Accessing the QCD Conformal Window with Perturbation Theory and Beyond

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based on 2003.01742, in collaboration with
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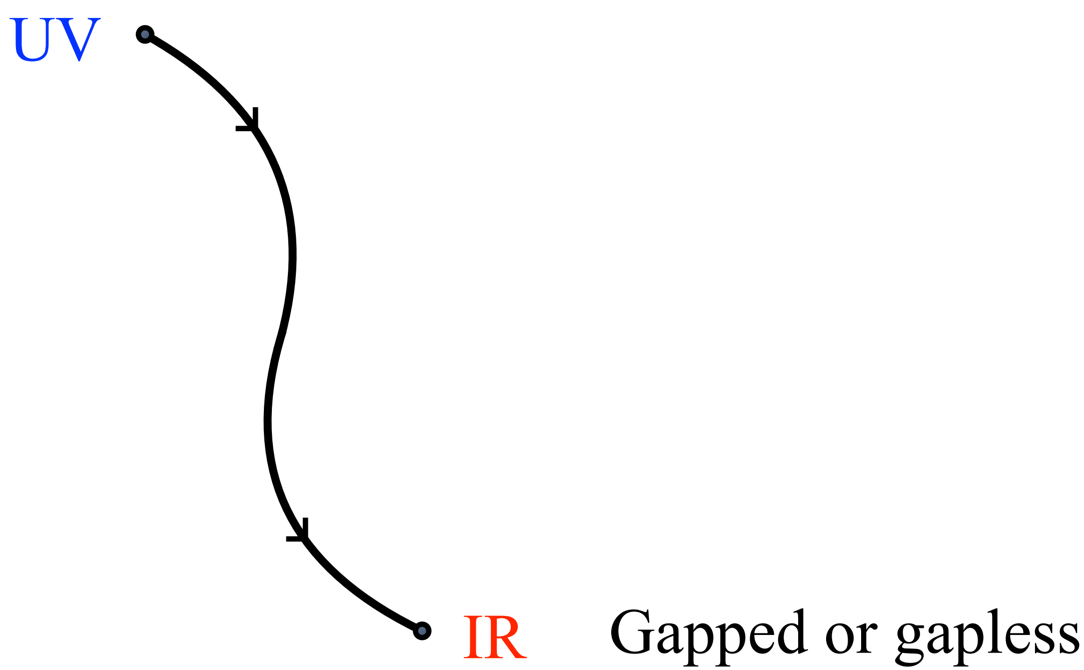
Introduction and Motivation

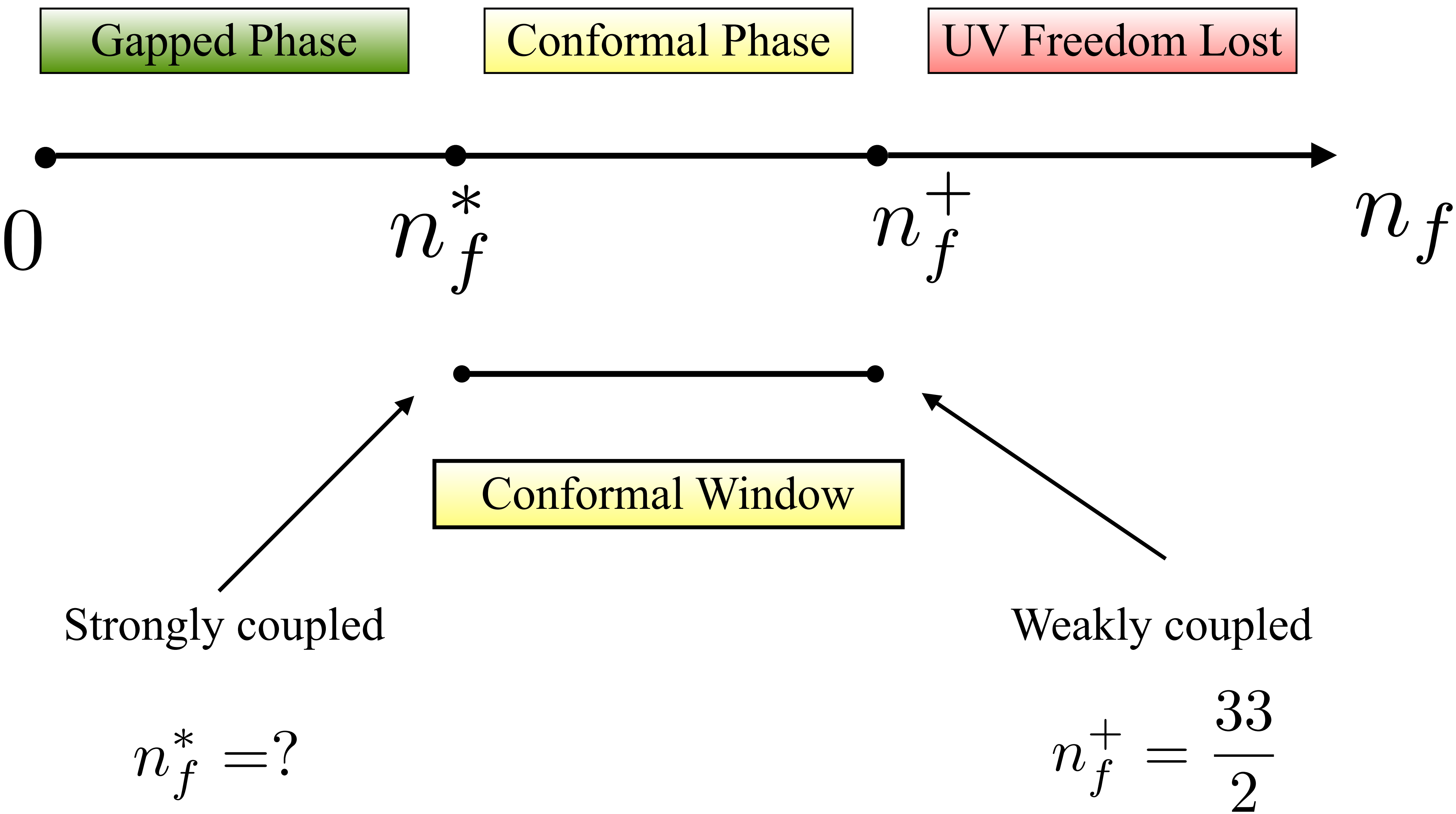
Understanding the phases of gauge theories is one of the most interesting problems in high energy and condensed matter physics

We focus on non-abelian gauge theories in d=4 dimensions with fermions in the fundamental representation of $SU(n_c)$, which we will denote as QCD

Depending on the number of flavours and colors, QCD can be UV-free or not, and flow in the IR to a gapped phase (confining and/or symmetry breaking) or to a gapless (Conformal Field Theory) phase



At fixed n_c (e.g. $n_c = 3$)



A way to detect a conformal behaviour is to look for fixed points of the beta-function of the theory

Our aim is to start from the upper edge of the conformal window and go down as much as possible using perturbation theory

We will be using Borel resummation techniques and other tools.

