Can negative kinetic energy be ok?

In classical mechanics, quantum mechanics, classical field theory, quantum field theory

Alessandro Strumia, Pisa U., 2020. Paper to appear with Gross, Teresi, Zirilli.

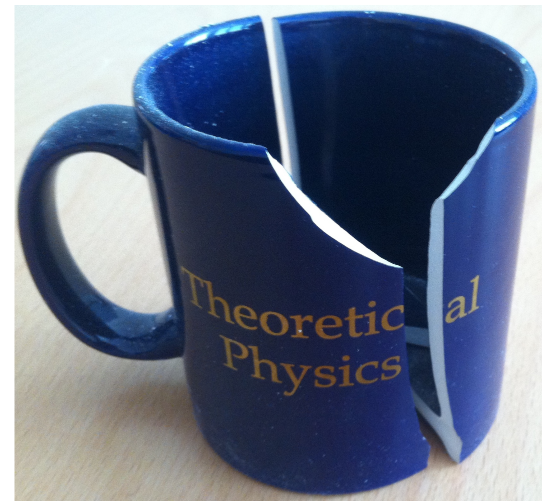
European Commission | Horizon 2020 European Union funding for Research & Innovation

erc European Research Council

Motivation: would give a theory of quantum gravity

To get a theory of quantum gravity, write the most generic renormalizable Lagrangian with the graviton $g_{\mu\nu}$: it has dimension 0 so 4 derivatives. $R^2_{\cdots}$ generated by loops, even starting from Einstein:

$$S = \int d^4x \sqrt{|\det g|} \left[\frac{R^2}{6f_0^2} + \frac{\frac{1}{3}R^2 - R^2_{\mu\nu}}{f_2^2} - \frac{1}{2}\bar{M}^2_{\rm Pl} R + \mathscr{L}_{\rm matter} \right]$$



Ghost with mass $M_2 \sim f_2 M_{\rm Pl}$ key to get renormalizable quantum gravity:

$$\left(\begin{array}{c}\text{4-derivative} \\ \text{graviton propagator}\end{array} \right) = \frac{1}{k^4 - M_2^2 k^2} = \frac{1}{M_2^2}\Big[\underbrace{\frac{1}{k^2}}_{\text{graviton}} - \underbrace{\frac{1}{k^2 - M_2^2}}_{\text{ghost}} \Big] \stackrel{?}{=}$$

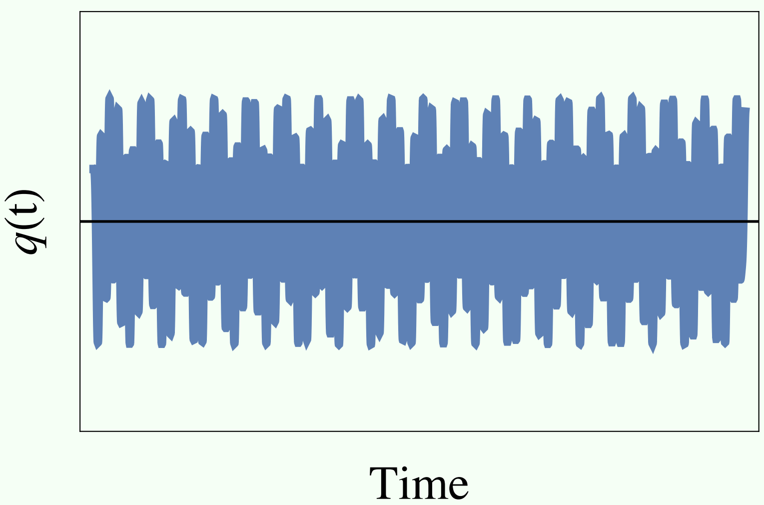
Ostrogradski: (more than 2 time derivatives) $\Rightarrow$ (classical energy down to $-\infty$). Then interactions give run-away evolution. The g-word makes many physicists run away. Various attempts to quantise as positive energy: Bender/Mannheim, Salvio/Strumia, Anselmi/Piva... Is negative kinetic energy hopeless? Or maybe it's similar to negative potential energy: meta-stability up to cosmological times?

1) Classical mechanics

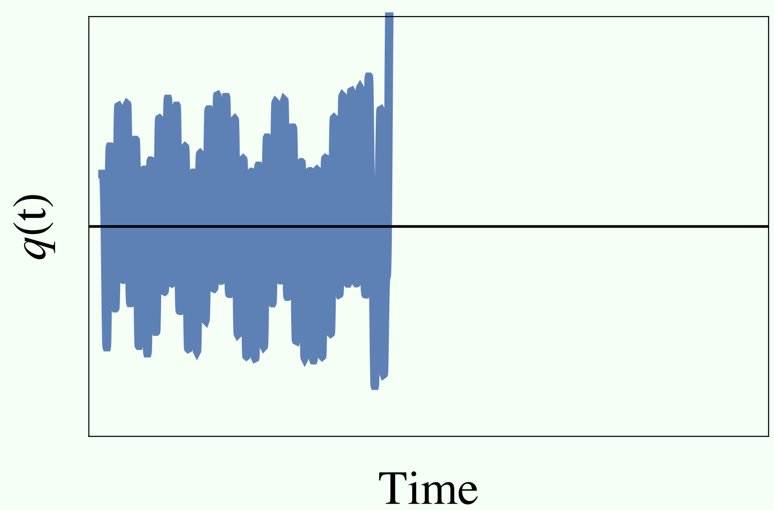
Ghost miracle?

To see: solve numerically $\ddot{\ddot{q}} + (\omega_1^2 + \omega_2^2)\ddot{q} + \omega_1^2\omega_2^2 q =$ interactions e.g. λq^3:

