Accessing the QCD Conformal Window with Perturbation Theory and Beyond

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Introduction and Motivation

Understanding the phases of gauge theories is one of the most interesting problems in high energy and condensed matter physics.

We focus on non-abelian gauge theories in d=4 dimensions with fermions in the fundamental representation of $SU(n_c)$, which we will denote as QCD.

Depending on the number of flavours and colors, QCD can be UV-free or not, and flow in the IR to a gapped phase (confining and/or symmetry breaking) or to a gapless (Conformal Field Theory) phase.
At fixed $n_c$ (e.g. $n_c = 3$)

Gapped Phase       Conformal Phase       UV Freedom Lost

At fixed $n_c$ (e.g. $n_c = 3$)

$$n_f^* = ?$$

$$n_f^+ = \frac{33}{2}$$
A way to detect a conformal behaviour is to look for fixed points of the beta-function of the theory.

Our aim is to start from the upper edge of the conformal window and go down as much as possible using perturbation theory.

We will be using Borel resummation techniques and other tools.

A large part of the talk will be a review of the main properties of the methods used, in particular about asymptotic series and their possible Borel resummation.
Asymptotic series and Borel resummations

Large order behaviour: instantons and renormalons

Nature of the series of RG functions in 4d non-abelian gauge theories

Go back to the QCD conformal window

Results

Conclusions
Asymptotic Series and Borel Resummations

Recall basic mathematical properties of power series of holomorphic functions.

If $f(\lambda)$ analytic at a point $\lambda_0$, in a small disc around $\lambda_0$

$$f(\lambda) = \sum_{n=0}^{\infty} c_n (\lambda - \lambda_0)^n$$

Radius of convergence

$$R = \lim_{n \to \infty} \left| \frac{c_n}{c_{n+1}} \right|$$

If $\lambda_0$ regular point of $f$  

$$R \neq 0$$

If $\lambda_0$ singular point, we generally expect $R = 0$
Perturbative expansions in QFT are generally asymptotic with zero radius of convergence [Dyson, 1952]

A modern generalization of Dyson’s argument can be obtained by considering QFT in euclidean signature in a path integral approach.

Consider the loopwise expansion in $\hbar$.

Observables of interest are the n-point correlation functions of local operators.

$$G^{(n)}(x_1, x_2, \ldots x_n; \hbar) = \int \mathcal{D}\phi \phi(x_1)\phi(x_2)\ldots\phi(x_n) e^{-S(\phi)/\hbar}$$

For $\hbar \to 0^-$ Green functions blow up $\rightarrow \hbar = 0$ is non-analytic

The loopwise expansion has zero radius of convergence!

Upon rescalings the loopwise expansion is equivalent to a proper expansion in the coupling constant: $S = S_0 + g\Delta S$
Mathematically perturbation theory does not make sense!

\[ G^{(n)} = \int \mathcal{D}\phi \phi_1 \ldots \phi_n \sum_{p=0}^{\infty} \frac{(-\Delta S)^p g^p}{p!} e^{-S_0} \neq \sum_{p=0}^{\infty} g^p \int \mathcal{D}\phi \phi_1 \ldots \phi_n \frac{(-\Delta S)^p}{p!} e^{-S_0} = \sum_{p=0}^{\infty} g^p G_p^{(n)} \]

The series is never uniformly convergent and not allowed to exchange sum and integration

Asymptotic series

\[ Z(\lambda) \sim \sum_{n=0}^{\infty} Z_n \lambda^n \quad \text{such that} \]

\[ Z(\lambda) - \sum_{n=0}^{N} Z_n \lambda^n = \mathcal{O}(\lambda^{N+1}), \quad \text{as} \quad \lambda \to 0 \]

Equality sign not allowed: in general \( \sum_{n=0}^{\infty} Z_n \lambda^n = \infty \)

Contrast with convergent series: \( \forall \lambda \in \mathcal{D} \)

\[ N \to \infty \quad Z(\lambda) - \sum_{n=0}^{N} Z_N \lambda_N \to 0 \]
If $\lambda$ is small enough, asymptotic series capture accurately the exact result. This explains the spectacular success of high precision computations in quantum field theories, such as the computation of $g$-2 in QED.