A large part of the talk will be a review of the main properties of the methods used, in particular about asymptotic series and their possible Borel resummation

Plan

Asymptotic series and Borel resummations

Large order behaviour: instantons and renormalons

Nature of the series of RG functions in
4d non-abelian gauge theories

Go back to the QCD conformal window

Results

Conclusions

Asymptotic Series and Borel Resummations

Recall basic mathematical properties of power series of holomorphic functions

If $f(\lambda)$ analytic at a point λ_0, in a small disc around λ_0

$$f(\lambda) = \sum_{n=0}^{\infty} c_n (\lambda - \lambda_0)^n$$

Radius of convergence $\quad R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|$

If λ_0 regular point of f $\Longleftrightarrow$ $R \neq 0$

If λ_0 singular point, we generally expect $R = 0$

Perturbative expansions in QFT are generally asymptotic with zero radius of convergence [Dyson, 1952]

A modern generalization of Dyson's argument can be obtained by considering QFT in euclidean signature in a path integral approach.

Consider the loopwise expansion in $\hbar$.

Observables of interest are the n-point correlation functions of local operators.

$$G^{(n)}(x_1, x_2, \ldots x_n; \hbar) = \int \mathcal{D}\phi \, \phi(x_1)\phi(x_2)\ldots\phi(x_n)\, e^{-S(\phi)/\hbar}$$

For $\hbar \to 0^-$ Green functions blow up $\longrightarrow$ $\hbar = 0$ is non-analytic

The loopwise expansion has zero radius of convergence!

Upon rescalings the loopwise expansion is equivalent to a proper expansion in the coupling constant: $S = S_0 + g\Delta S$

Mathematically perturbation theory does not make sense!

$$G^{(n)} = \int \mathcal{D}\phi\, \phi_1 \ldots \phi_n \sum_{p=0}^{\infty} \frac{(-\Delta S)^p g^p}{p!} e^{-S_0} \neq \sum_{p=0}^{\infty} g^p \int \mathcal{D}\phi\, \phi_1 \ldots \phi_n \frac{(-\Delta S)^p}{p!} e^{-S_0} = \sum_{p=0}^{\infty} g^p G_p^{(n)}$$

Exact answer

Perturbative answer

The series is **never** uniformly convergent and **not** allowed to exchange sum and integration

Asymptotic series

$Z(\lambda) \sim \sum_{n=0}^{\infty} Z_n \lambda^n$ such that

$$Z(\lambda) - \sum_{n=0}^{N} Z_n \lambda^n = \mathcal{O}(\lambda^{N+1}), \quad \text{as} \quad \lambda \to 0$$

Equality sign not allowed: in general $\sum_{n=0}^{\infty} Z_n \lambda^n = \infty$

Contrast with convergent series: $\forall \lambda \in \mathcal{D}$

$$N \to \infty \qquad Z(\lambda) - \sum_{n=0}^{N} Z_N \lambda_N \to 0$$

If λ is small enough, asymptotic series capture accurately the exact result

This explains the spectacular success of high precision computations in quantum field theories, such as the computation of g-2 in QED

Care has to be paid with asymptotic series, where summing more and more terms is **not** a good idea

Generally in QFT for $Z_n \sim n! a^n$ best accuracy for $Z(\lambda)$ is obtained by keeping

$$N_{\text{Best}} \approx \frac{1}{a\lambda} \quad \text{terms}$$

Optimal truncation

The higher the coupling the less terms you should compute

Intrinsic error associated to the asymptotic series $\sim e^{-\frac{1}{a\lambda}}$

E.g. $Z(\lambda)$ and $Z(\lambda) + f(\lambda)\exp(-c/\lambda^n)$ have **identical** asymptotic series

Extra input is needed to possibly resum the series using resummation methods

Most useful method is so called **Borel resummation**

Borel Resummation

Divide original series by a factorial term to get a convergent series

$$\mathcal{B}Z(t) = \sum_{n=0}^{\infty} \frac{Z_n}{n!} t^n \qquad\qquad Z_B(\lambda) = \int_0^\infty dt\, e^{-t} \mathcal{B}Z(t\lambda)$$

Warning: do not expand and exchange sum with integral

$$\int_0^\infty e^{-t} \mathcal{B}Z(\lambda t) = \int_0^\infty e^{-t} \sum_{n=0}^{\infty} \frac{Z_n}{n!} \lambda^n t^n \stackrel{?}{=} \sum_{n=0}^{\infty} Z_n \lambda^n \frac{1}{n!} \int_0^\infty e^{-t} t^n = \sum_{n=0}^{\infty} Z_n \lambda^n$$

Borel series has generally **finite** radius of convergence, so operation not allowed (and we would get back to the starting point!)

But $\mathcal{B}Z(t)$, if known, can be **analytically** continued over the whole complex t-plane (Borel plane) and $Z_B(\lambda)$ can then be calculable

The chances of success of this resummation procedure can be guessed from the form of the asymptotic series

Example: for $Z_n \sim n! a^n,$ $$\mathcal{B}Z(t) = \sum_n (at)^n \sim \frac{1}{1-at}$$

Singularity dangerous or not depending on the sign of a:

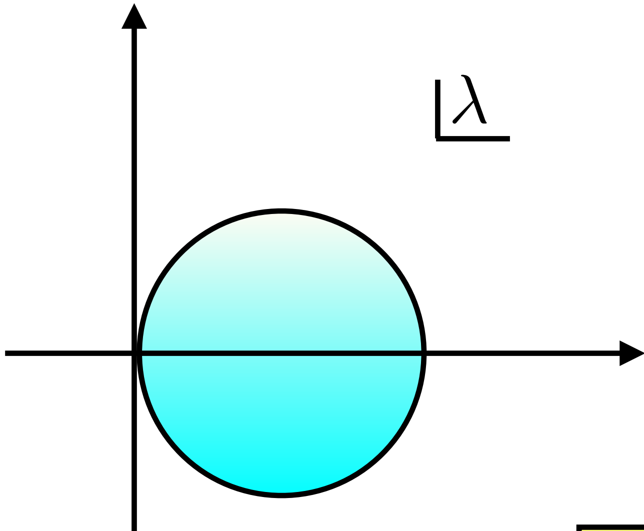
a<0 (alternating series) singularity for t < 0 😐

a>0 (same sign series) singularity for t>0 🙁

Sometimes an alternating series is said to be Borel resummable

This is misleading, because as we have seen there is no way to recover the original function with no further input, not being uniquely defined

Further input requires to know the analyticity properties of the function close to the origin



In general, if $Z(\lambda)$ is analytic in the light blue circle

[Early ‘900 theorem, rediscovered in 1980!]

$$Z_B(\lambda) = Z(\lambda)$$ 🙂

It is in general hard to argue the analyticity property of Z, which is unknown!