Smaller coupling: stable



Larger coupling: runaway



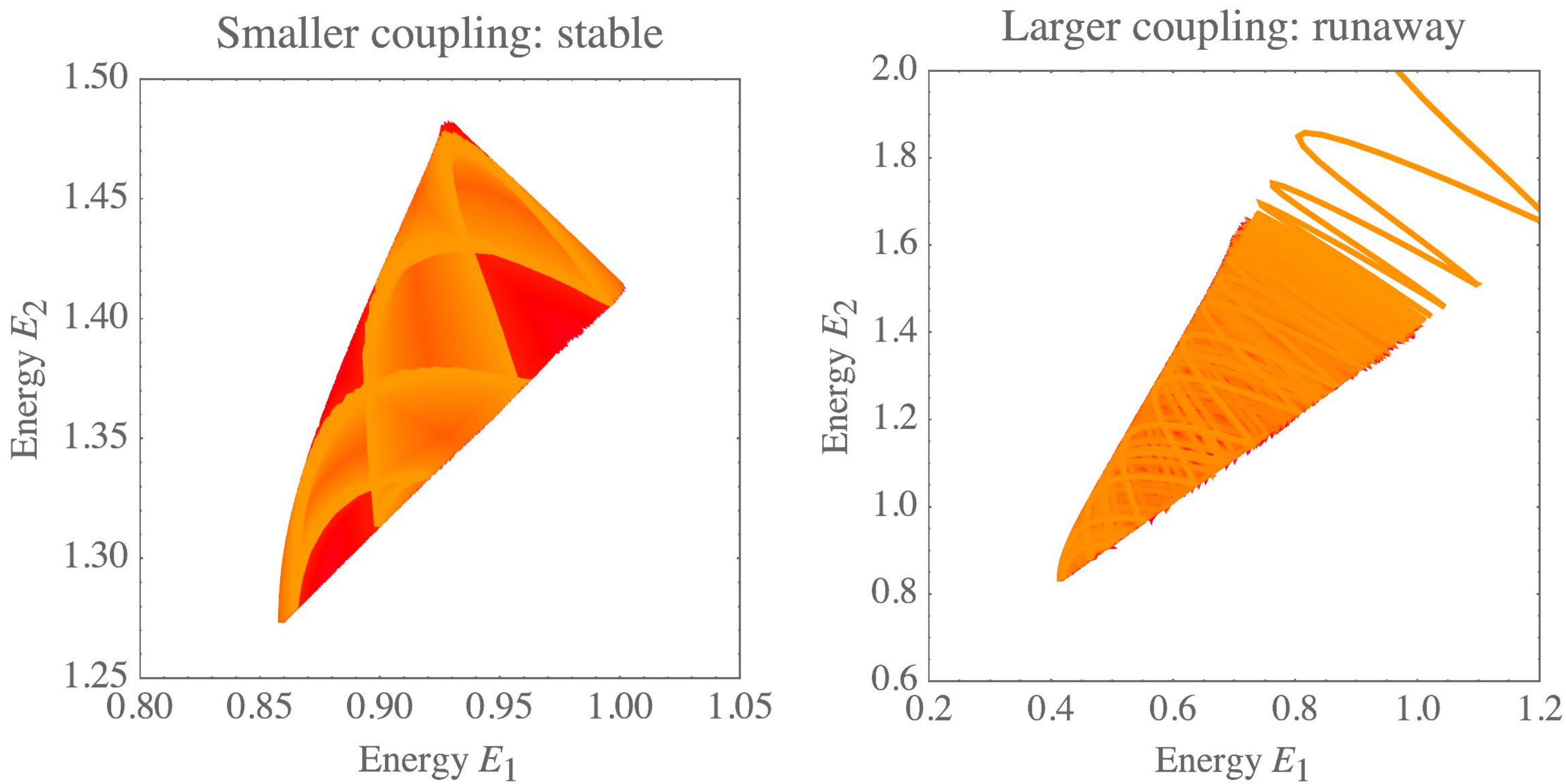
How long can be stable? And why?

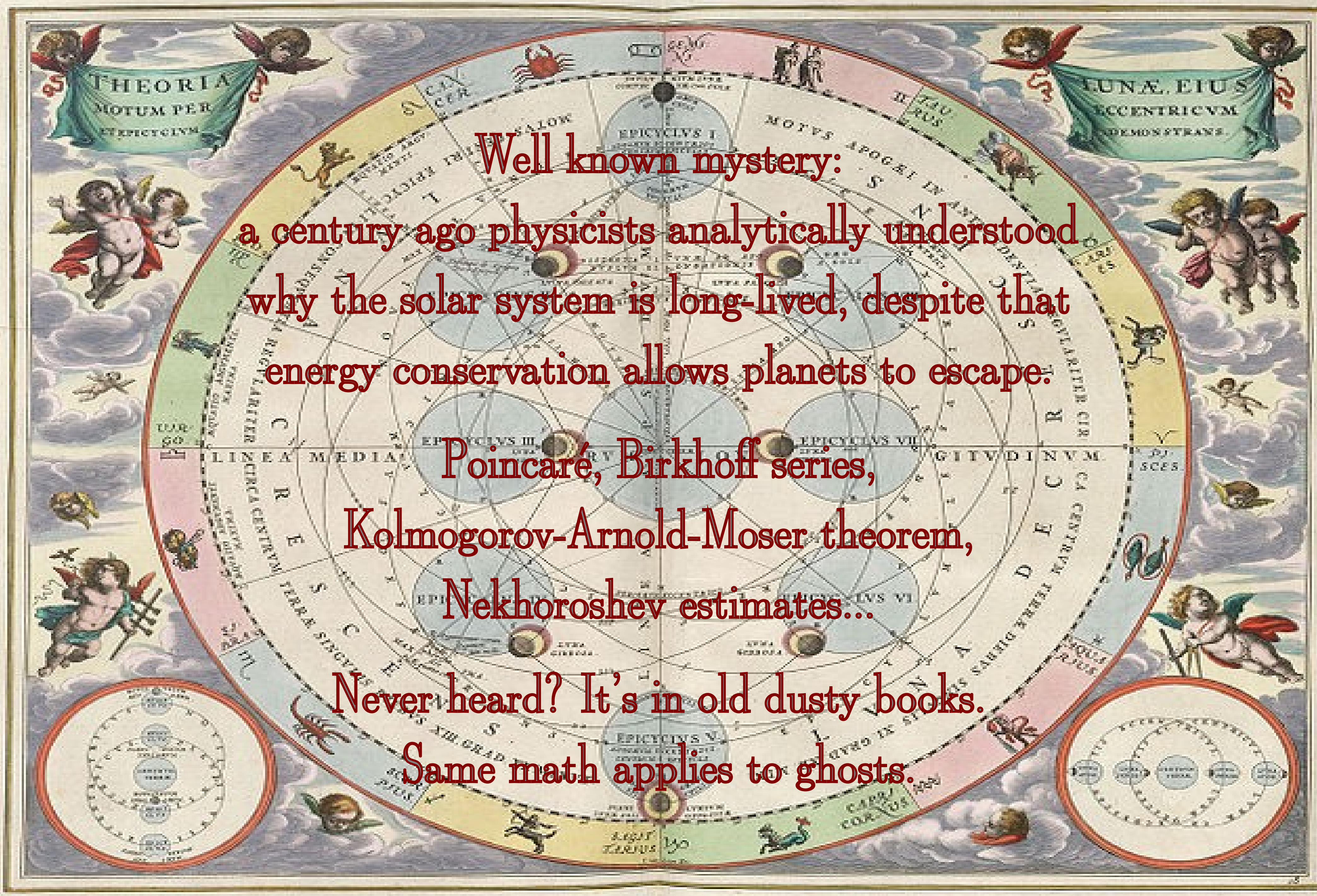
Ghost lockdown

One 4-derivative $q(t)$ can be rewritten as two 2-derivative $q_{1,2}(t)$, Ostrogradski-like:

$$H = E_1 - E_2 + V \qquad E_i = \omega_i \frac{p_i^2 + q_i^2}{2} \qquad V = \frac{\lambda}{2} q_1^2 q_2^2$$

H constant, no extra constant of motion prevents run-away. Energies E_1, E_2 grow and could go everywhere, but instead remain confined to a region if the coupling λ is small





Well known mystery:

a century ago physicists analytically understood why the solar system is long-lived, despite that energy conservation allows planets to escape.