Care has to be paid with asymptotic series, where summing more and more terms is not a good idea. Generally in QFT for $Z_n \sim n!a^n$ best accuracy for $Z(\lambda)$ is obtained by keeping $N_{\text{Best}} \approx \frac{1}{a\lambda}$ terms.

The higher the coupling the less terms you should compute. Intrinsic error associated to the asymptotic series $\sim e^{-\frac{1}{a\lambda}}$.

E.g. $Z(\lambda)$ and $Z(\lambda) + f(\lambda) \exp(-c/\lambda^n)$ have identical asymptotic series. Extra input is needed to possibly resum the series using resummation methods.

Most useful method is so called Borel resummation.
Borel Resummation

Divide original series by a factorial term to get a convergent series

$$BZ(t) = \sum_{n=0}^{\infty} \frac{Z_n}{n!} t^n \quad Z_B(\lambda) = \int_{0}^{\infty} dt \, e^{-t} BZ(t\lambda)$$

Warning: do not expand and exchange sum with integral

$$\int_{0}^{\infty} e^{-t} BZ(\lambda t) = \int_{0}^{\infty} e^{-t} \sum_{n=0}^{\infty} \frac{Z_n}{n!} \lambda^n t^n \neq \sum_{n=0}^{\infty} Z_n \lambda^n \frac{1}{n!} \int_{0}^{\infty} e^{-t} t^n = \sum_{n=0}^{\infty} Z_n \lambda^n$$

Borel series has generally finite radius of convergence, so operation not allowed (and we would get back to the starting point!)

But $BZ(t)$, if known, can be analytically continued over the whole complex t-plane (Borel plane) and $Z_B(\lambda)$ can then be calculable
The chances of success of this resummation procedure can be guessed from the form of the asymptotic series

Example: for \( Z_n \sim n!a^n \), \( BZ(t) = \sum_n (at)^n \sim \frac{1}{1 - at} \)

Singularity dangerous or not depending on the sign of \( a \):

- \( a<0 \) (alternating series) singularity for \( t < 0 \)
- \( a>0 \) (same sign series) singularity for \( t > 0 \)

Sometimes an alternating series is said to be Borel resummable

This is misleading, because as we have seen there is no way to recover the original function with no further input, not being uniquely defined

Further input requires to know the analyticity properties of the function close to the origin
It is in general hard to argue the analyticity property of $Z$, which is unknown!

In special cases, like in certain 2d or 3d scalar field theories, perturbation theory has been proved to be Borel resummable

[Early ‘900 theorem, rediscovered in 1980!]

$Z_B(\lambda) = Z(\lambda)$

Unfortunately interesting theories such as 4d gauge theories are not expected to be Borel resummable, at least in a simple way.
When a perturbative series is not Borel resummable because of a singularity in the real positive axis, one might deform the contour. The result however presents a non-perturbative ambiguity of order \( \exp(-a/\lambda^n) \).

If one is able to find a semi-classical instanton like configuration (and its whole series) leading to the same factors, one might hope to remove the ambiguity.

A systematic way to proceed along these lines uses the theory of resurgence [Ecalle, 1981].

Some progress has been achieved in the last years using the theory of resurgence to relate classical and instanton series among each other but it is unfortunately hard to pursue this direction in generic QFTs [Review by Aniceto, Basar, Schiappa, 1802.10441].

Moreover, non-ambiguous non-perturbative contributions might be present, which would be "invisible" in perturbation theory.

On the other hand, Borel resummation (no need of resurgence) can be useful even when we do not know if the series is resummable or not, and we do not know its large order behaviour.

Indeed, while optimal truncation is given by the singularity closest to the origin in the Borel plane (wherever it is), the non-Borel summability of a series is determined by its singularities on the positive real axis.
Optimal truncation error
\[ t_0 = \frac{1}{a} \]
\[ t_1 \]
\[ t_1 > |t_0| \]

Indeterminacy in the Borel function
\[ \sim e^{-\frac{|t_0|}{\lambda}} \]
\[ \sim e^{-\frac{t_1}{\lambda}} < e^{-\frac{|t_0|}{\lambda}} \]

It is useful to Borel resum a series even when it is not resummable!