In special cases, like in certain 2d or 3d scalar field theories, perturbation theory has been proved to be Borel resummable

[Eckmann, Magnen and Seneor, 1975; Magnen and Seneor, 1977; MS, Spada, Villadoro, 2018]

Unfortunately interesting theories such as 4d gauge theories are not expected to be Borel resummable, at least in a simple way.

When a perturbative series is not Borel resummable because of a singularity in the real positive axis, one might deform the contour. The result however presents a non-perturbative ambiguity of order $\exp(-a/\lambda^n)$

If one is able to find a semi-classical instanton like configuration (and its whole series) leading to the same factors, one might hope to remove the ambiguity

A systematic way to proceed along these lines uses the theory of resurgence [Ecalle, 1981]

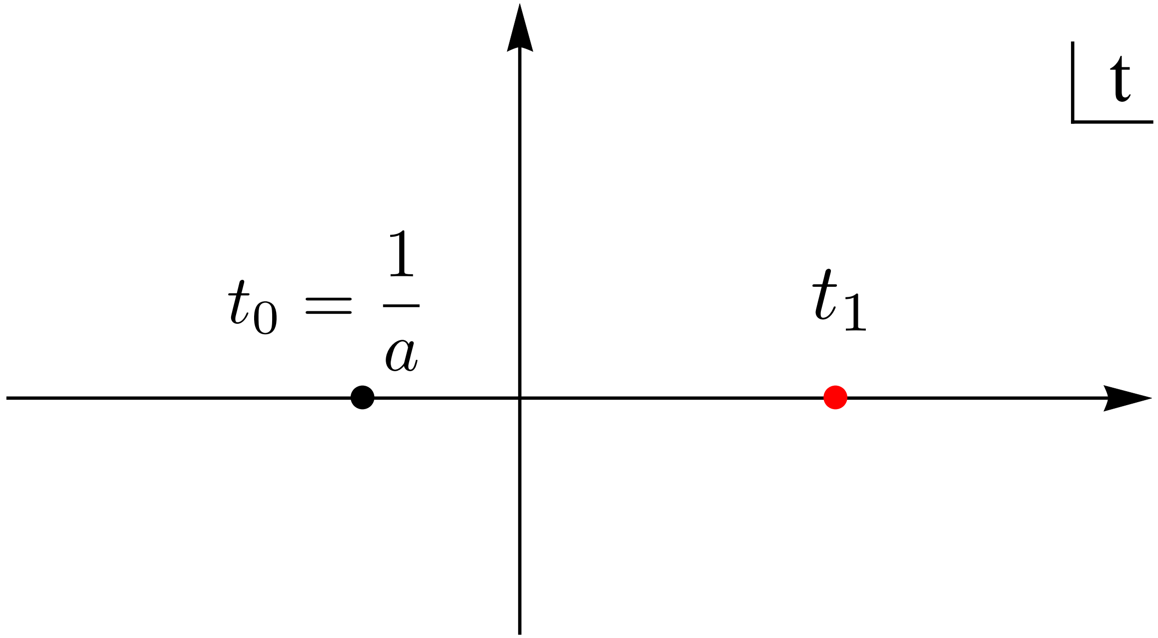
Some progress has been achieved in the last years using the theory of resurgence to relate classical and instanton series among each other but it is unfortunately hard to pursue this direction in generic QFTs

[Review by Aniceto, Basar, Schiappa, 1802.10441]

Moreover, non-ambiguous non-perturbative contributions might be present, which would be ``invisible” in perturbation theory

On the other hand Borel resummation (no need of resurgence) can be useful even when we do **not** know if the series is resummable or not, and we do not know its large order behaviour

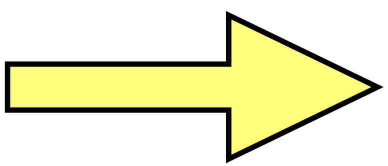
Indeed, while optimal truncation is given by the singularity closest to the origin in the Borel plane (wherever it is), the non-Borel summability of a series is determined by its singularities **on the positive real axis**



$$t_1 > |t_0|$$

Optimal truncation error $\sim e^{-\frac{|t_0|}{\lambda}}$

Indeterminacy in the Borel function $\sim e^{-\frac{t_1}{\lambda}} < e^{-\frac{|t_0|}{\lambda}}$



It is useful to Borel resum a series even when it is not resummable!

We will use this result when going back to the conformal window in QCD

Instantons and Renormalons

In QFT series are asymptotics because of factorial growth of their coefficients

There are two known sources for the factorial growth:

1) O(1) contributions of O(n!) Feynman diagrams

These are governed by **complex instantons** whose action determine the factor a. If **real instantons** are also present, perturbative series is not Borel resummable because of singularities appearing on the positive real axis of the Borel plane

[Vainhstein 1964, Lam 1968, Bender&Wu 1969, Lipatov 1976, …]

2) O(n!) contributions of O(1) Feynman diagrams

Generally expected in QFT with marginal couplings. Not known (if any) the real or complex semi-classical configurations associated. Singularities in the Borel function are called **renormalons** and are related to the RG flow of the marginal couplings

[Gross&Neveu 1974, Lautrup 1977,'t Hooft 1977]

Nature of the series of RG functions in 4d non-abelian gauge theories

Perturbative gauge coupling expansion of physical observables in 4d gauge theories is generally asymptotic and non-Borel resummable

Singularities in the positive real axis due both to instantons (or instanton-anti instanton pairs) and renormalons (more precisely IR renormalons)

Not much is known about RG functions such as beta-function or anomalous dimensions (i.e. of fermion bilinear)

These functions are renormalization scheme dependent so we expect that their large order behaviour depends on the scheme.

Established results only in the limit of large number of flavours

$$n_f \to \infty, \quad \alpha \to 0 \qquad \lambda = n_f \alpha \qquad \text{fixed}$$

In this limit the $\mathcal{O}(1/n_f)$ terms in $\beta(\lambda)$ and $\gamma(\lambda)$ can be computed exactly.

[Palanques-Mestre, Pascual, 1984; Gracey, 1996]

In particular in $\overline{\mathrm{MS}}$

$$\beta(\lambda) = \frac{2}{3}\lambda^2 + \frac{1}{n_f}\beta^{(1)}(\lambda) + \mathcal{O}\left(n_f^{-2}\right)$$

$$\gamma = \frac{1}{n_f}\gamma^{(1)}(\lambda) + \mathcal{O}\left(n_f^{-2}\right)$$

$\beta^{(1)}$, $\gamma^{(1)}$, known functions of λ

Interestingly enough, these are analytic at $\lambda = 0$, which implies a convergent series!