Poincaré, Birkhoff series,

Kolmogorov-Arnold-Moser theorem,

Nekhoroshev estimates...

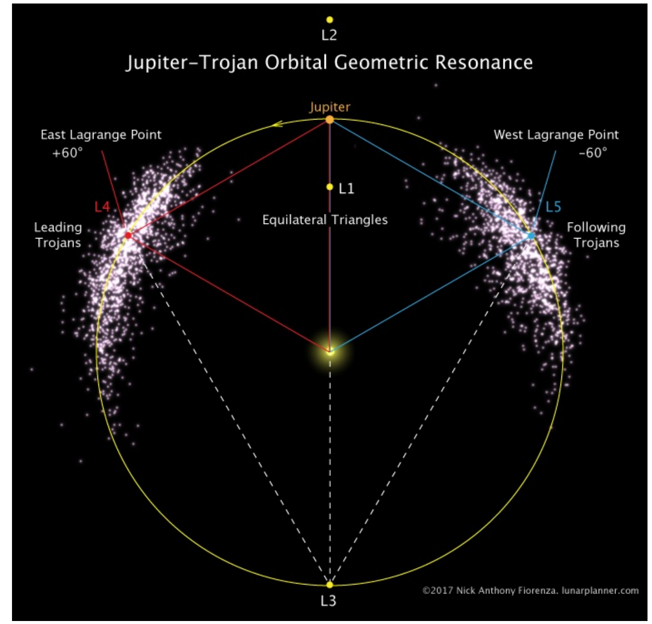
Never heard? It's in old dusty books.

Same math applies to ghosts.

Some physical systems are ghosts

Asteroid around the Lagrange point L4 e.g. Sun/Jupiter:

$$H = \frac{\vec{p}^{\,2}}{2m} + \omega(yp_x - xp_y) - \frac{GM_S m}{|\vec{x} - \vec{x}_S|} - \frac{GM_J m}{|\vec{x} - \vec{x}_J|}$$



in the rotating frame. Expand as quadratic + interaction around L4

$$H_2 = \frac{p_x^2 + p_y^2}{2} + yp_x - xp_y + \frac{x^2}{8} - \frac{5y^2}{8} + \frac{\sqrt{27}}{4}\left(\frac{2M_J}{M_{J+S}} - 1\right)xy$$

for $\omega = m = 1$. Diagonalise through a canonical Sp rotation:

$$H_2 = \omega_1 \frac{p_1^2 + q_1^2}{2} - \omega_2 \frac{p_2^2 + q_2^2}{2}.$$

Check, asteroids are still there!

Similarly for an electron rotating in a constant magnetic field B_z with a destabilizing potential ω_0^2

$$H = \frac{(\vec{p} - e\vec{A})^2}{2m} + e\varphi = \frac{\vec{p}^{\,2}}{2m} + \omega_B(yp_x - xp_y) + \frac{m}{2}(\omega_B^2 - \omega_0^2)(x^2 + y^2) \qquad \omega_B = \frac{eB_z}{2m}.$$

Birkhoff series

'Diagonalize' a classical Hamiltonian trough a canonical transformation from (q, p) to action-angle variables (J, Θ) such that

$$H(p_i, q_i) = H'(J_i)$$

makes motion trivial: $J_i = \text{cte}$ and $\Theta_i \propto t$. Harmonic oscillator:

$$q = \sqrt{\frac{2J}{m\omega}} \sin\Theta, \qquad p = \sqrt{2m\omega J} \cos\Theta.$$

Add small interactions, compute a perturbative Birkhoff series. Summarising books in 2 lines:

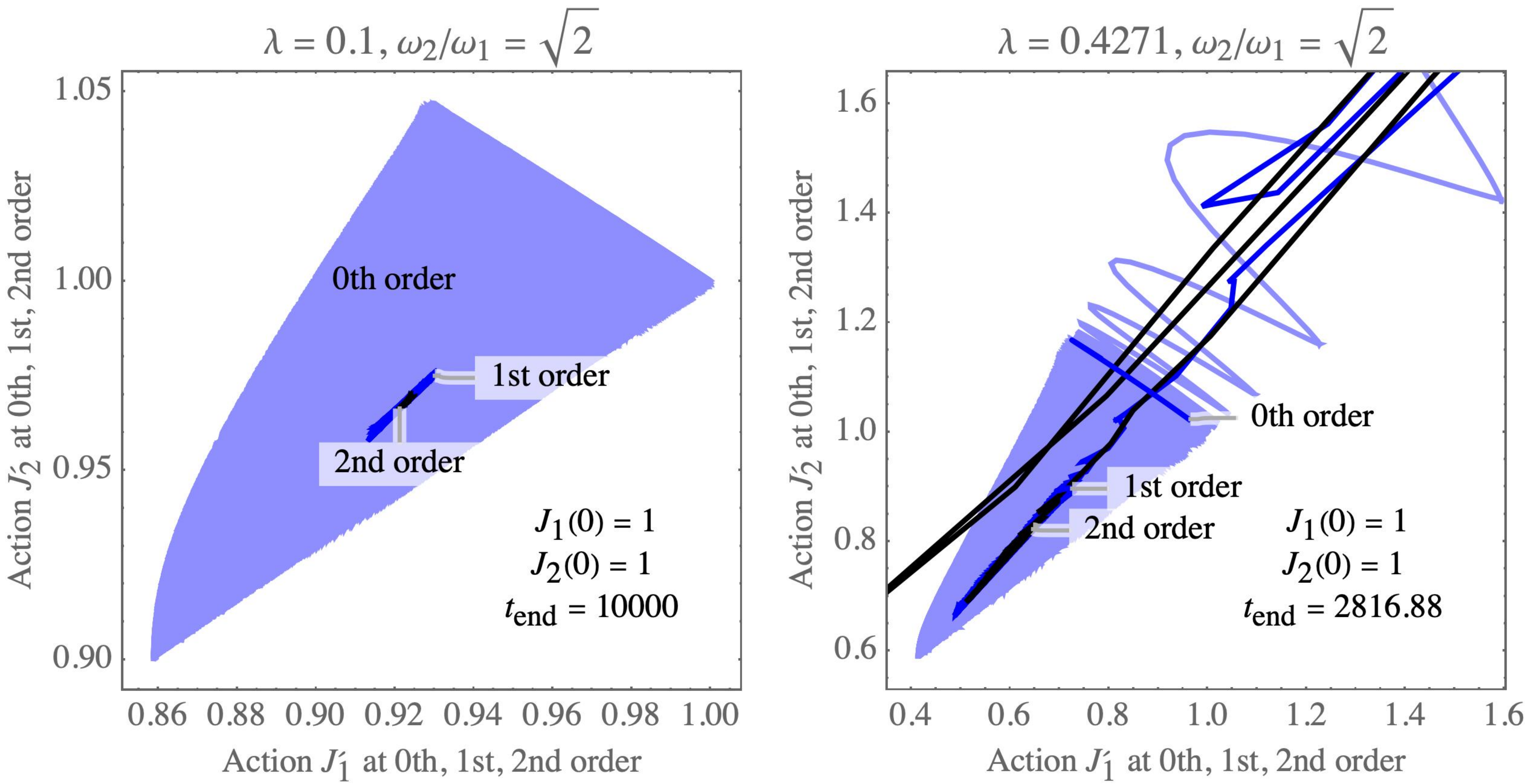
Smaller coupling $\tilde{\Rightarrow}$ Birkhoff series converges $\Rightarrow$ planets epycicle and stay, ghosts don't runaway.
Larger coupling $\tilde{\Rightarrow}$ Birkhoff series division divereges $\Rightarrow$ planets motion chaotic and escape, ghosts runaway.