We will use this result when going back to the conformal window in QCD.
Instantons and Renormalons

In QFT series are asymptotics because of factorial growth of their coefficients.

There are two known sources for the factorial growth:

1) **O(1) contributions of O(n!) Feynman diagrams**

These are governed by *complex instantons* whose action determine the factor $a$. If *real instantons* are also present, perturbative series is not Borel resummable because of singularities appearing on the positive real axis of the Borel plane.

[Vainhstein 1964, Lam 1968, Bender&Wu 1969, Lipatov 1976, …]

2) **O(n!) contributions of O(1) Feynman diagrams**

Generally expected in QFT with marginal couplings. Not known (if any) the real or complex semi-classical configurations associated. Singularities in the Borel function are called *renormalons* and are related to the RG flow of the marginal couplings.

Nature of the series of RG functions in 4d non-abelian gauge theories

Perturbative gauge coupling expansion of physical observables in 4d gauge theories is generally asymptotic and non-Borel resummable

Singularities in the positive real axis due both to instantons (or instanton-antiinstanton pairs) and renormalons (more precisely IR renormalons)

Not much is known about RG functions such as beta-function or anomalous dimensions (i.e. of fermion bilinear)

These functions are renormalization scheme dependent so we expect that their large order behaviour depends on the scheme.

Established results only in the limit of large number of flavours

\[ n_f \to \infty, \quad \alpha \to 0 \quad \lambda = n_f \alpha \quad \text{fixed} \]

In this limit the \( \mathcal{O}(1/n_f) \) terms in \( \beta(\lambda) \) and \( \gamma(\lambda) \) can be computed exactly.

[Palanques-Mestre, Pascual, 1984; Gracey, 1996]
In particular in $\overline{\text{MS}}$

$$\beta(\lambda) = \frac{2}{3} \lambda^2 + \frac{1}{n_f} \beta^{(1)}(\lambda) + O\left(n_f^{-2}\right)$$

$$\gamma = \frac{1}{n_f} \gamma^{(1)}(\lambda) + O\left(n_f^{-2}\right)$$

$\beta^{(1)}$, $\gamma^{(1)}$, known functions of $\lambda$

Interestingly enough, these are analytic at $\lambda = 0$, which implies a convergent series!

In other more physical schemes, such as on-shell or momentum subtraction, the series are divergent asymptotic

For finite number of flavours and colors, and in other large $n$ limits ('t Hooft and Veneziano) the nature of the $\overline{\text{MS}}$ series for $\beta$ and $\gamma$ is unknown

't Hooft:  $n_c \rightarrow \infty$  $\alpha \rightarrow 0$  $\lambda = n_c \alpha$  fixed

Veneziano:  $n_c \rightarrow \infty$  $n_f \rightarrow \infty$  $\alpha \rightarrow 0$  $x = \frac{n_f}{n_c}$  $\lambda = n_c \alpha$  fixed
QCD Conformal Window

After this long preparatory journey, we can come back to our original problem of studying the conformal window in QCD.

Of course we can’t have a conformal window in the large flavour or large color limit, but we can and do have a conformal window in the Veneziano limit. So we will also consider this limit of QCD.

Previous (non-lattice) results

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<tr>
<th>$n_f$</th>
<th>Method</th>
<th>References</th>
</tr>
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<tr>
<td>12</td>
<td>Schwinger-Dyson</td>
<td>[Appelquist et al, hep-ph/9602385+]</td>
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<tr>
<td>10</td>
<td>Exact RG</td>
<td>[Gies, Jaeckel, hep-ph/0507171+]</td>
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<tr>
<td>9</td>
<td>Perturbation Theory</td>
<td>[Ryttov,Shrock, 1607.06866]</td>
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<tr>
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<td>Conformal Expansion</td>
<td>[Kim, Hong, Lee, 2001.02690]</td>
</tr>
<tr>
<td>9</td>
<td>Padè/Resummations</td>
<td>[Ryttov,Shrock,2018; Antipin et al, 2019]</td>
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</table>
Veneziano

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<th>$x^*$</th>
<th>Method</th>
<th>References</th>
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<td>Schwinger-Dyson</td>
<td>[Appelquist et al, hep-ph/9602385+…]</td>
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</tr>
<tr>
<td>3</td>
<td>Padè Pert. Theory</td>
<td>[Ryttov, Shrock, 1710.06944]</td>
</tr>
<tr>
<td>3.7 ÷ 4.2</td>
<td>Bottom-up holographic</td>
<td>[Jarvinen, Kiritsis, 1112.1261+ …]</td>
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Previous results made predictions without an estimate of the associated error, so it is hard to assess their reliability.