In other more physical schemes, such as on-shell or momentum subtraction, the series are divergent asymptotic

For finite number of flavours and colors, and in other large n limits ('t Hooft and Veneziano) the nature of the $\overline{\mathrm{MS}}$ series for β and γ is **unknown**

't Hooft: $n_c \to \infty$ $\quad \alpha \to 0 \quad \lambda = n_c \alpha$ fixed

Veneziano: $n_c \to \infty \quad n_f \to \infty \quad \alpha \to 0 \quad x = \frac{n_f}{n_c} \quad \lambda = n_c \alpha$ fixed

QCD Conformal Window

After this long preparatory journey, we can come back to our original problem of studying the conformal window in QCD

Of course we can't have a conformal window in the large flavour or large color limit, but we can and do have a conformal window in the Veneziano limit. So we will also consider this limit of QCD.

Previous (non-lattice) results

QCD

n_f^*	Method	References
12	Schwinger-Dyson	[Appelquist et al, hep-ph/9602385+…]
10	Exact RG	[Gies, Jaeckel, hep-ph/0507171+ …]
9	Perturbation Theory	[Ryttov,Shrock, 1607.06866]
10	Conformal Expansion	[Kim, Hong, Lee, 2001.02690]
9	Padè/Resummations	[Ryttov,Shrock,2018; Antipin et al, 2019]

Veneziano

x^*	Method	References
4	Schwinger-Dyson	[Appelquist et al,hep-ph/9602385+…]
4	Exact RG	[Gies, Jaeckel, hep-ph/0507171+ …]
3	Padè Pert. Theory	[Ryttov,Shrock,1710.06944]
$3.7 \div 4.2$	Bottom-up holographic	[Jarvinen, Kiritsis, 1112.1261+ …]

Previous results made predictions without an estimate of the associated error, so it is hard to assess their reliability.

We will argue that it is too optimistic to hope to reach the lower end of the conformal window using perturbation theory techniques only

Waiting for the conformal bootstrap, lattice is the only first principle method to address the QCD conformal window.

This is a hard problem in the lattice and no consensus yet on the value of n_f^*

[See e.g. review by DeGrand, 1510.05018]

In particular it is still debated whether QCD with $n_f = 12$ flavours is conformal or not

[Fodor et al, 1811.05024]

No lattice results are available in the Veneziano limit

Our approach

We start from the available 5-loop coefficients of the $\overline{\mathrm{MS}}$ β-function

[Baikov, Chetyrkin, Kuhn,1606.08659; Herzog et al, 1701.01404]
[Luthe et al, 1709.07718; Chetyrkin et al, 1709.08541]

We assume the worst case scenario in which the series diverges and is not-Borel resummable. Then we perform a Borel resummation using Padè approximants to reconstruct the whole Borel function (Padè-Borel method)

We then look for zeros of the Borel resummed beta-function

Remarks:

- It has been conjectured that conformality is lost because two nearby fixed points merge and disappear into the complex plane. In the Veneziano limit this implies that one (or more) double trace operator (presumably four fermion operators) approaches marginality

[Kaplan et al, 0905.4752]

- This is in agreement with earlier results using gap equations according to which in the IR $|\gamma_{\bar{\psi}\psi}| \approx 1$ for $n_f = n_f^*$

Recall that $\Delta_{\psi^4} \approx 2\Delta_{\psi^2}$ at large n

[Yamawaki, Bando, Matumoto,1986; Appelquist, Lane, Mahanta,1988; Cohen, Georgi,1989]

- For this reason and also because this is an observable measured on the lattice, we will also compute the mass anomalous dimensions at the IR fixed point (γ^*)

A fixed point can also be found using the **Banks-Zaks conformal expansion**

Assume that the coefficient of the first available order is accidentally small so that a "violation" of perturbation theory is allowed

$$\beta(\alpha) = \beta_0\alpha^2 + \beta_1\alpha^3 + O(\alpha^4) \qquad \alpha^* \approx -\frac{\beta_0}{\beta_1} \propto \epsilon$$

If $\beta_0 < 0$ and $\beta_1 > 0$ fixed point is IR stable [Caswell,1974; Banks,Zaks,1982]

$\epsilon \propto (n_f^+ - n_f)$ (QCD) $\epsilon \propto (x^+ - x)$ (Veneziano)

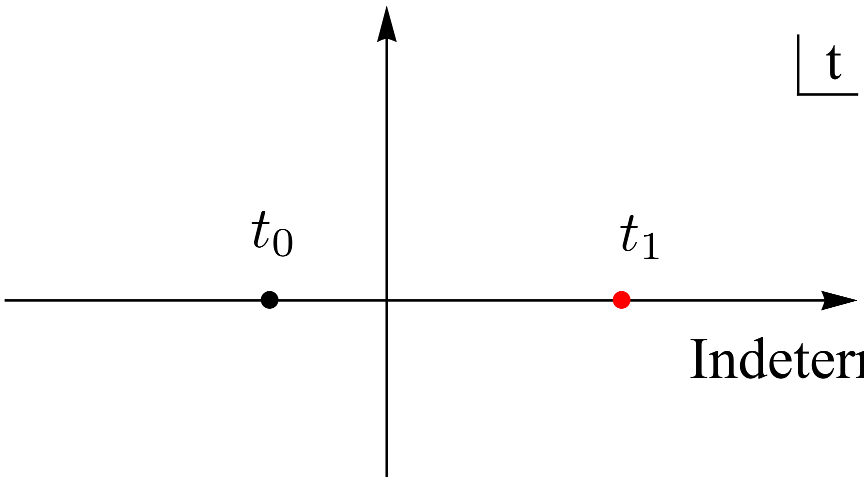
$$\alpha^* = \sum_{n=1}^{\infty} b_n \epsilon^n \qquad \gamma^* = \sum_{n=1}^{\infty} g_n \epsilon^n$$

g_n are scheme-independent coefficients

Conformal expansion series seems to be better behaved than ordinary coupling expansions, but its nature (convergent or not) follows that of the ordinary expansion

As double check, we also Borel resum the conformal expansion and compare the results with those obtained using the ordinary coupling series

Even if both the conformal and the ordinary series expansions would be non-Borel resummable, we expect an improvement with respect to optimal truncation



$t_1 > |t_0|$

Optimal truncation error $\sim e^{-\frac{|t_0|}{\lambda}}$

Indeterminacy in the Borel function $\sim e^{-\frac{t_1}{\lambda}} < e^{-\frac{|t_0|}{\lambda}}$

Perturbation theory of the 5-loop beta-function breaks down for $n_f \leq 13$

Loop Order	$n_f = 12$	$n_f = 13$	$n_f = 14$	$n_f = 15$	$n_f = 16$
2	$6 \cdot 10^{-2}$	$3.7 \cdot 10^{-2}$	$2.2 \cdot 10^{-2}$	$1.1 \cdot 10^{-2}$	$3.3 \cdot 10^{-3}$
3	$3.5 \cdot 10^{-2}$	$2.5 \cdot 10^{-2}$	$1.7 \cdot 10^{-2}$	$9.8 \cdot 10^{-3}$	$3.2 \cdot 10^{-3}$
4	$3.5 \cdot 10^{-2}$	$2.7 \cdot 10^{-2}$	$1.8 \cdot 10^{-2}$	$1.0 \cdot 10^{-2}$	$3.2 \cdot 10^{-3}$
5	$-5.5 \cdot 10^{-6}$	$3.2 \cdot 10^{-2}$	$1.9 \cdot 10^{-2}$	$1.0 \cdot 10^{-2}$	$3.2 \cdot 10^{-3}$

Table 1: Approximate values of the QCD Caswell-Banks-Zaks fixed point coupling a^* as a function of n_f obtained using different loop orders. The red color indicates values that should be taken with care, because of a possible breakdown of perturbation theory.