A free ghost is good. A weakly coupled ghost remains good.

In practice: compute more and more orders making the residual interaction smaller $\lambda \to \lambda^2 \to \lambda^3 \ldots$

$$H(p_i, q_i) = H'(J_i) + \lambda^N H_{\text{int}}(\Theta, J)$$

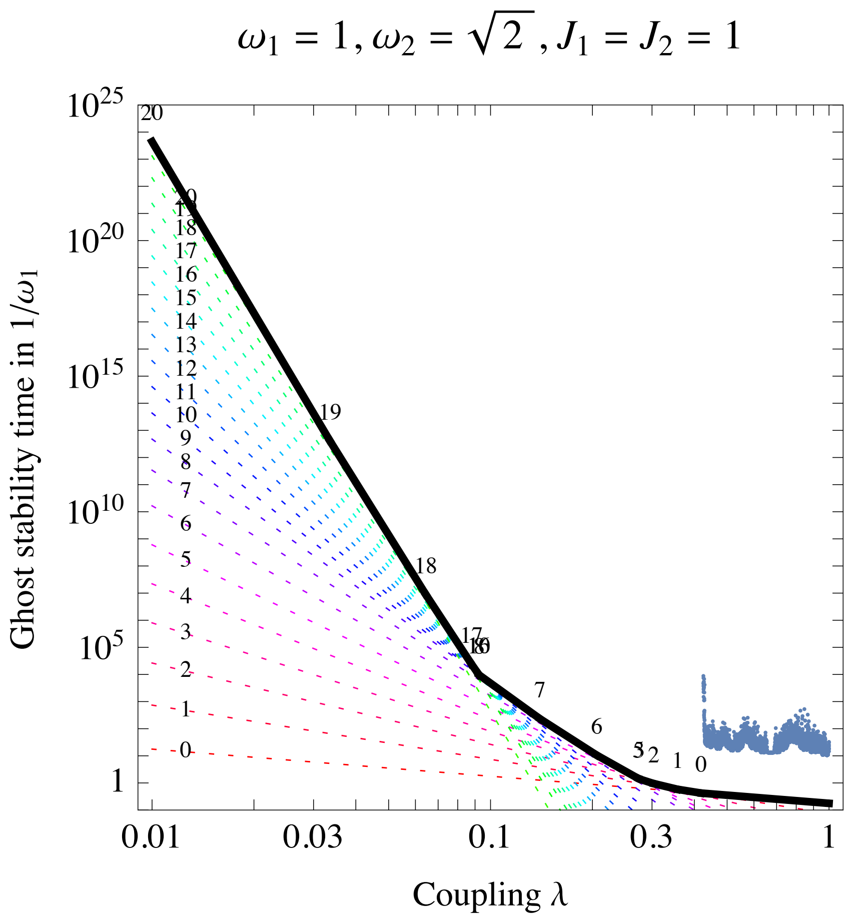
Example

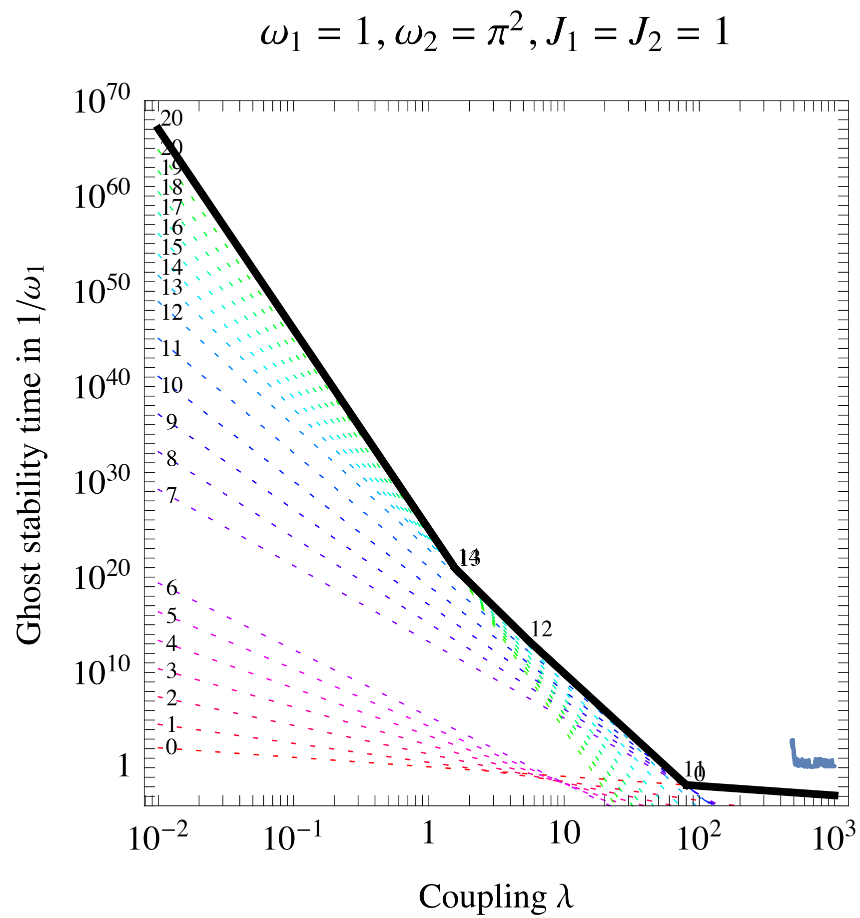


Like a hidden integral of motion

NNNNNNNNNNNNNNNNNNNNNNNNLO

Compute up to order n, small residual interaction λ^{n+1} allows escape after τ_n, compute $\tau > \max_n \tau_n$.
Asymptotic series: bound strongest for finite n depending on coupling λ: non-trivial function of λ.