We will argue that it is too optimistic to hope to reach the lower end of the conformal window using perturbation theory techniques only
Waiting for the conformal bootstrap, lattice is the only first principle method to address the QCD conformal window.

This is a hard problem in the lattice and no consensus yet on the value of $n_f^*$

[See e.g. review by DeGrand, 1510.05018]

In particular it is still debated whether QCD with $n_f = 12$ flavours is conformal or not

[Fodor et al, 1811.05024]

No lattice results are available in the Veneziano limit

**Our approach**

We start from the available 5-loop coefficients of the $\overline{\text{MS}}$ $\beta$-function

[Baikov, Chetyrkin, Kuhn, 1606.08659; Herzog et al, 1701.01404]

[Luthe et al, 1709.07718; Chetyrkin et al, 1709.08541]

We assume the worst case scenario in which the series diverges and is not-Borel resummable. Then we perform a Borel resummation using Padè approximants to reconstruct the whole Borel function (Padè-Borel method)

We then look for zeros of the Borel resummed beta-function
Remarks:

- It has been conjectured that conformality is lost because two nearby fixed points merge and disappear into the complex plane. In the Veneziano limit this implies that one (or more) double trace operator (presumably four fermion operators) approaches marginality

  \[ \text{[Kaplan et al, 0905.4752]} \]

- This is in agreement with earlier results using gap equations according to which in the IR \(|\gamma \bar{\psi} \psi| \approx 1\) for \(n_f = n_f^*\)

  Recall that \(\Delta_{\psi^4} \approx 2\Delta_{\psi^2}\) at large \(n\)

  [Yamawaki, Bando, Matumoto,1986; Appelquist, Lane, Mahanta,1988; Cohen, Georgi,1989 ]

- For this reason and also because this is an observable measured on the lattice, we will also compute the mass anomalous dimensions at the IR fixed point \((\gamma^*)\)
A fixed point can also be found using the Banks-Zaks conformal expansion

Assume that the coefficient of the first available order is accidentally small so that a “violation” of perturbation theory is allowed

\[ \beta(\alpha) = \beta_0 \alpha^2 + \beta_1 \alpha^3 + O(\alpha^4) \quad \alpha^* \approx -\frac{\beta_0}{\beta_1} \propto \epsilon \]

If \( \beta_0 < 0 \) and \( \beta_1 > 0 \) fixed point is IR stable \quad [\text{Caswell,1974; Banks,Zaks,1982}]

\[ \epsilon \propto (n_f^+ - n_f) \quad (\text{QCD}) \quad \epsilon \propto (x^+ - x) \quad (\text{Veneziano}) \]

\[ \alpha^* = \sum_{n=1}^{\infty} b_n \epsilon^n \]

\[ \gamma^* = \sum_{n=1}^{\infty} g_n \epsilon^n \]

\( g_n \) are scheme-independent coefficients

Conformal expansion series seems to be better behaved than ordinary coupling expansions, but its nature (convergent or not) follows that of the ordinary expansion

As double check, we also Borel resum the conformal expansion and compare the results with those obtained using the ordinary coupling series
Even if both the conformal and the ordinary series expansions would be non-Borel resummable, we expect an improvement with respect to optimal truncation

\[
\begin{align*}
\text{Optimal truncation error} & \sim e^{-\frac{|t_0|}{\lambda}} \\
\text{Indeterminacy in the Borel function} & \sim e^{-\frac{t_1}{\lambda}} < e^{-\frac{|t_0|}{\lambda}}
\end{align*}
\]

Perturbation theory of the 5-loop beta-function breaks down for \( n_f \leq 13 \)