[from 2003.01742]

We do not expect that the perturbative $\overline{\mathrm{MS}}$ β -function (independently of whether it is convergent or divergent) can capture well the IR physics for any $n_f^* \leq n_f \leq n_f^+$

$\overline{\mathrm{MS}}$ is a mass-independent scheme and no mass scale can enter in RG functions

to all orders in perturbation theory

Non-perturbatively, however, contributions proportional to

$$\frac{\Lambda_{\mathrm{QCD}}}{\mu} \approx e^{\frac{1}{\beta_0 \alpha}} \qquad \text{could appear}$$

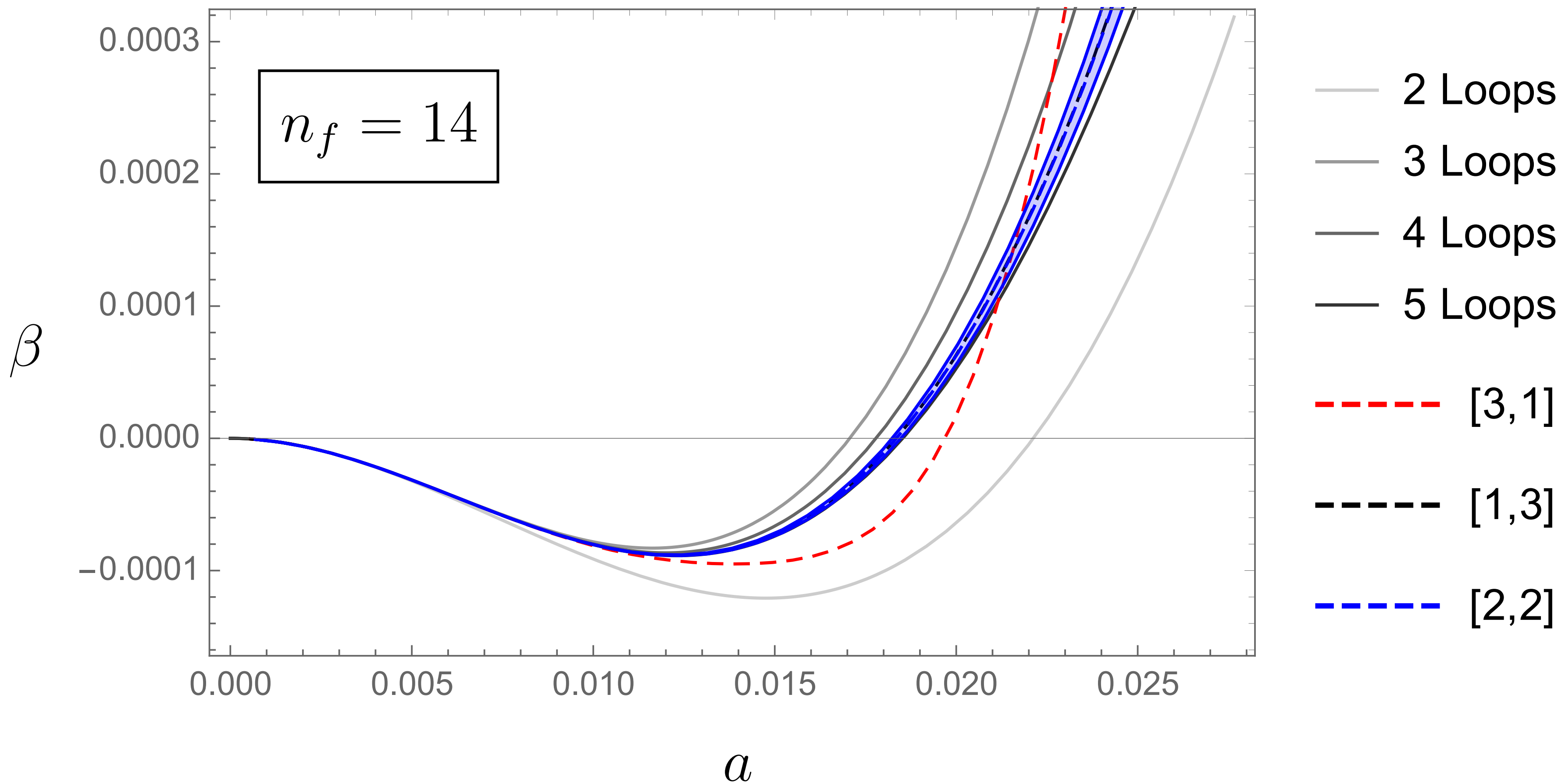
By dimensional analysis an irrelevant operator with UV dimension $4+k$ and dimensionless $\overline{\mathrm{MS}}$ coupling $\hat{h}$ can appear in β function as

$$\delta\beta \sim \hat{h} \left(\frac{\Lambda_{\mathrm{QCD}}}{\mu} \right)^k = \hat{h}\, e^{\frac{k}{\beta_0 \alpha}}$$