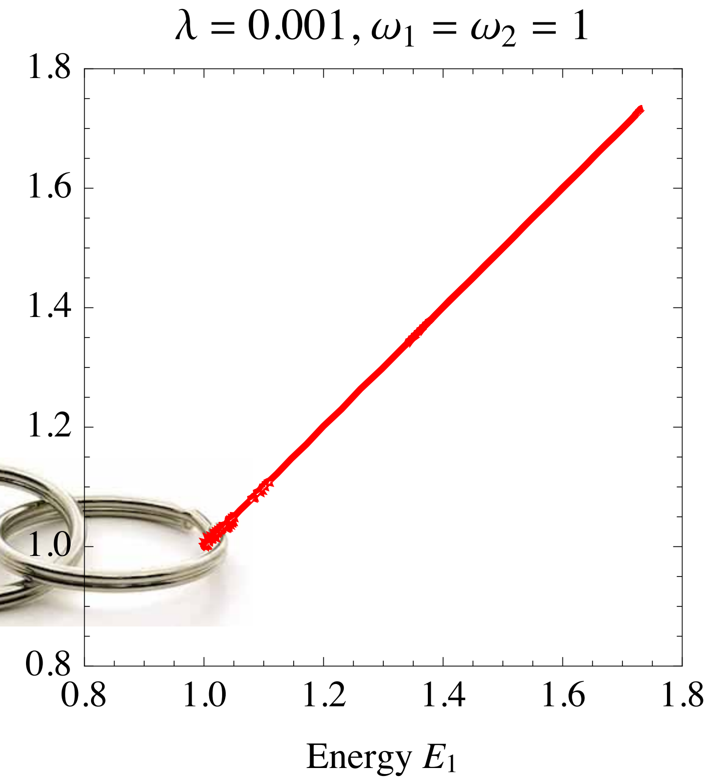




Order jumps in the figure: Birkhoff series contains $\lambda^{\cdots}/(N_1\omega_1 - N_2\omega_2)$ where $N_{1,2}$ are any integers.
It can get accidentally big even at small λ when **resonances** happen: planets escape, ghosts runaway.

Resonances

Worst resonance: leading order $N_1 = N_2 = 1$ i.e. $\omega_1 = \omega_2$. Hidden integrals of motion lost if $\Delta\omega \lesssim \lambda J/\omega_{1,2}^2$. To compute what happens: *resonant normal forms* i.e. another Arnold book i.e. diagonalise everything but the resonance. Leading order decides. A resonant $\lambda q_1^2 q_2^2$ interaction behaves as a **loose chain**



$$H' = \omega_1 J_1' - \omega_2 J_2' + \frac{\lambda J_1' J_2'}{2\omega_1\omega_2}\left[1 + \frac{1}{2}\cos 2(\Theta_1' + \Theta_2')\right] + \cdots =$$

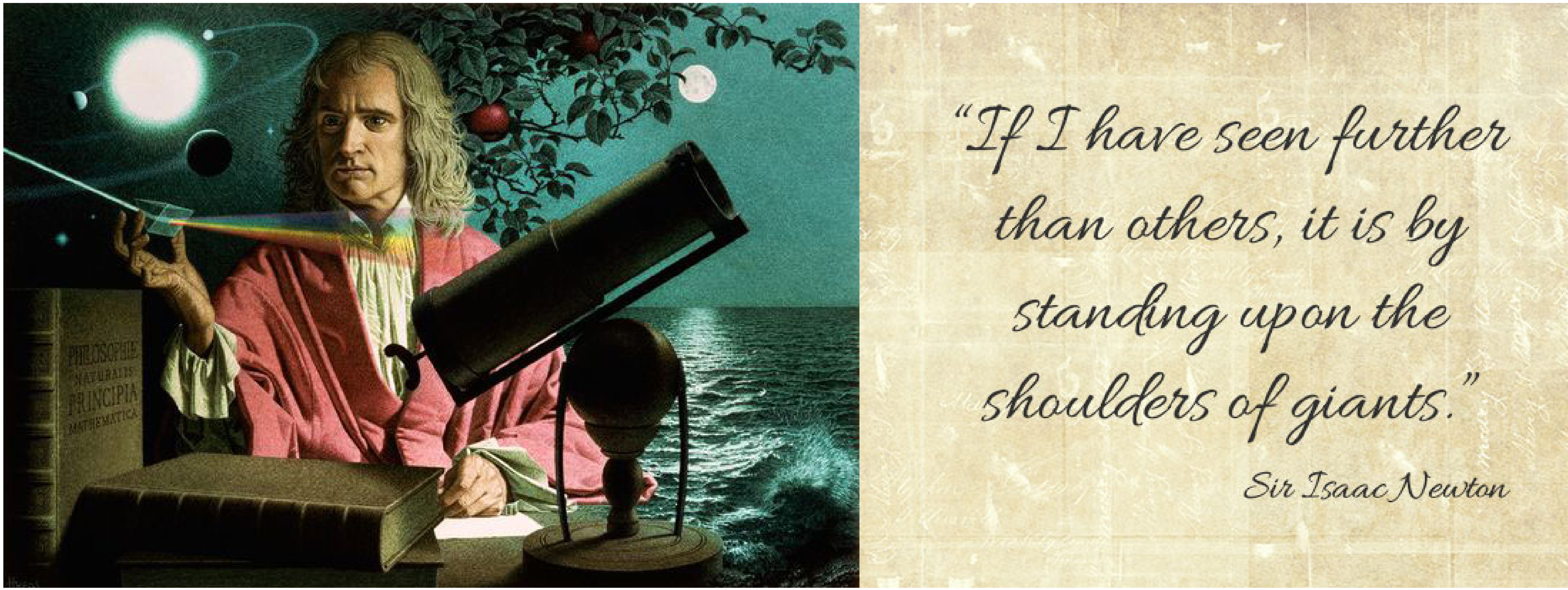
where $1/2 < 1$ means that $E_{1,2}$ vary by $\mathcal{O}(1)$ and come back.

A resonant $q_1^2 q_2$ interaction behaves as a **broken chain** i.e. run-away

$$H' = \omega_1 J_1' - \omega_2 J_2' - \frac{\epsilon}{4} J_1 \sqrt{J_2}\sin(2\Theta_1' + \Theta_2') \quad =$$

On resonance the fate depends on the model. Will be relevant in field theory.

Classical mechanics: ghosts can be meta-stable and exist



Now we get off and explore terra incognita.
Possibly without stomping on their graves.

2) Quantum mechanics

Quantum mechanics

Adding a ghost the sign of $E - V$ no longer tells if the wave function $\psi(q_1, q_2)$ oscillates or damps

$$H = \frac{p_1^2}{2} - \frac{p_2^2}{2} + V, \qquad V = \omega_1^2 \frac{q_1^2}{2} - \omega_2^2 \frac{q_2^2}{2} + \frac{\lambda}{2} q_1^2 q_2^2, \qquad -\frac{\hbar^2}{2}\frac{\partial^2 \psi}{\partial q_1^2} + \frac{\hbar^2}{2}\frac{\partial^2 \psi}{\partial q_2^2} = (E - V)\psi.$$

Does any ψ spread into $E_1 - E_2 \approx 0$ up to large E i.e. runaway?