<table>
<thead>
<tr>
<th>Loop Order</th>
<th>( n_f = 12 )</th>
<th>( n_f = 13 )</th>
<th>( n_f = 14 )</th>
<th>( n_f = 15 )</th>
<th>( n_f = 16 )</th>
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<tr>
<td>2</td>
<td>( 6 \cdot 10^{-2} )</td>
<td>( 3.7 \cdot 10^{-2} )</td>
<td>( 2.2 \cdot 10^{-2} )</td>
<td>( 1.1 \cdot 10^{-2} )</td>
<td>( 3.3 \cdot 10^{-3} )</td>
</tr>
<tr>
<td>3</td>
<td>( 3.5 \cdot 10^{-2} )</td>
<td>( 2.5 \cdot 10^{-2} )</td>
<td>( 1.7 \cdot 10^{-2} )</td>
<td>( 9.8 \cdot 10^{-3} )</td>
<td>( 3.2 \cdot 10^{-3} )</td>
</tr>
<tr>
<td>4</td>
<td>( 3.5 \cdot 10^{-2} )</td>
<td>( 2.7 \cdot 10^{-2} )</td>
<td>( 1.8 \cdot 10^{-2} )</td>
<td>( 1.0 \cdot 10^{-2} )</td>
<td>( 3.2 \cdot 10^{-3} )</td>
</tr>
<tr>
<td>5</td>
<td>( -5.5 \cdot 10^{-6} )</td>
<td>( 3.2 \cdot 10^{-2} )</td>
<td>( 1.9 \cdot 10^{-2} )</td>
<td>( 1.0 \cdot 10^{-2} )</td>
<td>( 3.2 \cdot 10^{-3} )</td>
</tr>
</tbody>
</table>

Table 1: Approximate values of the QCD Caswell-Banks-Zaks fixed point coupling \( a^* \) as a function of \( n_f \) obtained using different loop orders. The red color indicates values that should be taken with care, because of a possible breakdown of perturbation theory.

[from 2003.01742]
We do not expect that the perturbative \( \overline{\text{MS}} \) \( \beta \)-function (independently of whether it is convergent or divergent) can capture well the IR physics for any \( n_f^* \leq n_f \leq n_f^+ \)

\( \overline{\text{MS}} \) is a mass-independent scheme and no mass scale can enter in RG functions

**to all orders in perturbation theory**

Non-perturbatively, however, contributions proportional to

\[
\frac{\Lambda_{\text{QCD}}}{\mu} \approx e^{\frac{1}{\beta_0 \alpha}}
\]

could appear

By dimensional analysis an irrelevant operator with UV dimension \( 4 + k \) and dimensionless \( \overline{\text{MS}} \) coupling \( \hat{h} \) can appear in \( \beta \) function as

\[
\delta \beta \sim \hat{h} \left( \frac{\Lambda_{\text{QCD}}}{\mu} \right)^k = \hat{h} e^{\frac{k}{\beta_0 \alpha}}
\]

We expect leading contributions arise from \( k=2 \), due to four fermion operators. When these are sizable our analysis break down.

We add this systematic uncertainty to our error estimate, that includes the numerical error due to the reconstruction of the Borel function out of few perturbative coefficients
Large $n_f$ exact results are better reproduced by [2/2] approximant.
We see conformality at $n_f = 12$
Banks-Zaks conformal expansion

Evidence of conformality also for $n_f = 11$
Table 2: Comparison between the results of our Padé-Borel (PB) resummation for the mass-anomalous dimension for QCD with $n_f = 12$—both using the coupling expansion and the Banks-Zaks conformal expansion, and averaging over all available Padé approximants in each case—and lattice results.

| Ref. | $|\gamma^*|$ | Method       |
|------|-------------|--------------|
| This work | 0.320(85) | PB coupling |
|        | 0.345(47) | PB conformal |
| [44]  | 0.23(6)    | Gradient flow |
| [45]  | 0.47(10)   | Top. susceptibility |
| [46]  | 0.33(6)    | Dirac eigenmodes |
| [47]  | 0.32(3)    |               |
| [48]  | 0.235(46)  |               |
| [49]  | 0.235(15)  |               |
| [50]  | 0.45(5)    |               |
|       | 0.403(13)  |               |