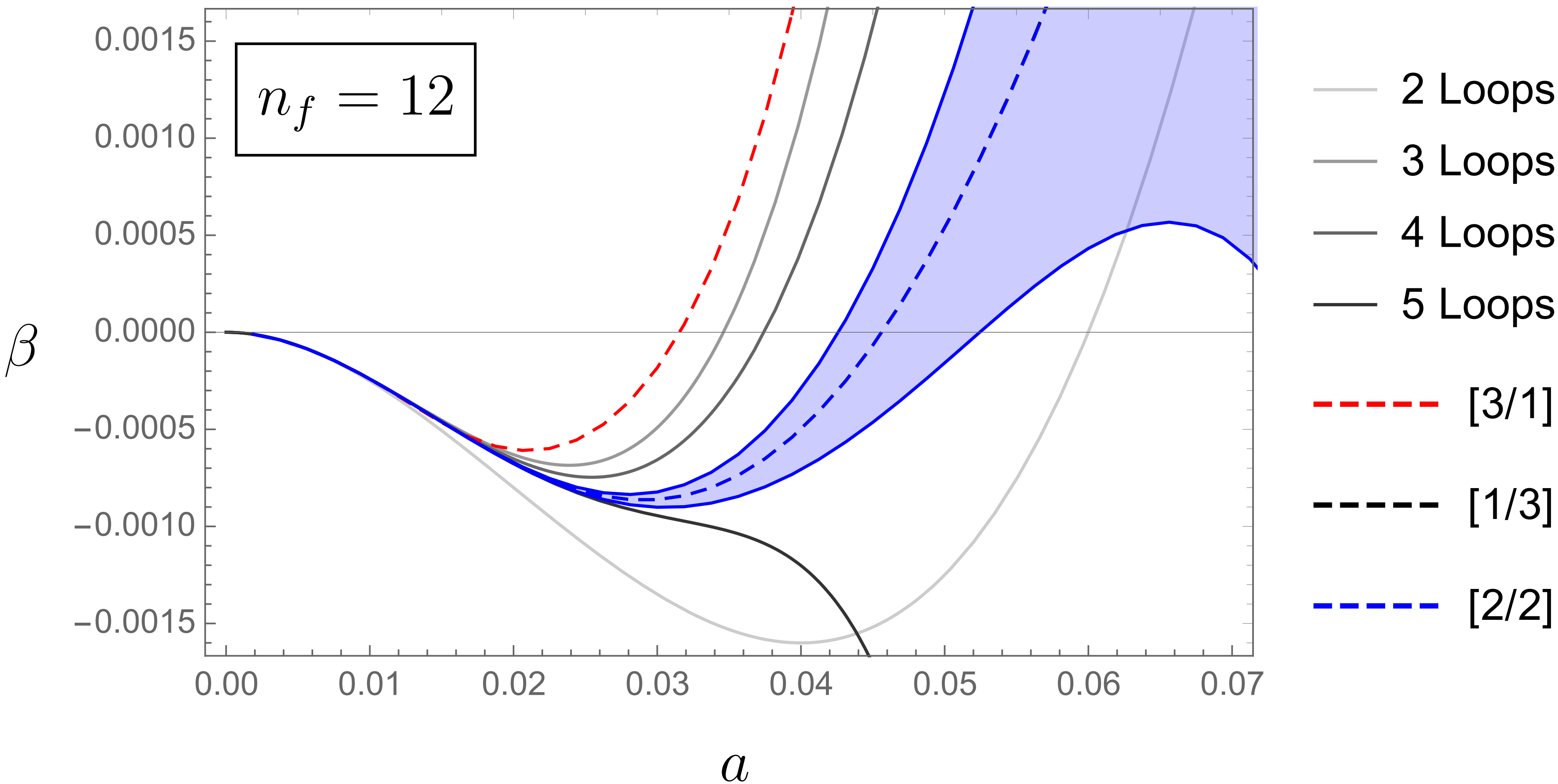
We expect leading contributions arise from k=2, due to four fermion operators. When these are sizable our analysis break down.

We add this systematic uncertainty to our error estimate, that includes the numerical error due to the reconstruction of the Borel function out of few perturbative coefficients

Results



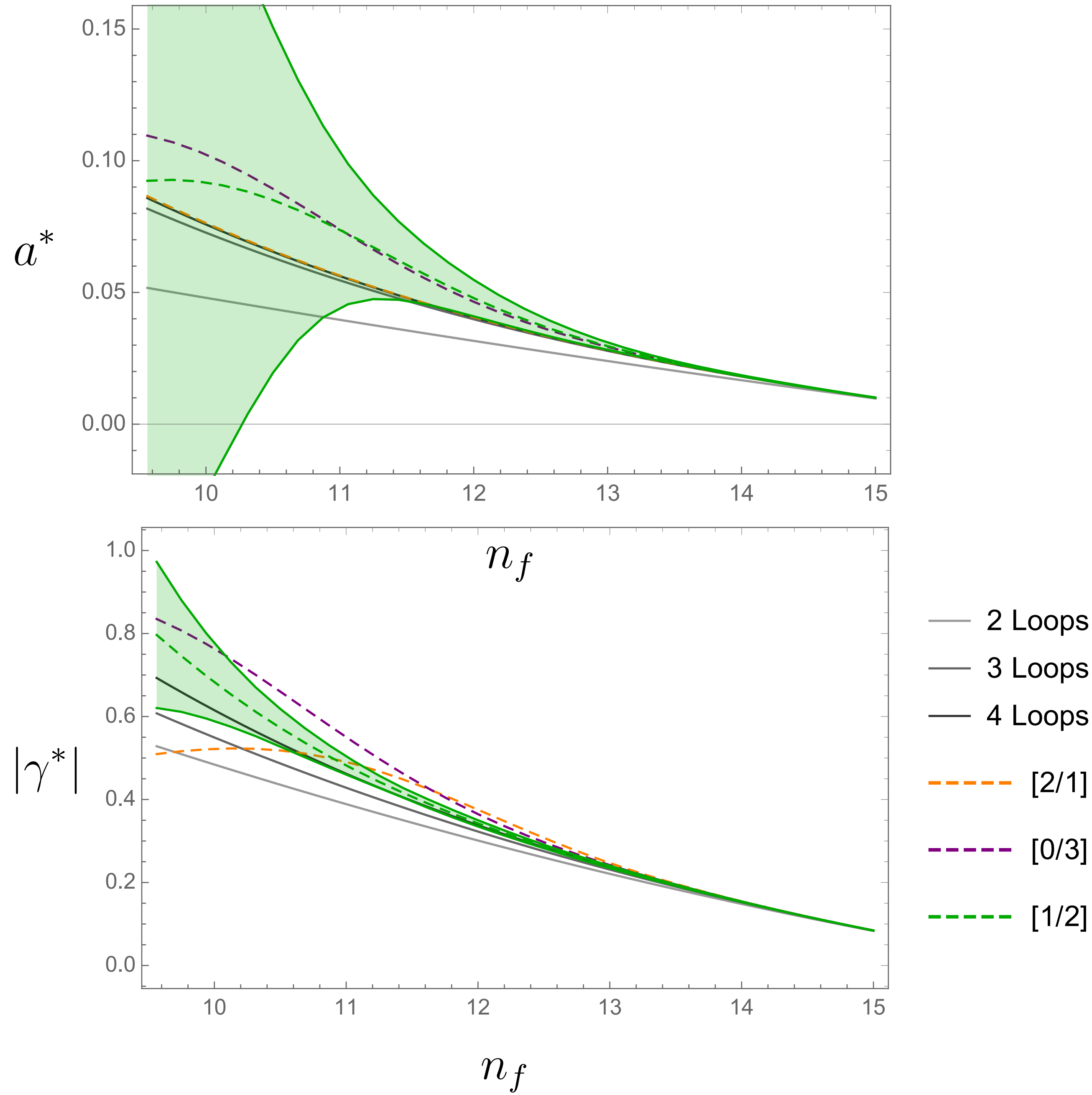
Large n_f exact results are better reproduced by [2/2] approximant