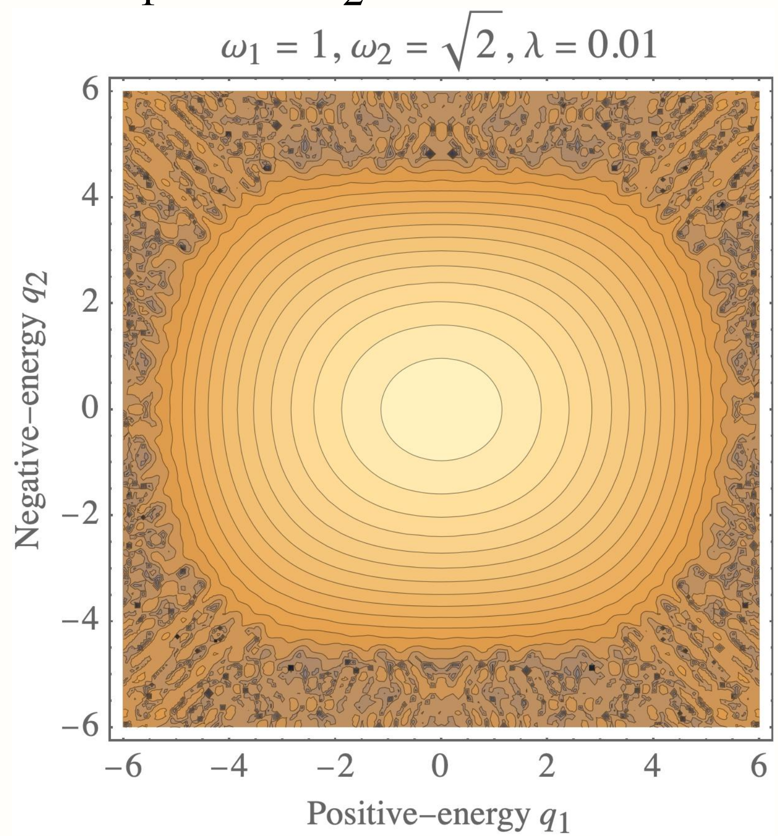
Compute numerically the bound ground state with no nodes

$$\text{around } q_{1,2} \sim 0: \ \psi(q_1, q_2) \sim e^{-(\omega_1 q_1^2 + \omega_2 q_2^2)/2\hbar}$$

Exponentially suppressed out-flowing probability current.

Meta-stable like a normal particle trapped in a potential barrier:

$$\text{e.g.} \quad H = \frac{p_1^2}{2} + \frac{p_2^2}{2} + V, \qquad V = \omega_1^2 \frac{q_1^2}{2} + \omega_2^2 \frac{q_2^2}{2} - \frac{\lambda}{2} q_1^2 q_2^2.$$



K-instability (ghost run-away) is exponentially suppressed like V-instability (tunnelling).

Differences appear in the resonant case $\omega_1 \neq \omega_2$.

Quantum mechanics: why?

Tunnelling can be approximated semi-classically as $\psi = e^{iS/\hbar}$, such that

$$\text{Schroedinger} \overset{\hbar\to 0}{=} \text{Hamilton-Jacobi} \quad \frac{\partial S}{\partial t} = -H(q_i, p_i = \frac{\partial W}{\partial q_i})$$

The classical action S and thereby the classical hidden integrals of motion still play a role, and keep ψ close to $q_{1,2} \sim 0$. Energy eigenstates: $S(q,t) = W(q) - Et$.

WKB $\approx$ HJ approximates multi-dof $q_i(t)$ tunnelling simpler than Schroedinger:

find the classical trajectory that minimises the barrier, $W = \min_{\vec{q}(t)} \int_0^{\vec{q}_{\rm release}} dq\,\sqrt{2V}$, rate $\propto e^{-2W}$.

But $(\partial W/\partial q)^2 = 2(E-V)$ has two solutions with opposite signs of W. Ground-like state at $\lambda = 0$:

- Usual HJ solution for q_1, bounce $W_1(q_1) = \lim_{t_E\to\infty} S$.
- Other sign for normalizable ghost $\psi(q_2)$: classical motion backwards in time, $W_2(q_2) = \lim_{t_E\to-\infty} S$

For $\lambda \neq 0$ we don't know how to use WKB to compute efficiently. And it's needed in QFT.

3) Classical Field Theory

Classical field theory

For example two scalars $\varphi_{1,2}$, simpler than $g^{\mu\nu}$ and its ghost $g_2^{\mu\nu}$. Typical theory:

$$\mathscr{L} = \frac{(\partial_\mu \varphi_1)^2 - m_1^2 \varphi_1^2}{2} - \frac{(\partial_\mu \varphi_2)^2 - m_2^2 \varphi_2^2}{2} - \frac{\lambda}{2} \varphi_1^2 \varphi_2^2 \, .$$

Classical field theory is sick even without ghosts. Wants to equipartition energy among infinite modes giving black body divergence cut at $\omega \lesssim T/\hbar$. Ghost is co-morbidity. Classical can be computed:

Numeric. Classical field theory can be computed numerically on smart light-cone lattice.

Analytic. Expand field $\varphi(\vec{x}, t)$ as Fourier modes $q_{\vec{n}}(t)$ to use Birkhoff & co

$$\varphi(\vec{x}, t) = \frac{1}{L^{d/2}} \sum_{\vec{n}=-\infty}^{\infty} q_{\vec{n}}(t) e^{i\vec{k}\cdot\vec{x}} \qquad \vec{k} = \frac{2\pi\vec{n}}{L} \qquad \omega_n^2 = m^2 + k^2$$

Resonances

Off-shell processes don't runaway. But lots of q_n with frequencies ω_n allow for lots of resonances. These on-shell processes are the usual decays, scatterings, etc.

- Resonant normal form of the complicated interaction $q_{n_1} q_{n'_1} q_{n_2} q_{n'_2}$ shows that local interactions like $\lambda\, \varphi_1^2 \varphi_2^2$ keep ghosts in chain

$$H \simeq \omega_{n_1} J_{n_1} + \omega_{n'_1} J_{n'_1} - \omega_{n_2} J_{n_2} - \omega_{n'_2} J_{n'_2} + \frac{\epsilon}{4}\Big[\Big(\frac{J_{n_1} J_{n_2}}{\omega_{n_1}\omega_{n_2}} + \frac{J_{n_1} J_{n'_2}}{\omega_{n_1}\omega_{n'_2}} + \frac{J_{n'_1} J_{n_2}}{\omega_{n'_1}\omega_{n_2}} + \frac{J_{n'_1} J_{n'_2}}{\omega_{n'_1}\omega_{n'_2}}\Big) + 2\sqrt{\frac{J_{n_1} J_{n'_1} J_{n'_2} J_{n'_2}}{\omega_{n_1}\omega_{n'_1}\omega_{n'_2}\omega_{n'_2}}}\cos(\Theta_{n_1} + \Theta_{n'_1} + \Theta_{n_2} + \Theta_{n'_2})\Big].$$

  Each resonance is benign: does not allow run-away, but violates one hidden constant of motion at $\mathcal{O}(1)$.

