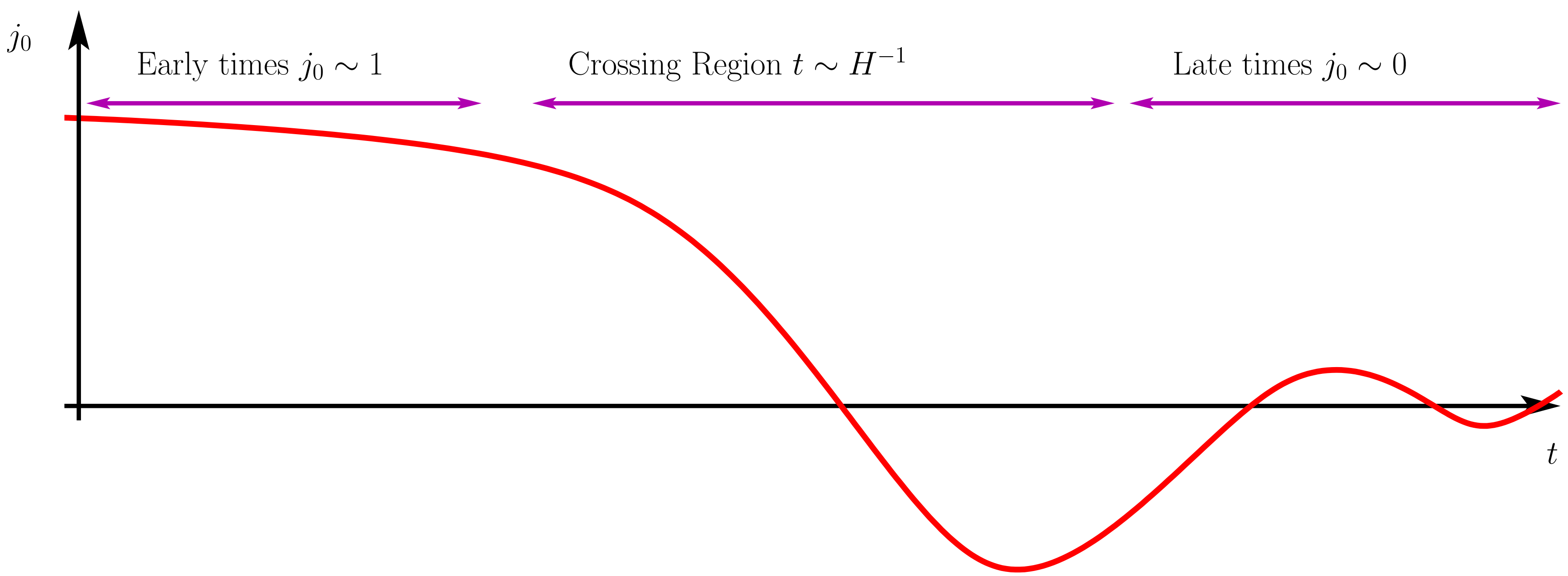


– Solution is non-perturbative and de Sitter invariant.

$$\langle \phi(\vec{x}_1, t_1)^n \phi(\vec{x}_2, t_2)^m \rangle = f_{nm}(z)$$

$$\text{where} \quad z^2 = \cosh(t_1 + t_2) - H^2 e^{H(t_1+t_2)} |\vec{x}_1 - \vec{x}_2|$$

de Sitter invariance

– Correlation functions at different spacetime-points

– under perturbative control: decay at long distances,

$$\langle \phi(\vec{x}_1, t_1)^n \phi(\vec{x}_2, t_2)^m \rangle = f_{nm}(z) \sim z^{-\sqrt{\lambda}}, \qquad z \to \infty$$

$$\text{where} \quad z^2 = \cosh(t_1 + t_2) - H^2 e^{H(t_1+t_2)} |\vec{x}_1 - \vec{x}_2|^2$$

- In general

$$\langle \phi(\vec{x}_1, t_1)^{n_1} \ldots \phi(\vec{x}_n, t_n)^{n_n} \rangle \ \to \ \langle \phi(\vec{x}_1)^{n_1} \rangle_{eq} \cdot \ldots \cdot \langle \phi(\vec{x}_n)^{n_n} \rangle_{eq}$$

- *signaling stability of dS*

- compute subleading $\sqrt{\lambda}$ corrections, finding again that $\log(\epsilon)$ cancelled.

State Independence

– Our construction so far uses the $\Psi_{BD}[\phi, \eta]$

– What happens for other states? We can consider states of the form

$$\psi_{\mathcal{O}_{1,\ldots,n}} = N_\Psi^{-1} \int \mathcal{D}\phi\, \mathcal{O}(x^\mu_{s,1}, x^\mu_{s,2}, \ldots)\, e^{iS} \ ,$$

- with BD boundary conditions.