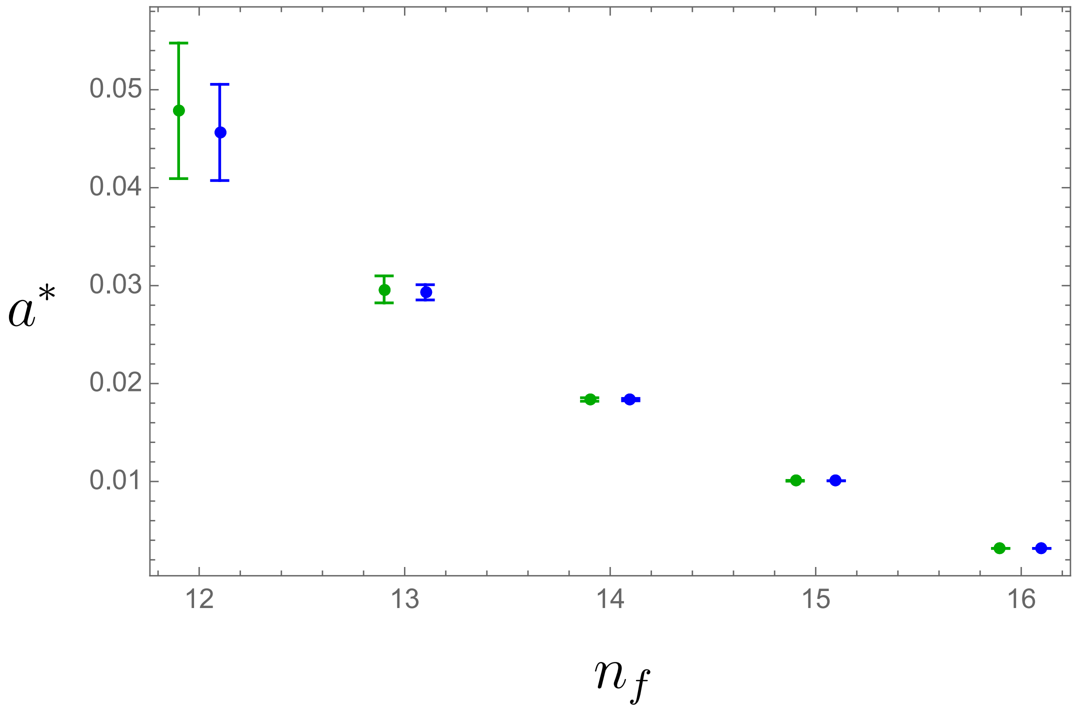


We see conformality at $n_f = 12$

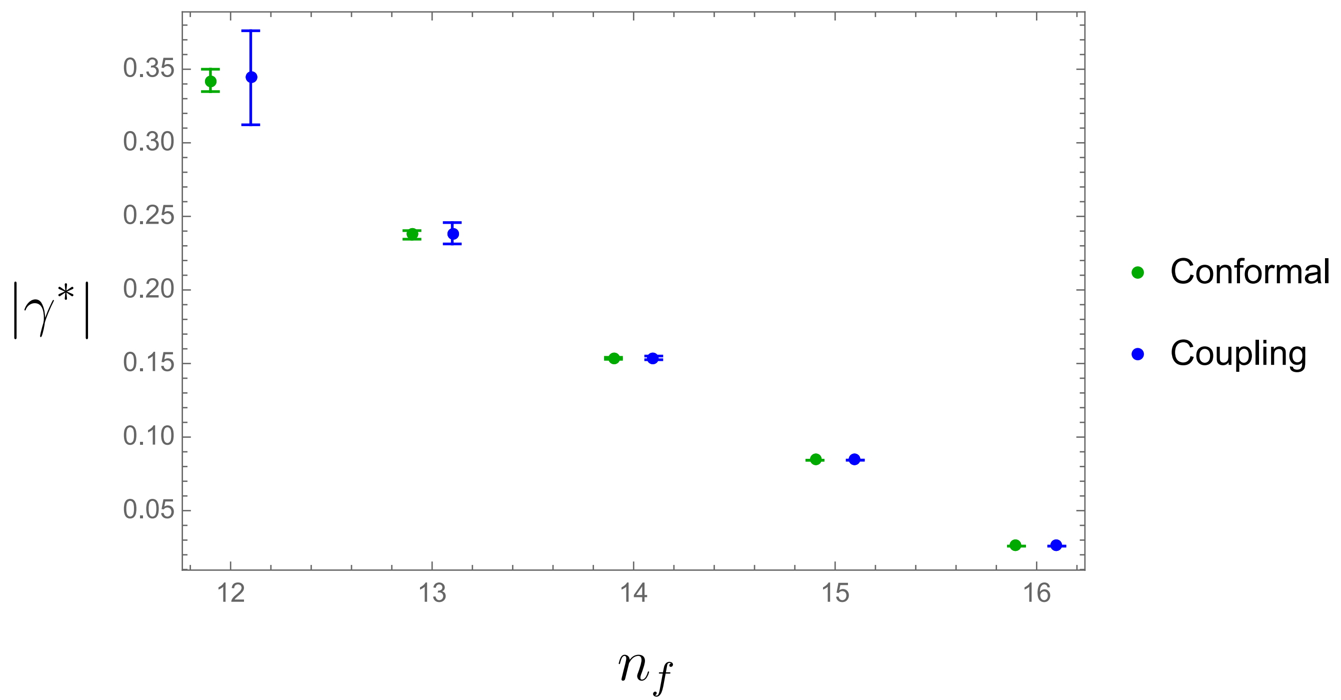
Banks-Zaks conformal expansion

Evidence of conformality also for $n_f = 11$

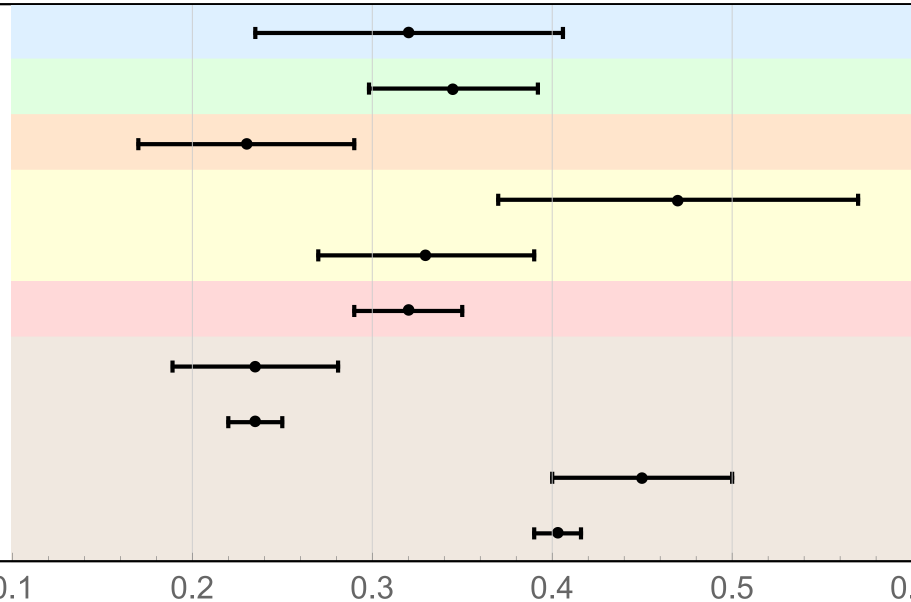




0.05
0.04
0.03
a^*
0.02
0.01
12
13
14
15
16
n_f



0.35
0.30
0.25
$|\gamma^*|$
0.20
0.15
0.10
0.05
12
13
14
15
16
n_f
Conformal
Coupling