- One field has $N = L/a$ dof, there are $2N$ hidden constants of motion, $\sim N^2$ resonances. In the continuum limit $N^2 \gg N$: energy can flow $\varphi_1 \leftrightarrow \varphi_2$. Pictorially, **too many ghosts escape from lockdown because locked by a loose chain**.

Ambiguous situation: something good happens but not good enough?

Ghost entropy

Assume worst case scenario: mess wins, runaway possible, entropy $S_2 = N_{\rm dof} \ln |T_2|$ decides what happens. $S = S_1 + S_2$ is maximal for run-away $T_1 \to \infty$ and $T_2 \to -\infty$. Intuition: ghost with $T_2 < 0$ cannot thermalise with normal $T_1 > 0$ towards same T. Heat flows in the direction where $|T_{1,2}|$ grow.

To see how fast solve *classical* Boltzmann eq.s for generic $f(E)$ beyond the thermal limit.

(To start: $\varphi_{1,2}$ positive-energy; *quantum* Boltzmann equations are well known e.g. for $12 \leftrightarrow 1'2'$

$$\dot\rho_1 = -\int d\vec{k}_1 d\vec{k}_2 d\vec{k}'_1 d\vec{k}'_2\, E_1\, (2\pi)^{d+1} \delta(K_1 + K_2 - K'_1 - K'_2) |\mathscr{A}|^2 F \qquad \mathscr{A} = 2\hbar\lambda$$

$$F = f_1(E'_1) f_2(E'_2)[1 + f_1(E_1)][1 + f_2(E_2)] - f_1(E_1) f_2(E_2)[1 + f_1(E'_1)][1 + f_2(E'_2)]$$

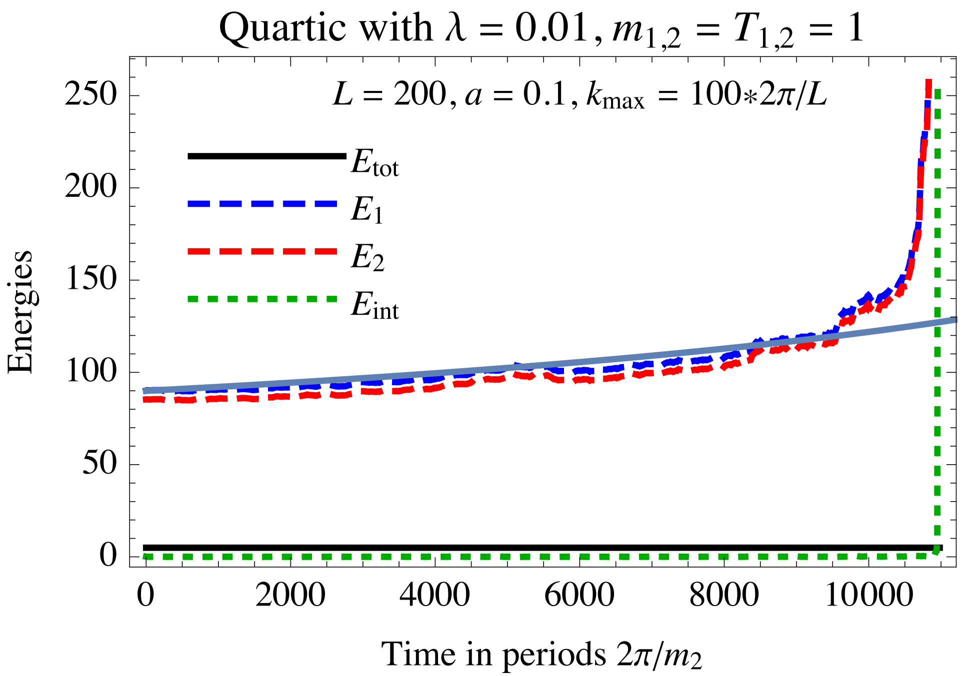
Bose-Einstein $f = 1/[e^{E/T} - 1]$ at equilibrium. Two classical limits: particle ($f \simeq e^{-E/T} \ll 1$, ignore) and wave ($f \simeq T/E \gg 1$). Thermalization rate $\dot T_1 \propto \lambda^2 T_1 T_2 (T_2 - T_1)$ agrees with numerics).

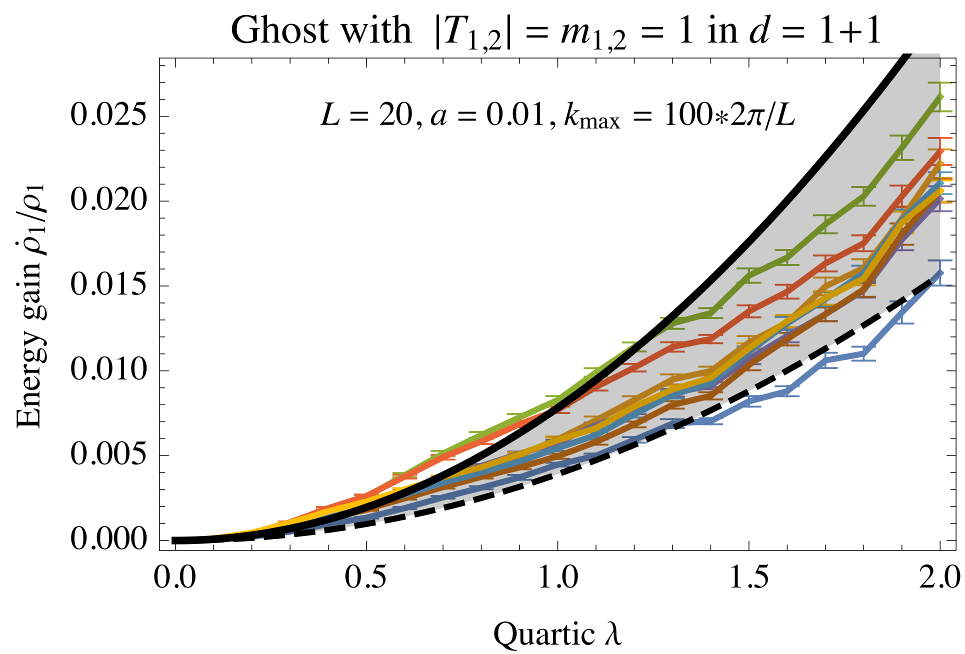
Ghost runs away in classical field theory

Next compute the ghost. Kinematics with $E < 0$ looks unusual. Trick: $\dot{\rho}_1$ remains the same using

$$\tilde{K}_\mu = -K_\mu \qquad f(E/T) = -[1 + f(\tilde{E}/T)]$$

i.e. (emission of negative energy) $\leftrightarrow$ (absorption of positive energy). No thermal equilibrium, run-away rate equals the heat flow rate $\propto \lambda^2$, not exponentially suppressed $e^{-1/\lambda}$. Analytic $\approx$ numerics:





Not a problem in 4∂ gravity: $\dot{T}/T \sim T^3/M_{\rm Pl}^2 \ll H \sim T^2/M_{\rm Pl}$. And $T \sim H_{\rm infl}$ during inflation

4) Relativistic Quantum Field Theory

Relativistic Quantum Field Theory?