Some care is needed before jumping too quickly to a conclusion. When $n_f < 12$ the contribution to the full error, which is sub-leading for higher values of $n_f$, becomes sizable. The results shown have $cnp = 10$. In both the ordinary coupling and conformal expansions, for $n_f > 13$ the choice of $cnp$ is essentially irrelevant, unless one considers unreasonably large values of this parameter. For $n_f = 12$ we have to take $cnp \sim 50$ to enlarge the error so that this is compatible with no fixed point in the coupling expansion. In the conformal expansion this value reaches $cnp \sim 5 \times 10^4$. We think that a non-perturbative correction of this size is unreasonable and that the evidence for a fixed point at $n_f = 12$ is overwhelming. On the other hand, the fixed point for $n_f = 11$ in the conformal expansion is compatible with no fixed point for $cnp \sim 50$, while in the ordinary coupling expansion the error is dominated by the contribution associated to the numerical reconstruction of the Borel function, namely it is compatible with no fixed point even if one takes $cnp = 0$. We take these results as evidence that $n_f = 11$ is conformal, but we do not consider it enough to make a strong claim.

If we trust this evidence, we can use the resummation of the conformal expansion to obtain $\gamma^* = 0.485(143)$ and $g^* = 0.36(19)$ (averaging all the available Padé's weighted by their errors and combining the errors in quadrature). Needless to say, we do not commit ourselves with an estimate for $n_f$.

We would finally like to conclude with a general observation about the use of Padé-Borel versus simple Padé approximants. In the former case, we would not expect a gain in considering Borel-resummation, because $a$ would be analytic at $a = 0$ and an ordinary Padé approximant on $a$ should suffice. On the contrary, for a convergent series the Borel function is analytic everywhere and expected to have an exponential behaviour at infinity, and functions of this kind are not well approximated by low-order Padé approximants. We have verified that by taking ordinary Padé approximants our results remain qualitatively unchanged, though Padé-Borel approximants give slightly more accurate results. This can be seen as a sort of indirect numerical evidence of the non-convergence of the $\text{MS}_\text{bar}$-function. See appendix D for a more

[44] - Carosso, Hasenfratz, Neil 1806.01385
[45] - Aoki et al. 1601.04687
[46] - Cheng, Hasenfratz, Petropoulos, Schaich, 1301.1355
[47] - Lombardo, Miura, Nunes da Silva, Pallante, 1410.0298
[48] - Cheng et al, 1401.0195
[49] - Aoki et al, 1207.3060
[50] - Appelquist et al, 1106.2148

[from 2003.01742]
Similar results have been derived in the Veneziano limit of QCD.

![Graph showing data points for \( \lambda^* \) and \( |\gamma^*| \) against \( x \).]
Conclusions and future perspectives

Perturbation theory is the most notable analytical tool to study QFTs.

Its natural regime of validity, by definition, is weak coupling, where it allows us to do accurate predictions, despite the generic non-convergence of the associated series.

However, computing a sufficient number of higher order terms allows us to estimate the Borel transform of the function and resum the series.

If the series is Borel resummable we get an approximation of the exact result valid beyond perturbation theory.

This is not the case if the series is **not** Borel resummable, yet we might do **better** than perturbation theory.
We have shown this in the study of the conformal window of QCD-like theories. In particular, we have ample evidence that QCD with $n_f = 12$ flavours (a debated case in the lattice community) flows in the IR to a CFT $n^{\ast} f = 12$

Note that we are not claiming that $n_f = 12$. Indeed we have some evidence that $n_f = 11$ is also conformal

Below that, perturbative methods break down and a genuine non-perturbative approach, like lattice or conformal bootstrap, is required
Future perspectives

1. Although our results point toward a divergent asymptotic nature of the \( \overline{\text{MS}} \) RG functions, it would be nice to firmly establish their nature.

2. Keep going in computing higher and higher loops, necessary for a better and better resummation and to get insights for point 1.

3. Generalizations to theories with other gauge theories or fermion representations (straightforward).
Thank You