Ref.	$\lvert\gamma^*\rvert$	Method
This work	0.320(85)	PB coupling
	0.345(47)	PB conformal
[44]	0.23(6)	Gradient flow
[45]	0.47(10)	Top. susceptibility
	0.33(6)	
[46]	0.32(3)	Dirac eigenmodes
[47]	0.235(46)	Finite-size scaling
[48]	0.235(15)	
[49]	0.45(5)	
[50]	0.403(13)	

Table 2: Comparison between the results of our Padé-Borel (PB) resummation for the mass-anomalous dimension for QCD with $n_f = 12$ –both using the coupling expansion and the Banks-Zaks conformal expansion, and averaging over all available Padé approximants in each case– and lattice results.

[44] - Carosso, Hasenfratz, Neil 1806.01385

[45] - Aoki et al. 1601.04687

[46] - Cheng, Hasenfratz, Petropoulos, Schaich, 1301.1355

[47] - Lombardo, Miura, Nunes da Silva, Pallante, 1410.0298

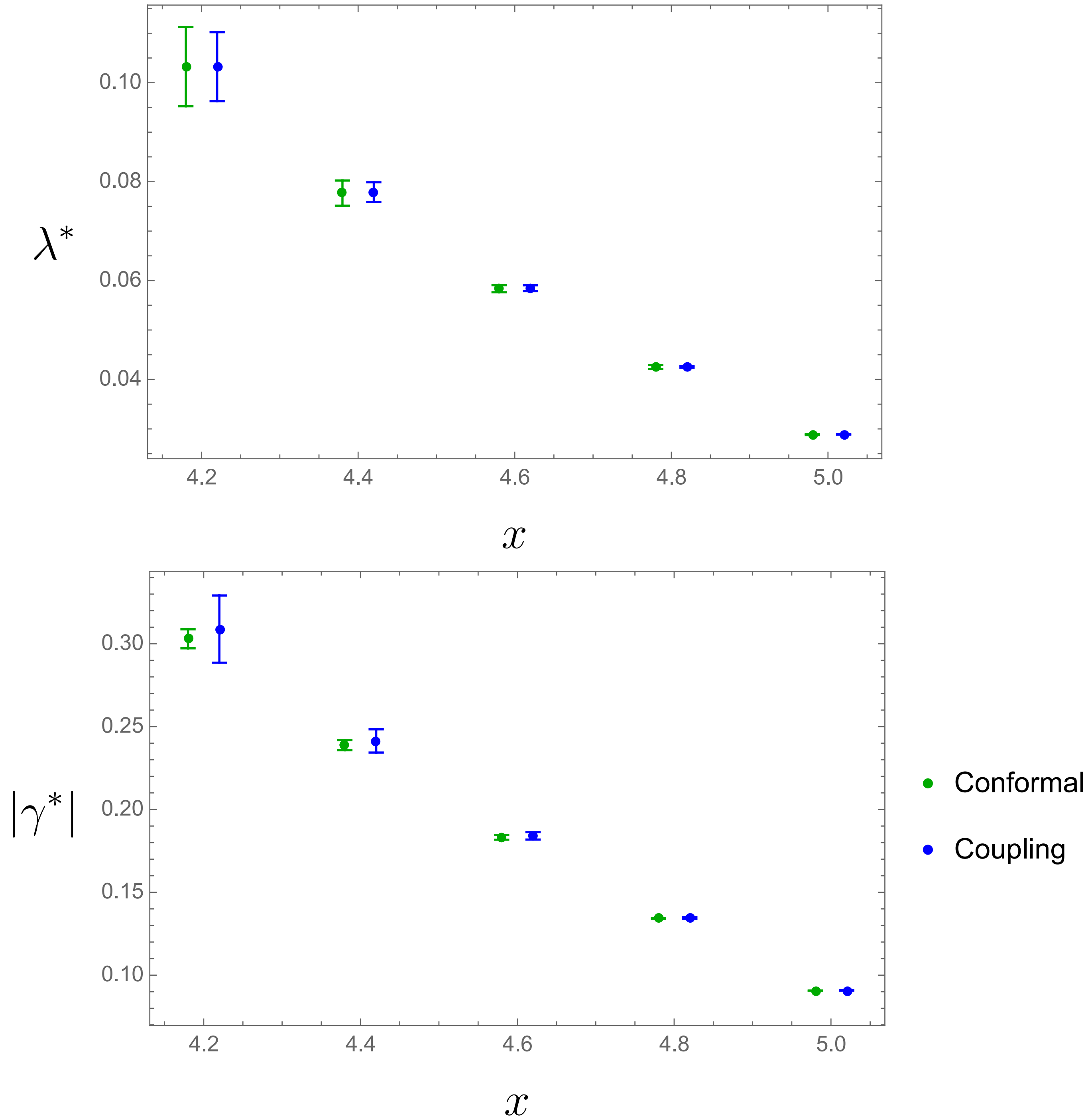
[48] - Cheng et al, 1401.0195

[49] - Aoki et al, 1207.3060

[50] - Appelquist et al, 1106.2148

[from 2003.01742]

Similar results have been derived in the Veneziano limit of QCD



Conclusions and future perspectives

Perturbation theory is the most notable analytical tool to study QFTs.

Its natural regime of validity, by definition, is weak coupling, where it allows us to do accurate predictions, despite the generic non-convergence of the associated series.

However, computing a sufficient number of higher order terms allows us to estimate the Borel transform of the function and resum the series.

If the series is Borel resummable we get an approximation of the exact result valid beyond perturbation theory.

This is not the case if the series is **not** Borel resummable, yet we might do **better** than perturbation theory.

We have shown this in the study of the conformal window of QCD-like theories. In particular, we have ample evidence that QCD with $n_f = 12$ flavours (a debated case in the lattice community) flows in the IR to a CFT

Note that we are not claiming that $n_f^* = 12$. Indeed we have some evidence that $n_f = 11$ is also conformal

Below that, perturbative methods break down and a genuine non-perturbative approach, like lattice or conformal bootstrap, is required

Future perspectives

1. Although our results point toward a divergent asymptotic nature of the $\overline{\mathrm{MS}}$ RG functions, it would be nice to firmly establish their nature.

2. Keep going in computing higher and higher loops, necessary for a better and better resummation and to get insights for point 1.

3. Generalizations to theories with other gauge theories or fermion representations (straightforward).

Thank You