Rate for $\emptyset \leftrightarrow 11'22'$ etc from Bolztmann equation in the limit $T_1 \to 0^+$, $T_2 \to 0^-$: $F \to -1 \neq 0$ so

$$\dot{\rho}_1 = \int d\vec{k}_{\rm all}\, E_1\, (2\pi)^{d+1} \delta(K_1 + K_1' - \tilde{K}_2 - \tilde{K}_2')|\mathscr{A}|^2 = \text{coupling}^2 \cdots \int_{\sqrt{s}}^{\infty} dE\, E\, (E^2 - s)^{\frac{d}{2}-1}$$

contains a divergent dE integral over the Lorentz group, $E = (K_1 + K_1')_0$. Needed because $\emptyset$ is Lorentz invariant. Ghost production rate is infinite: large enough that K-instability is excluded?

Same problem in old computations of vacuum decay [Okun et al.]: the critical bubble can be produced with any initial speed. And $V < 0$ allows ghost bubbles with $m < 0$. Is V-instability excluded?

Later, Coleman argued that $\Gamma_{V\text{-tunnelling}} \propto \exp(-W)$ of O(4) invariant bounce action, $W \sim 1/\lambda$.

Could K-instability similarly be exponentially suppressed?

- Dvali [1107.0956] doubts about V meta-stability.
- Coleman extended MQ to QFT using a simple WKB formula that does not generalize to ghosts. We don't yet know, not enough brute force to compute QFT $\to$ MQ as $\varphi_i(r)$ checking resonances.

Conclusions: can negative kinetic energy be ok?

Newton stopped at 2 derivatives: $F = ma$. More make quantum gravity renormalizable, give ghosts.

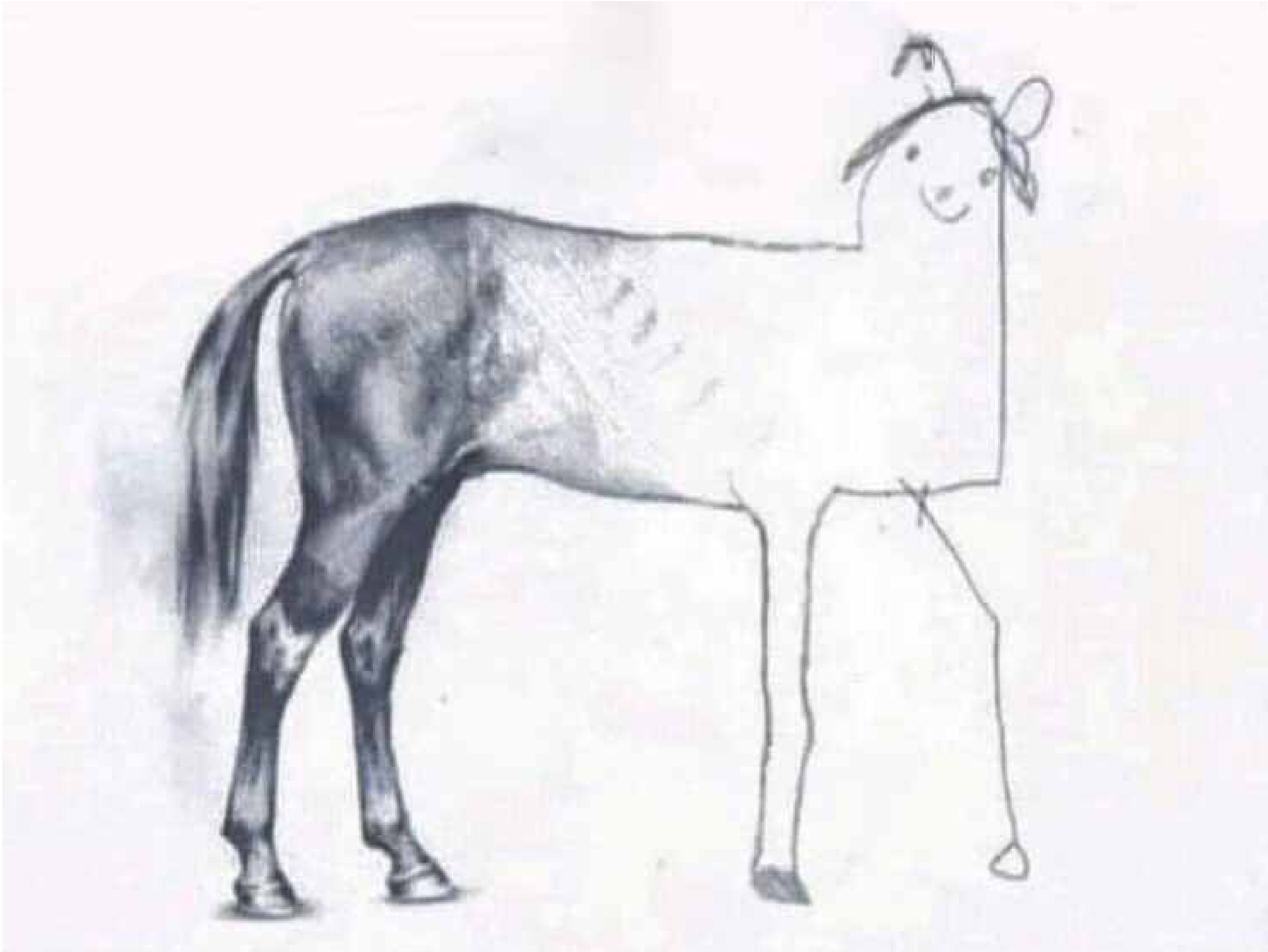
- **Classical mechanics**. Weakly-coupled ghosts could runaway but instead undergo lockdown, can be meta-stabile up to cosmologically large time. Seen in physical systems like Trojan asteroids. Resonances allow for partial or total energy flow.

- **Classical field theory**. Too many ghosts escape lockdown: infinite benign resonances give energy flow. No thermal state, heat flows from negative $T_{\rm ghost}$ to positive $T_{\rm normal}$ because entropy maximal for $|T| \to \infty$. Rate $\propto$ coupling^2, small in agravity where coupling $\sim E/M_{\rm Pl}$.

- **Quantum mechanics**: K-instability exponentially suppressed like V-instability if $\omega_1 \neq \omega_2$. But we don't know how to simply compute *à la* WKB.

- **Quantum field theory**: ghost runaway rate $\propto \lambda^2 \times$ (divergent integral over Lorentz group). Might signal analogous of Coleman O(4) bubbles?

Answer so far: $|\text{yes}\rangle + |\text{no}\rangle$



What emerges seems a less dangerous animal than expected. Lockdown is over, picture to be completed, we got the back and so far it smells good. It's quiet but could get wild and kick.