- $\Rightarrow$ correlation functions in these states are higher-points correlation functions in BD.
  - $\Rightarrow$ their decay implies *stability of dS* in these states.
  - $\Rightarrow$ if we keep the dS distance among the points of the original correlation function fixed while sending the insertion of the operators at earlier times, correlation functions converge towards the ones computed in BD:

$$\langle \phi(\vec{x}_1, t_1)^{n_1} \ldots \phi(\vec{x}_n, t_n)^{n_n} \rangle \underset{t_i - t_0 \to -\infty}{\longrightarrow} \langle \phi(\vec{x}_1, t_1)^{n_1} \ldots \phi(\vec{x}_n, t_n)^{n_n} \rangle_{eq}$$

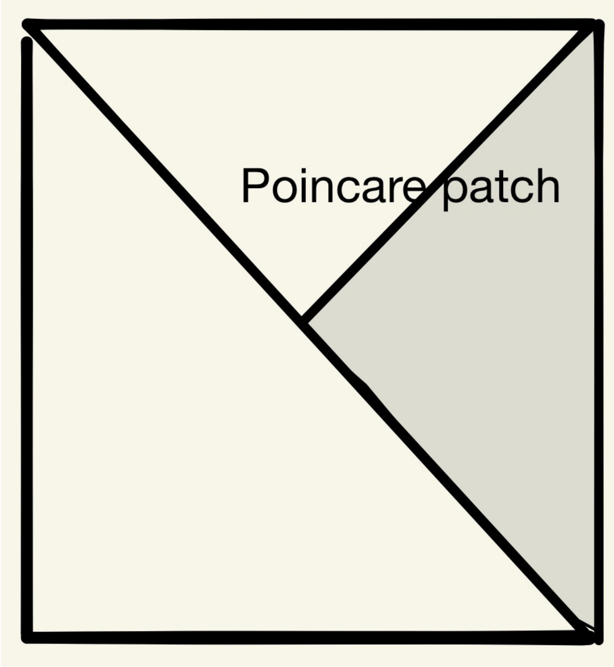
- $\Rightarrow$ *BD vacuum is an attractor*

Thermality

– Correlation functions at different spacetime-points

$$\langle \phi(\vec{x}_1, t_1)^n \phi(\vec{x}_2, t_2)^m \rangle = f_{nm}(z) \sim z^{-\sqrt{\lambda}}, \quad z \to \infty$$

– Restricted to static patch, they satisfy thermality with $T_{\rm dS} = \frac{H}{2\pi}$



- i.e. the KMS condition - certain periodicity in imaginary time

$$\langle \phi(x_1, t_1) \phi(x_2, t_2 + iT_{dS}^{-1}) \rangle = \langle \phi(x_2, t_2) \phi(x_1, t_1) \rangle$$

- Not obviously true, since the leading term by itself does not satisfy it. KMS condition requires particular coefficient of the subleading term:
- $$f_{nm}(z) \sim z^{-\sqrt{\lambda}}(1 - i\pi\sqrt{\lambda})$$, which we also computed.

Conclusion

– We have developed a formalism to compute correlation functions of $\lambda\,\phi^4$ in dS

– manifest expansion in $\sqrt{\lambda}$ & ϵ & δ

– the solution is remarkably non-perturbative, and yet we can solve it:

– *Equilibrium & Stability*:

$$\langle\phi(\vec{x}_1,t_1)^{n_1}\ldots\phi(\vec{x}_n,t_n)^{n_n}\rangle \;\to\; \langle\phi(\vec{x}_1)^{n_1}\rangle_{eq}\cdot\ldots\cdot\langle\phi(\vec{x}_n)^{n_n}\rangle_{eq}, \quad \text{for } t_i\to\infty,\ \text{fixed } x_i$$

– *State-independence:*

$$\langle\phi(\vec{x}_1,t_1)^{n_1}\ldots\phi(\vec{x}_n,t_n)^{n_n}\rangle \;\to\; \langle\phi(\vec{x}_1,t_1)^{n_1}\ldots\phi(\vec{x}_n,t_n)^{n_n}\rangle_{eq}, \text{ for } t_i - t_0\to\infty, z_{jk}=\text{fixed}$$

– *de Sitter invariant*

– *Thermal in static patch*

- all radiative corrections in rigid dS and inflation are understood, and well behaved

– no instability in the rigid limit

with Zaldarriaga **JHEP 2010, JHEP 2012, JCAP 2012, JHEP 2013**
with Pimentel and Zaldarriaga **JHEP 2012**
& this work

- Existence of slow-roll eternal inflation is close to rigorously established.